Noname manuscript No.
(will be inserted by the editor)

A mixed FEM for the quad-curl eigenvalue problem

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Received: date / Accepted: date

Abstract The quad-curl problem arises in the study of the electromagnetic interior transmission problem and magnetohydrodynamics (MHD). In this paper, we study the quad-curl eigenvalue problem and propose a mixed method using edge elements. Assuming stringent regularity of the solution of the quad-curl source problem, we prove the convergence and show that the divergence-free condition can be bypassed.

Keywords quad-curl eigenvalue problem · spectral approximation · mixed finite element method · edge element

Mathematics Subject Classification (2000) 65N30 · 35Q60

1 Introduction

The quad-curl problem arises in inverse electromagnetic scattering theory for inhomogeneous media [13] and magnetohydrodynamics (MHD) equations [15]. To compute eigenvalues, one usually starts with the corresponding source problem. The construction of conforming finite elements with suitable regularity for the quad-curl problem is extremely technical and prohibitively expensive even if such finite elements exist.

In this paper, we propose a mixed finite element method for the quad-curl source problem. The major advantage of this approach lies in the fact that only curl-conforming edge elements are needed [14]. Then we employ the method to compute quad-curl eigenvalues. We prove convergence following [1]. Unlike the Maxwell eigenvalue problem, which has been studied extensively in the literature (see, for example [7] and [3]), there are few results on the quad-curl

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source problem. Recently Zheng et al. [15] propose a non-conforming finite element method. To the author's knowledge, this paper is the first numerical treatment of the quad-curl eigenvalue problem.

The rest of the paper is organized as follows. Section 2 contains some preliminaries. In Section 3, we propose a mixed finite element method for the the quad-curl problem and prove its convergence. In Section 4, we employ the mixed method for the quad-curl eigenvalue problem. In addition, we show that the divergence-free condition, which is usually treated using Lagrange multipliers, can be ignored for the eigenvalue problem.

2 Preliminaries

2.1 Function Spaces

Let $D \subset \mathbb{R}^3$ be a convex, simply connected Lipschitz polyhedral domain. The boundary of D is assumed to be connected with unit outward normal $\boldsymbol{\nu}$. We denote by $(\cdot,\cdot)$ the $L^2(D)$ inner product and by $\|\cdot\|$ the $L^2(D)$ norm. The variational approach we shall describe for the quad-curl problem requires several Hilbert spaces. We define

$$H^s(\mathrm{curl}; D) := \left\{\mathbf{u} \in L^2(D)^3 \mid \mathrm{curl}^j\, \mathbf{u} \in L^2(D)^3,\ 1 \le j \le s\right\}$$

equipped with the scalar product

$$(\mathbf{u},\mathbf{v})_{H^s(\mathrm{curl};D)} = (\mathbf{u},\mathbf{v}) + \sum_{j=1}^{s} (\mathrm{curl}^j\, \mathbf{u}, \mathrm{curl}^j\, \mathbf{v})$$

and the corresponding norm $\|\cdot\|_{H^s(\mathrm{curl};D)}$. We shall use the standard notation $H(\mathrm{curl}; D)$ when $s = 1$. Next we define

$$\begin{aligned} H_0(\mathrm{curl}; D) &:= \{\mathbf{u} \in H(\mathrm{curl}; D) \mid \mathbf{u} \times \boldsymbol{\nu} = 0 \text{ on } \partial D\}\,, \\ H_0^2(\mathrm{curl}; D) &:= \left\{\mathbf{u} \in H^2(\mathrm{curl}; D) \mid \mathbf{u} \times \boldsymbol{\nu} = 0 \text{ and } (\mathrm{curl}\mathbf{u}) \times \boldsymbol{\nu} = 0 \text{ on } \partial D\right\}. \end{aligned}$$

We also need the space $H(\mathrm{div}; D)$ of functions with square-integrable divergence defined by

$$H(\mathrm{div}; D) = \{\mathbf{u} \in L^2(D)^3 \mid \mathrm{div}\, \mathbf{u} \in L^2(D)\},$$

equipped with the scalar product

$$(\mathbf{u},\mathbf{v})_{H(\mathrm{div};D)} = (\mathbf{u},\mathbf{v}) + (\mathrm{div}\, \mathbf{u}, \mathrm{div}\, \mathbf{v})$$

and the corresponding norm $\|\cdot\|_{H(\mathrm{div},D)}$.

Taking the divergence-free condition into account, we define

$$\begin{aligned} X &= \{\mathbf{u} \in H(\mathrm{curl}; D) \cap H(\mathrm{div}; D) | \mathrm{div}\, \mathbf{u} = 0 \text{ in } D\}\,, \\ Y &= \{\mathbf{u} \in H_0(\mathrm{curl}; D) \cap H(\mathrm{div}; D) | \mathrm{div}\, \mathbf{u} = 0 \text{ in } D\}\,, \\ H(\mathrm{div}^0; D) &= \{\mathbf{u} \in H(\mathrm{div}; D) | \mathrm{div}\, \mathbf{u} = 0 \text{ in } D\}\,. \end{aligned}$$

For functions in Y, the following Friedrichs inequality holds.

Lemma 1 *(see, for example, Corollary 3.51 of [12]) Suppose that D is a bounded Lipschitz domain. If D is simply connected, and has a connected boundary, there is a constant $C \geq 0$ such that for every $\mathbf{u} \in Y$,*

$$\|\mathbf{u}\| \leq C\|\operatorname{curl} \mathbf{u}\|. \tag{1}$$

2.2 The edge element

We give a short introduction of edge elements [14]. We assume that the domain D is covered by a regular tetrahedral mesh. We denote the mesh by $\mathcal{T}_h$ where h is the maximum diameter of the elements in $\mathcal{T}_h$. Let P_k be the space of polynomials of maximum total degree k and $\tilde{P}_k$ the space of homogeneous polynomials of degree k. We define

$$R_k = (P_{k-1})^3 \oplus \{\mathbf{p} \in (\tilde{P}_k)^3 \mid \mathbf{x} \cdot \mathbf{p}(\mathbf{x}) = 0 \text{ for all } \mathbf{x} \in \mathbb{R}^3\}.$$

The curl-conforming edge element space [14] is given by

$$U_h = \{\mathbf{v} \in H(\text{curl}; D) \mid \mathbf{v}|_K \in R_k \text{ for all } K \in \mathcal{T}_h\}.$$

The $H_0(\text{curl}; D)$ conforming edge element space is simply

$$U_{0,h} = \{\mathbf{u}_h \in U_h \mid \boldsymbol{\nu} \times \mathbf{u}_h = 0 \quad \text{on } \partial D\}, \tag{2}$$

which can be easily obtained by setting the degrees of freedom associated with edges and faces on ∂D to zero. Let $\mathbf{r}_h \mathbf{u} \in U_h$ be the global interpolant. The following result holds.

Lemma 2 *(Lemma 5.38 of [12]) Suppose there are constants $\delta > 0$ and $p > 2$ such that $\mathbf{u} \in H^{1/2+\delta}(K)^3$ and $\operatorname{curl} \mathbf{u} \in L^p(K)^3$ for each $K \in \mathcal{T}_h$. Then $\mathbf{r}_h \mathbf{u}$ is well-defined and bounded.*

The following result provides error estimates for the interpolant.

Lemma 3 *(Theorem 5.41 of [12]) Let $\mathcal{T}_h$ be a regular mesh on D. Then*

(1) If $\mathbf{u} \in H^s(D)^3$ and $\operatorname{curl} \mathbf{u} \in H^s(D)^3$ for $1/2 + \delta \leq s \leq k$ for $\delta > 0$ then

$$\begin{aligned} &\|\mathbf{u} - \mathbf{r}_h \mathbf{u}\|_{L^2(D)^3} + \|\operatorname{curl}(\mathbf{u} - \mathbf{r}_h \mathbf{u})\|_{L^2(D)^3} \\ &\qquad \leq Ch^s \left(\|\mathbf{u}\|_{H^s(D)^3} + \|\operatorname{curl} \mathbf{u}\|_{H^s(D)^3}\right). \end{aligned} \tag{3}$$

(2) If $\mathbf{u} \in H^{1/2+\delta}(K)^3$, $0 < \delta \leq 1/2$ and $\operatorname{curl} \mathbf{u}|_K \in P_k$, then

$$\|\mathbf{u} - \mathbf{r}_h \mathbf{u}\|_{L^2(D)^3} \leq C \left(h_K^{1/2+\delta} \|\mathbf{u}\|_{H^{1/2+\delta}(K)^3} + h_K \|\operatorname{curl} \mathbf{u}\|_{L^2(K)^3}\right).$$

(3) If $\mathbf{u} \in H^s(D)^3$ and $\operatorname{curl} \mathbf{u} \in H^s(D)^3$ for $1/2 + \delta \leq s \leq k$ and $\delta > 0$, the following result holds

$$\|\operatorname{curl}(\mathbf{u} - \mathbf{r}_h \mathbf{u})\|_{L^2(D)^3} \leq Ch^s \|\operatorname{curl} \mathbf{u}\|_{H^s(D)^3}.$$

The following inverse inequality for edge elements will be useful in the forthcoming error analysis (see Section 3.6 of [10]).

Lemma 4 *Let $\mathcal{T}_h$ be a regular mesh for D. Then for $\mathbf{u}_h \in U_h$,*

$$\|\mathbf{u}_h\|_{H(\mathrm{curl};D)} \le Ch^{-1}\|\mathbf{u}_h\|$$

for some constant C independent of $\mathbf{u}_h$ and h.

2.3 The curl-curl problem

The curl-curl problem has been studied extensively in the literature. We just collect some results for later use and refer the readers to [10] and [11] for details. The problem is stated as follows. For $\mathbf{f} \in H(\mathrm{div}^0; D)$, find $\mathbf{u}$ such that

$$\begin{aligned}
\mathrm{curl}\,\mathrm{curl}\,\mathbf{u} &= \mathbf{f} && \text{in } D, && (4a)\\
\mathrm{div}\,\mathbf{u} &= 0 && \text{in } D, && (4b)\\
\mathbf{u}\times\boldsymbol{\nu} &= 0 && \text{on } \partial D. && (4c)
\end{aligned}$$

The mixed form is to find $(\mathbf{u}, p) \in H_0(\mathrm{curl}; D) \times H_0^1(D)$ such that

$$\begin{aligned}
(\mathrm{curl}\,\mathbf{u}, \mathrm{curl}\,\boldsymbol{\phi}) + (\mathrm{grad}\,p, \boldsymbol{\phi}) &= (\mathbf{f}, \boldsymbol{\phi}) && \text{for all } \boldsymbol{\phi} \in H_0(\mathrm{curl}; D), && (5a)\\
(\mathbf{u}, \mathrm{grad}\,q) &= 0 && \text{for all } q \in H_0^1(D). && (5b)
\end{aligned}$$

Let the finite element space for $H_0^1(D)$ be given by

$$S_h = \left\{p_h \in H_0^1(D) \mid p_h|_K \in P_k \text{ for all } K \in \mathcal{T}_h\right\}.$$

It follows that $\mathrm{grad} S_h \subset U_{0,h}$. The discrete Helmholtz decomposition can be defined via

$$U_{0,h} = Y_h \oplus \mathrm{grad} S_h$$

where Y_h is given by

$$Y_h = \{\mathbf{u}_h \in U_{0,h} \mid (\mathbf{u}_h, \mathrm{grad}\,\xi_h) = 0 \quad \text{for all } \xi_h \in S_h\}. \tag{6}$$

Then the discrete problem for (5) is to find $(\mathbf{u}_h, p_h) \in U_{0,h} \times S_h$ such that

$$\begin{aligned}
(\mathrm{curl}\,\mathbf{u}_h, \mathrm{curl}\,\boldsymbol{\phi}_h) + (\mathrm{grad}\,p_h, \boldsymbol{\phi}_h) &= (\mathbf{f}, \boldsymbol{\phi}_h) && \text{for all } \boldsymbol{\phi}_h \in U_{0,h}, && (7a)\\
(\mathbf{u}_h, \mathrm{grad}\,q_h) &= 0 && \text{for all } q_h \in S_h. && (7b)
\end{aligned}$$

Lemma 5 *(discrete Friedrichs inequality of Y_h) Let D be a bounded simply connected Lipschitz domain. There exists a positive constant C independent of h such that, for h small enough,*

$$\|\mathbf{u}_h\| \le C\|\mathrm{curl}\,\mathbf{u}_h\| \quad \textit{for all } \mathbf{u}_h \in Y_h.$$

Lemma 6 *The discrete problem (7) has a unique solution $(\mathbf{u}_h, p_h) \in U_{0,h} \times S_h$ with $p_h = 0$. In addition, if $(\mathbf{u}, p) \in H_0(\mathrm{curl}; D) \times H_0^1(D)$ is the solution of (4) with $p = 0$, there exists a constant C independent of h, $\mathbf{u}$, and $\mathbf{u}_h$ such that*

$$\|\mathbf{u} - \mathbf{u}_h\|_{H(\mathrm{curl})} \le C \inf_{\mathbf{v}_h \in U_{0,h}} \|\mathbf{u} - \mathbf{v}_h\|_{H(\mathrm{curl})}.$$

3 The Quad-curl Problem

The quad-curl problem is defined as follows. For $\mathbf{f} \in H(\mathrm{div}^0; D)$, find $\mathbf{u}$ such that

$$(\mathrm{curl})^4\, \mathbf{u} = \mathbf{f} \qquad \text{in } D, \tag{8a}$$

$$\mathrm{div}\, \mathbf{u} = 0 \qquad \text{in } D, \tag{8b}$$

$$\mathbf{u} \times \boldsymbol{\nu} = 0 \qquad \text{on } \Gamma, \tag{8c}$$

$$(\mathrm{curl}\, \mathbf{u}) \times \boldsymbol{\nu} = 0 \qquad \text{on } \Gamma. \tag{8d}$$

In this section, we first prove the well-posedness of the quad-curl problem. Then we propose a mixed formulation. To this end, we let V and W be given by

$$V := \left\{ \mathbf{u} \in H_0^2(\mathrm{curl}; D) \cap H(\mathrm{div}; D) \mid \mathrm{div}\, \mathbf{u} = 0 \right\}, \tag{9}$$

$$W := \{ \mathbf{u} \in H^2(\mathrm{curl}; D) \cap H(\mathrm{div}; D) \mid \mathrm{div}\mathbf{u} = 0 \}. \tag{10}$$

We define a bilinear form $\mathcal{C} : V \times V \to \mathbb{R}$ by

$$\mathcal{C}(\mathbf{u}, \mathbf{v}) := (\mathrm{curl}\,\mathrm{curl}\, \mathbf{u}, \mathrm{curl}\,\mathrm{curl}\, \mathbf{v}) \quad \text{ for all } \mathbf{u}, \mathbf{v} \in V. \tag{11}$$

Let $\mathbf{f} \in H(\mathrm{div}^0; D)$. The weak formulation for the quad-curl problem is to find $\mathbf{u} \in V$ such that

$$\mathcal{C}(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \quad \text{for all } \mathbf{v} \in V. \tag{12}$$

Theorem 1 *There exists a unique solution* $\mathbf{u} \in V$ *to* (12).

Proof Due to the fact that functions in V are divergence-free, using the Friedrichs inequality twice, we see that the bilinear form $\mathcal{C}$ is elliptic on V. Then Lax-Milgram lemma implies that there exists a unique solution $\mathbf{u}$ of (12) in V. $\square$

To the author's knowledge, there are no regularity results for the quad-curl problem in the literature. For Maxwell's equations, it is well-known that non-convexity leads to singularities, see [7] and [8]. For the biharmonic equation with clamped plate boundary conditions, convexity is sufficient for the solution to be in H^3 [9]. Therefore the mixed finite element method given in [6] for the corresponding biharmonic eigenvalue problem does not produce spurious modes. However, whether convexity is sufficient for the quad-curl solution to be in $H^3(\mathrm{curl}; D)$ is a non-trivial open problem. On the other hand, for biharmonic eigenvalue problems on non-convex domains, mixed finite methods compute spurious modes [4]. Thus non-convexity might lead to the failure of the mixed method for the quad-curl eigenvalue problem. For simplicity, we shall make the following assumption in the rest of the paper.

Assumption: The solution $\mathbf{u}$ of (12) belongs to $H^3(\mathrm{curl}; D)$.

Let $\boldsymbol{\phi} = \operatorname{curl}\operatorname{curl}\mathbf{u}$. We define

$$a(\mathbf{u},\mathbf{v}) = (\mathbf{u},\mathbf{v}), \quad b(\mathbf{u},\mathbf{v}) = -(\operatorname{curl}\mathbf{u},\operatorname{curl}\mathbf{v}).$$

The mixed formulation for the quad-curl problem can be stated as follows. For $\mathbf{f} \in H(\operatorname{div}^0; D)$, find $(\mathbf{u},\boldsymbol{\phi}) \in Y \times X$ such that

$$a(\mathbf{f},\mathbf{v}) + b(\boldsymbol{\phi},\mathbf{v}) = 0, \qquad \text{for all } \mathbf{v} \in Y, \tag{13a}$$

$$b(\mathbf{u},\boldsymbol{\psi}) = -(\boldsymbol{\phi},\boldsymbol{\psi}), \qquad \text{for all } \boldsymbol{\psi} \in X. \tag{13b}$$

In the following, we derive the equivalence of the above mixed formulation to the quad-curl problem. We employ a technique similar to that in Section 7.1 of [6] for the biharmonic equation.

The solution of the quad-curl problem is the solution of the following unconstrained minimization problem: Find $\mathbf{u}$ such that

$$J(\mathbf{u}) = \inf_{\mathbf{v}\in V} J(\mathbf{v}) \tag{14}$$

where

$$J(\mathbf{v}) = \frac{1}{2}\int_D |\operatorname{curl}^2\mathbf{v}|^2 \,\mathrm{d}x - \int_D \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}x. \tag{15}$$

This follows because (12) is the Euler-Lagrange equation for the minimization problem. Equivalently we consider the constrained minimization problem associated with the quadratic form

$$\mathcal{J}(\mathbf{v},\boldsymbol{\psi}) = \frac{1}{2}\int_D |\boldsymbol{\psi}|^2 \,\mathrm{d}x - \int_D \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}x \tag{16}$$

for $(\mathbf{v},\boldsymbol{\psi}) \in V \times L^2(D)^3$ such that $\operatorname{curl}^2\mathbf{v} = \boldsymbol{\psi}$.

We define

$$\mathcal{V} := \left\{(\mathbf{v},\boldsymbol{\psi}) \in Y \times L^2(D)^3 \mid \beta((\mathbf{v},\boldsymbol{\psi}),\boldsymbol{\mu}) = 0 \text{ for all } \boldsymbol{\mu} \in X\right\},$$

where

$$\beta((\mathbf{v},\boldsymbol{\psi}),\boldsymbol{\mu}) = \int_D \operatorname{curl}\mathbf{v}\cdot\operatorname{curl}\boldsymbol{\mu}\,\mathrm{d}x - \int_D \boldsymbol{\psi}\cdot\boldsymbol{\mu}\,\mathrm{d}x. \tag{17}$$

Thus the problem can be stated as: Find $(\mathbf{u},\boldsymbol{\phi}) \in \mathcal{V}$ such that

$$\int_D \boldsymbol{\phi}\cdot\boldsymbol{\psi}\,\mathrm{d}x = \int_D \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}x \quad \text{for all } (\mathbf{v},\boldsymbol{\psi}) \in \mathcal{V}.$$

Lemma 7 *The mapping*

$$(\mathbf{v},\boldsymbol{\psi}) \in \mathcal{V} \to \|\boldsymbol{\psi}\|$$

is a norm over $\mathcal{V}$. Furthermore, we have

$$\mathcal{V} := \left\{(\mathbf{v},\boldsymbol{\psi}) \in V \times L^2(D)^3 \mid \operatorname{curl}^2\mathbf{v} = \boldsymbol{\psi}\right\}.$$

Proof The lemma follows directly from the Friedrichs inequality. □

Theorem 2 *If* $\mathbf{u} \in V$ *is the solution of* (14)*, we have that*

$$\mathcal{J}(\mathbf{u}, \operatorname{curl}^2 \mathbf{u}) = \inf_{(\mathbf{v},\boldsymbol{\psi})\in\mathcal{V}} \mathcal{J}(\mathbf{v}, \boldsymbol{\psi}). \tag{18}$$

and $(\mathbf{u}, \operatorname{curl}^2 \mathbf{u}) \in \mathcal{V}$ *is the unique solution of* (18)*.*

Proof Since

$$((\mathbf{u}, \boldsymbol{\phi}), (\mathbf{v}, \boldsymbol{\psi})) \in \mathcal{V} \times \mathcal{V} \to \int_D \boldsymbol{\phi} \cdot \boldsymbol{\psi} \, \mathrm{d}x$$

is continuous and $\mathcal{V}$-elliptic,

$$(\mathbf{v}, \boldsymbol{\psi}) \in \mathcal{V} \to \int_D \mathbf{f} \cdot \mathbf{v} \, \mathrm{d}x$$

is continuous, the minimization problem of finding $(\mathbf{u}^*, \boldsymbol{\phi}) \in \mathcal{V}$ such that

$$\mathcal{J}(\mathbf{u}^*, \boldsymbol{\phi}) = \inf_{(\mathbf{v},\boldsymbol{\psi})\in\mathcal{V}} \mathcal{J}(\mathbf{v}, \boldsymbol{\psi})$$

has a unique solution that satisfies

$$\int_D \boldsymbol{\phi} \cdot \boldsymbol{\psi} \, \mathrm{d}x = \int_D \mathbf{f} \cdot \mathbf{v} \, \mathrm{d}x \quad \text{for all } (\mathbf{v}, \boldsymbol{\psi}) \in \mathcal{V}.$$

From Lemma 7, we see that $\mathbf{u}^* \in V$ and that $\operatorname{curl}^2 \mathbf{u}^* = \boldsymbol{\phi}$. We have

$$\int_D \operatorname{curl}^2 \mathbf{u} \cdot \operatorname{curl}^2 \mathbf{v} \, \mathrm{d}x = \int_D \mathbf{f} \cdot \mathbf{v} \, \mathrm{d}x.$$

Consequently $\mathbf{u}^*$ is the solution $\mathbf{u}$ of (14). □

Based on the above theorem, we define two solution operators $A : H(\operatorname{div}^0; D) \to X$ and $B : H(\operatorname{div}^0; D) \to Y$ for (13) such that

$$A\mathbf{f} = \boldsymbol{\phi}, \quad B\mathbf{f} = \mathbf{u}. \tag{19}$$

We can write (13) as

$$a(A\mathbf{f}, \mathbf{v}) + b(\mathbf{v}, B\mathbf{f}) = 0, \qquad \text{for all } \mathbf{v} \in X, \tag{20a}$$

$$b(A\mathbf{f}, \mathbf{q}) = -(\mathbf{f}, \mathbf{q}), \qquad \text{for all } \mathbf{q} \in Y. \tag{20b}$$

Next we consider the edge element method for the minimization problem. Let

$$\mathcal{V}_h = \{(\mathbf{v}_h, \boldsymbol{\psi}_h) \in Y_h \times X_h \mid \beta((\mathbf{v}_h, \boldsymbol{\psi}_h), \boldsymbol{\mu}_h) = 0 \quad \text{for all } \boldsymbol{\mu}_h \in X_h\},$$

where X_h is such that

$$U_h = X_h \oplus \operatorname{grad} S_h.$$

Note that $X_h \not\subset X$. Functions in X_h are said to be discrete divergence-free. The discrete problem corresponding to (16) is to find $(\mathbf{u}_h, \boldsymbol{\phi}_h) \in \mathcal{V}_h$ such that

$$\mathcal{J}(\mathbf{u}_h, \boldsymbol{\phi}_h) = \inf_{(\mathbf{v}_h,\boldsymbol{\psi}_h)\in\mathcal{V}_h} \mathcal{J}(\mathbf{v}_h, \boldsymbol{\psi}_h). \tag{21}$$

It is easy to see that the discrete problem (21) has a unique solution and the element $(\mathbf{u}_h, \boldsymbol{\phi}_h) \in \mathcal{V}_h$ satisfies

$$\int_D \boldsymbol{\phi}_h \cdot \boldsymbol{\psi}_h \,\mathrm{d}x = \int_D \mathbf{f} \cdot \mathbf{v}_h \,\mathrm{d}x \quad \text{for all } (\mathbf{v}_h, \boldsymbol{\psi}_h) \in \mathcal{V}_h. \tag{22}$$

Theorem 3 *Let* $(\mathbf{u}, \boldsymbol{\phi})$ *and* $(\mathbf{u}_h, \boldsymbol{\phi}_h)$ *be solutions of* (18) *and* (21)*, respectively, and assume that* $\mathbf{u} \in H^3(\mathrm{curl}; D)$*. There exists a constant* C *independent of* h *such that*

$$\begin{aligned}
&\|\mathrm{curl}\,\mathbf{u} - \mathrm{curl}\,\mathbf{u}_h\| + \|\mathrm{curl}^2\mathbf{u} - \boldsymbol{\phi}_h\| \le \\
&\quad C\Bigg(\inf_{(\mathbf{v}_h, \boldsymbol{\psi}_h) \in \mathcal{V}_h} \left(\|\mathrm{curl}\,\mathbf{u} - \mathrm{curl}\,\mathbf{v}_h\| + \|\mathrm{curl}^2\mathbf{u} - \boldsymbol{\psi}_h\| \right) \\
&\qquad + \inf_{\boldsymbol{\mu}_h \in X_h} \|\mathrm{curl}^2\,\mathbf{u} - \boldsymbol{\mu}_h\|_{H(\mathrm{curl})} \Bigg).
\end{aligned} \tag{23}$$

Proof Assuming that $\mathbf{u} \in H^3(\mathrm{curl}; D)$, we have

$$\int_D \mathrm{curl}(\mathrm{curl}^2\mathbf{u}) \cdot \mathrm{curl}\,\mathbf{v} \,\mathrm{d}x = \int_D \mathrm{curl}^2\mathbf{u} \cdot \mathrm{curl}^2\mathbf{v} \,\mathrm{d}x = \int_D \mathbf{f} \cdot \mathbf{v} \,\mathrm{d}x$$

for all $\mathbf{v} \in \mathcal{D}(D)^3$, the space of smooth functions with compact support in D. Hence for all $\mathbf{v} \in H_0(\mathrm{curl}; D)$, the following holds

$$\int_D \mathrm{curl}(\mathrm{curl}^2\mathbf{u}) \cdot \mathrm{curl}\,\mathbf{v} \,\mathrm{d}x = \int_D \mathbf{f} \cdot \mathbf{v} \,\mathrm{d}x. \tag{24}$$

Thus for any $\mathbf{v} \in H_0(\mathrm{curl}; D)$ and $\boldsymbol{\psi} \in L^2(D)^3$, we have

$$\beta\left((\mathbf{v}, \boldsymbol{\psi}), \mathrm{curl}^2\mathbf{u}\right) = \int_D \mathbf{f} \cdot \mathbf{v} \,\mathrm{d}x - \int_D \boldsymbol{\psi} \cdot \mathrm{curl}^2\mathbf{u} \,\mathrm{d}x.$$

For any $(\mathbf{v}_h, \boldsymbol{\psi}_h) \in \mathcal{V}_h$ and $\boldsymbol{\mu}_h \in X_h$, using the fact that $\beta((\mathbf{v}_h, \boldsymbol{\psi}_h), \boldsymbol{\mu}_h) = 0$, (24), and (22), we have

$$\begin{aligned}
&\beta\left((\mathbf{u}_h - \mathbf{v}_h, \boldsymbol{\phi}_h - \boldsymbol{\psi}_h), \mathrm{curl}^2\mathbf{u} - \boldsymbol{\mu}_h\right) \\
&= \int_D \mathrm{curl}(\mathbf{u}_h - \mathbf{v}_h) \cdot \mathrm{curl}(\mathrm{curl}^2\mathbf{u} - \boldsymbol{\mu}_h)\,\mathrm{d}x \\
&\qquad - \int_D (\boldsymbol{\phi}_h - \boldsymbol{\psi}_h) \cdot (\mathrm{curl}^2\mathbf{u} - \boldsymbol{\mu}_h)\,\mathrm{d}x \\
&= \int_D \mathrm{curl}(\mathbf{u}_h - \mathbf{v}_h) \cdot \mathrm{curl}\,\mathrm{curl}^2\mathbf{u}\,\mathrm{d}x - \int_D \mathrm{curl}(\mathbf{u}_h - \mathbf{v}_h) \cdot \mathrm{curl}\,\boldsymbol{\mu}_h\,\mathrm{d}x \\
&\qquad - \int_D (\boldsymbol{\phi}_h - \boldsymbol{\psi}_h) \cdot \mathrm{curl}^2\mathbf{u}\,\mathrm{d}x + \int_D (\boldsymbol{\phi}_h - \boldsymbol{\psi}_h) \cdot \boldsymbol{\mu}_h\,\mathrm{d}x \\
&= \int_D \mathrm{curl}(\mathbf{u}_h - \mathbf{v}_h) \cdot \mathrm{curl}\,\mathrm{curl}^2\mathbf{u}\,\mathrm{d}x - \int_D (\boldsymbol{\phi}_h - \boldsymbol{\psi}_h) \cdot \mathrm{curl}^2\mathbf{u}\,\mathrm{d}x \\
&= \int_D \mathbf{f} \cdot (\mathbf{u}_h - \mathbf{v}_h)\,\mathrm{d}x - \int_D (\boldsymbol{\phi}_h - \boldsymbol{\psi}_h) \cdot \mathrm{curl}^2\mathbf{u}\,\mathrm{d}x \\
&= \int_D \boldsymbol{\phi}_h \cdot (\boldsymbol{\phi}_h - \boldsymbol{\psi}_h)\,\mathrm{d}x - \int_D (\boldsymbol{\phi}_h - \boldsymbol{\psi}_h) \cdot \mathrm{curl}^2\mathbf{u}\,\mathrm{d}x \\
&= -\int_D (\mathrm{curl}^2\mathbf{u} - \boldsymbol{\phi}_h) \cdot (\boldsymbol{\phi}_h - \boldsymbol{\psi}_h)\,\mathrm{d}x.
\end{aligned} \tag{25}$$

On the other hand, for all $\boldsymbol{\mu}_h \in X_h$, one has

$$\begin{aligned}
\int_D \mathrm{curl}\mathbf{u}_h \cdot \mathrm{curl}\boldsymbol{\mu}_h \mathrm{d}\,x &= \int_D \boldsymbol{\phi}_h \cdot \boldsymbol{\mu}_h \mathrm{d}x, \\
\int_D \mathrm{curl}\mathbf{v}_h \cdot \mathrm{curl}\boldsymbol{\mu}_h \mathrm{d}\,x &= \int_D \boldsymbol{\psi}_h \cdot \boldsymbol{\mu}_h \mathrm{d}x.
\end{aligned}$$

Taking the difference and letting $\boldsymbol{\mu}_h = \mathbf{u}_h - \mathbf{v}_h$, we have

$$\int_D \mathrm{curl}(\mathbf{u}_h - \mathbf{v}_h) \cdot \mathrm{curl}(\mathbf{u}_h - \mathbf{v}_h)\mathrm{d}x = \int_D (\boldsymbol{\phi}_h - \boldsymbol{\psi}_h) \cdot (\mathbf{u}_h - \mathbf{v}_h)\mathrm{d}x,$$

which implies

$$\|\mathrm{curl}(\mathbf{u}_h - \mathbf{v}_h)\| \le C\|\boldsymbol{\phi}_h - \boldsymbol{\psi}_h\|, \tag{26}$$

where C is the constant in the discrete Friedrichs inequality.

Using the above inequality and (25), we have

$$\begin{aligned}
&\left|\int_D (\mathrm{curl}^2\mathbf{u} - \boldsymbol{\phi}_h) \cdot (\boldsymbol{\phi}_h - \boldsymbol{\psi}_h)\,\mathrm{d}x\right| \\
&= \left|\beta\left((\mathbf{u}_h - \mathbf{v}_h, \boldsymbol{\phi}_h - \boldsymbol{\psi}_h), \mathrm{curl}^2\mathbf{u} - \boldsymbol{\mu}_h\right)\right| \\
&\le \|\mathrm{curl}(\mathbf{u}_h - \mathbf{v}_h)\|\,\|\mathrm{curl}(\mathrm{curl}^2\mathbf{u} - \boldsymbol{\mu}_h)\| + \|\boldsymbol{\phi}_h - \boldsymbol{\psi}_h\|\,\|\mathrm{curl}^2\mathbf{u} - \boldsymbol{\mu}_h\| \\
&\le C\|\boldsymbol{\phi}_h - \boldsymbol{\psi}_h\|\,\|\mathrm{curl}(\mathrm{curl}^2\mathbf{u} - \boldsymbol{\mu}_h)\| + \|\boldsymbol{\phi}_h - \boldsymbol{\psi}_h\|\,\|\mathrm{curl}^2\mathbf{u} - \boldsymbol{\mu}_h\| \\
&\le C_1\|\boldsymbol{\phi}_h - \boldsymbol{\psi}_h\|\,\|\mathrm{curl}^2\mathbf{u} - \boldsymbol{\mu}_h\|_{H(\mathrm{curl})},
\end{aligned}$$

where $C_1 = \max\{C, 1\}$. Thus we have that

$$
\begin{aligned}
&\|\boldsymbol{\phi}_h - \boldsymbol{\psi}_h\|^2 \\
&= -\int_D (\boldsymbol{\phi}_h - \boldsymbol{\psi}_h) \cdot (\mathrm{curl}^2\mathbf{u} - \boldsymbol{\phi}_h)\,\mathrm{d}x + \int_D (\boldsymbol{\phi}_h - \boldsymbol{\psi}_h) \cdot (\mathrm{curl}^2\mathbf{u} - \boldsymbol{\psi}_h)\,\mathrm{d}x \\
&\le C_1\|\boldsymbol{\phi}_h - \boldsymbol{\psi}_h\|\,\|\mathrm{curl}^2\mathbf{u} - \boldsymbol{\mu}_h\|_{H(\mathrm{curl})} + \|\boldsymbol{\phi}_h - \boldsymbol{\psi}_h\|\,\|\mathrm{curl}^2\mathbf{u} - \boldsymbol{\psi}_h\|
\end{aligned}
$$

and hence

$$
\|\boldsymbol{\phi}_h - \boldsymbol{\psi}_h\| \le C_1\|\mathrm{curl}^2\mathbf{u} - \boldsymbol{\mu}_h\|_{H(\mathrm{curl})} + \|\mathrm{curl}^2\mathbf{u} - \boldsymbol{\psi}_h\|.
$$

Moreover, we have that

$$
\begin{aligned}
&\|\mathrm{curl}\,\mathbf{u} - \mathrm{curl}\,\mathbf{u}_h\| + \|\mathrm{curl}^2\mathbf{u} - \boldsymbol{\phi}_h\| \\
&\le \|\mathrm{curl}\,\mathbf{u} - \mathrm{curl}\,\mathbf{v}_h\| + \|\mathrm{curl}\,\mathbf{v}_h - \mathrm{curl}\,\mathbf{u}_h\| + \|\mathrm{curl}^2\mathbf{u} - \boldsymbol{\psi}_h\| + \|\boldsymbol{\psi}_h - \boldsymbol{\phi}_h\| \\
&\le \|\mathrm{curl}\,\mathbf{u} - \mathrm{curl}\,\mathbf{v}_h\| + \|\mathrm{curl}^2\mathbf{u} - \boldsymbol{\psi}_h\| + (1 + C)\|\boldsymbol{\psi}_h - \boldsymbol{\phi}_h\|,
\end{aligned}
$$

where we have used (26). Combining the above inequalities, we obtain

$$
\begin{aligned}
&\|\mathrm{curl}\,\mathbf{u} - \mathrm{curl}\,\mathbf{u}_h\| + \|\mathrm{curl}^2\mathbf{u} - \boldsymbol{\phi}_h\| \\
&\le \|\mathrm{curl}\,\mathbf{u} - \mathrm{curl}\,\mathbf{v}_h\| + \|\mathrm{curl}^2\mathbf{u} - \boldsymbol{\psi}_h\| \\
&\qquad + (1 + C)\left(C_1\|\mathrm{curl}^2\mathbf{u} - \boldsymbol{\mu}_h\|_{H(\mathrm{curl})} + \|\mathrm{curl}^2\mathbf{u} - \boldsymbol{\psi}_h)\|\right) \\
&\le \|\mathrm{curl}\,\mathbf{u} - \mathrm{curl}\,\mathbf{v}_h\| + (2 + C)\|\mathrm{curl}^2\mathbf{u} - \boldsymbol{\psi}_h\| + (1 + C)C_1\|\mathrm{curl}^2\,\mathbf{u} - \boldsymbol{\mu}_h\|_{H(\mathrm{curl})}.
\end{aligned}
$$

The proof is complete by taking the infimum of the right side over all $(\mathbf{v}_h, \boldsymbol{\psi}_h) \in \mathcal{V}_h$ and $\boldsymbol{\mu}_h \in X_h$. $\square$

Theorem 4 *Let $(\mathbf{u}, \boldsymbol{\phi})$ and $(\mathbf{u}_h, \boldsymbol{\phi}_h)$ solve (18) and (21), respectively. Let $\alpha(h) = C_1/h$ where C_1 is the constant in Lemma 4. Then there exists a constant C independent of the mesh size h such that*

$$
\begin{aligned}
&\|\mathrm{curl}\,\mathbf{u} - \mathrm{curl}\,\mathbf{u}_h\| + \|\mathrm{curl}^2\,\mathbf{u} - \boldsymbol{\phi}_h\| \\
&\quad \le C\left\{(1 + \alpha(h)) \inf_{\mathbf{v}_h \in Y_h} \|\mathrm{curl}\,\mathbf{u} - \mathrm{curl}\,\mathbf{v}_h\| + \inf_{\boldsymbol{\mu}_h \in X_h} \|\mathrm{curl}^2\,\mathbf{u} + \boldsymbol{\mu}_h\|_{H(\mathrm{curl})}\right\}.
\end{aligned}
$$

Proof Let $(\mathbf{v}_h, \boldsymbol{\psi}_h) \in \mathcal{V}_h$ and $\boldsymbol{\mu}_h \in X_h$. Writing $\mathbf{w}_h = \boldsymbol{\mu}_h + \boldsymbol{\psi}_h$, we have that $\beta((\mathbf{v}_h, \boldsymbol{\psi}_h), \mathbf{w}_h) = 0$, i.e.,

$$
\int_D \mathrm{curl}\,\mathbf{v}_h \cdot \mathrm{curl}\,\mathbf{w}_h\,\mathrm{d}x - \int_D \boldsymbol{\psi}_h \cdot \mathbf{w}_h\,\mathrm{d}x = 0.
$$

Using the fact that $\boldsymbol{\nu} \times \mathrm{curl}\,\mathbf{u} = 0$ on ∂D, we obtain

$$
\int_D \mathrm{curl}\,\mathrm{curl}\,\mathbf{u} \cdot \mathbf{w}_h\,\mathrm{d}x = \int_D \mathrm{curl}\,\mathbf{u} \cdot \mathrm{curl}\,\mathbf{w}_h\,\mathrm{d}x.
$$

Combining the above two equations, we have

$$
\int_D (\mathrm{curl}\,\mathrm{curl}\,\mathbf{u} - \boldsymbol{\psi}_h) \cdot \mathbf{w}_h\,\mathrm{d}x = \int_D \mathrm{curl}(\mathbf{u} - \mathbf{v}_h) \cdot \mathrm{curl}\,\mathbf{w}_h\,\mathrm{d}x.
$$

Hence we have that

$$\begin{aligned}\left|\int_D (\operatorname{curl}\operatorname{curl}\mathbf{u}-\boldsymbol{\psi}_h)\cdot\mathbf{w}_h\,\mathrm{d}x\right| &\le \|\operatorname{curl}\mathbf{u}-\operatorname{curl}\mathbf{v}_h\|\,\|\operatorname{curl}\mathbf{w}_h\|\\ &\le \alpha(h)\|\operatorname{curl}\mathbf{u}-\operatorname{curl}\mathbf{v}_h\|\,\|\mathbf{w}_h\|\end{aligned}$$

and

$$\begin{aligned}\|\mathbf{w}_h\|^2 &= \int_D (\boldsymbol{\mu}_h+\operatorname{curl}\operatorname{curl}\mathbf{u})\cdot\mathbf{w}_h\,\mathrm{d}x+\int_D(\boldsymbol{\psi}_h-\operatorname{curl}\operatorname{curl}\mathbf{u})\cdot\mathbf{w}_h\,\mathrm{d}x\\ &\le \|\boldsymbol{\mu}_h+\operatorname{curl}\operatorname{curl}\mathbf{u}\|\,\|\mathbf{w}_h\|+\alpha(h)\|\operatorname{curl}\mathbf{u}-\operatorname{curl}\mathbf{v}_h\|\,\|\mathbf{w}_h\|.\end{aligned}$$

From this inequality, we have that

$$\begin{aligned}\|\operatorname{curl}\operatorname{curl}\mathbf{u}-\boldsymbol{\psi}_h\| &\le \|\operatorname{curl}\operatorname{curl}\mathbf{u}+\boldsymbol{\mu}_h\|+\|\mathbf{w}_h\|\\ &\le 2\|\operatorname{curl}\operatorname{curl}\mathbf{u}+\boldsymbol{\mu}_h\|+\alpha(h)\|\operatorname{curl}\mathbf{u}-\operatorname{curl}\mathbf{v}_h\|,\end{aligned}$$

and thus,

$$\begin{aligned}&\inf_{(\mathbf{v}_h,\boldsymbol{\psi}_h)\in\mathcal{V}_h}\left(\|\operatorname{curl}\mathbf{u}-\operatorname{curl}\mathbf{v}_h\|+\|\operatorname{curl}\operatorname{curl}\mathbf{u}-\boldsymbol{\psi}_h\|\right)\\ &\quad\le (1+\alpha(h))\inf_{\mathbf{v}_h\in Y_h}\|\operatorname{curl}\mathbf{u}-\operatorname{curl}\mathbf{v}_h\|+2\inf_{\boldsymbol{\mu}_h\in X_h}\|\operatorname{curl}\operatorname{curl}\mathbf{u}+\boldsymbol{\mu}_h\|.\end{aligned}$$

Combination of this inequality and (23) completes the proof. □

Theorem 5 *Let* $(\mathbf{u},\boldsymbol{\phi})$ *and* $(\mathbf{u}_h,\boldsymbol{\phi}_h)$ *be the solutions of* (18) *and* (21), *respectively. Furthermore, assume that* $\operatorname{curl}^i\mathbf{u}\in H^s(D)^3, i=1,2,3$ *and* s *is the same as in Lemma 3. Then there exists a constant* C *independent of the mesh size* h *such that*

$$\|\operatorname{curl}\mathbf{u}-\operatorname{curl}\mathbf{u}_h\|+\|\operatorname{curl}^2\mathbf{u}-\boldsymbol{\phi}_h\|\le Ch^{s-1}\left(\|\mathbf{u}\|_{H^s(D)^3}+\|\operatorname{curl}\mathbf{u}\|_{H^s(D)^3}\right). \tag{27}$$

Proof We define the Fortin operator $\Pi_h: Y\to Y_h$ such that $\Pi_h\mathbf{u}$ is the first component $\mathbf{u}_h$ of (7) with $(\mathbf{f},\boldsymbol{\phi}_h)$ replaced by $(\operatorname{curl}\mathbf{u},\operatorname{curl}\boldsymbol{\phi}_h)$ (see Sec. 3 of [2]). According to Lemma 6, we have that

$$\|\mathbf{u}-\Pi_h\mathbf{u}\|_{H(\mathrm{curl})}\le C\inf_{\mathbf{v}_h\in U_{0,h}}\|\mathbf{u}-\mathbf{v}_h\|_{H(\mathrm{curl})}.$$

Using Lemma 3, we have that

$$\|\operatorname{curl}\mathbf{u}-\operatorname{curl}\Pi_h\mathbf{u}\|\le Ch^s\left(\|\mathbf{u}\|_{H^s(D)^3}+\|\operatorname{curl}\mathbf{u}\|_{H^s(D)^3}\right).$$

For $\mathbf{w}=\operatorname{curl}\operatorname{curl}\mathbf{u}$, we define the $H(\mathrm{curl};D)$ orthogonal projection $P_h: H(\mathrm{curl};D)\to U_h$ such that

$$(\operatorname{curl}(\mathbf{w}-P_h\mathbf{w}),\operatorname{curl}\boldsymbol{\phi}_h)+(\mathbf{w}-P_h\mathbf{w},\boldsymbol{\phi}_h)=0\quad\text{for all }\boldsymbol{\phi}_h\in U_h.$$

Then Cea's Lemma leads to the following estimate (see Sec. 7.2 of [12])

$$\|\mathbf{w}-P_h\mathbf{w}\|_{H(\mathrm{curl})}=\inf_{\boldsymbol{\mu}_h\in U_h}\|\mathbf{w}-\boldsymbol{\mu}_h\|_{H(\mathrm{curl})}.$$

Letting $\boldsymbol{\phi}_h = \operatorname{grad} \xi_h$ for $\xi_h \in S_h$, we find that $P_h\mathbf{w}$ is discrete divergence-free, i.e., $P_h\mathbf{w} \in X_h$. Thus we have that

$$\inf_{\boldsymbol{\mu}_h \in X_h} \|\operatorname{curl}^2 \mathbf{u} - \boldsymbol{\mu}_h\|_{H(\operatorname{curl})} \leq \inf_{\boldsymbol{\mu}_h \in U_h} \|\operatorname{curl}^2 \mathbf{u} - \boldsymbol{\mu}_h\|_{H(\operatorname{curl})}.$$

From Lemma 3, we have that

$$\inf_{\boldsymbol{\mu}_h \in U_h} \|\operatorname{curl}^2 \mathbf{u} - \boldsymbol{\mu}_h\|_{H(\operatorname{curl})} \leq Ch^s \left(\|\operatorname{curl}^2 \mathbf{u}\|_{H^s(D)^3} + \|\operatorname{curl}^3 \mathbf{u}\|_{H^s(D)^3}\right)$$

for some constants C independent of h. Using Theorem 4, we obtain that

$$\begin{aligned}
&\|\operatorname{curl} \mathbf{u} - \operatorname{curl} \mathbf{u}_h\| + \|\operatorname{curl}^2 \mathbf{u} - \boldsymbol{\phi}_h\| \\
&\leq C\left(1 + \frac{C_1}{h}\right) h^s \left(\|\mathbf{u}\|_{H^s(D)^3} + \|\operatorname{curl} \mathbf{u}\|_{H^s(D)^3}\right) \\
&\qquad + Ch^s \left(\|\operatorname{curl}^2 \mathbf{u}\|_{H^s(D)^3} + \|\operatorname{curl}^3 \mathbf{u}\|_{H^s(D)^3}\right) \\
&\leq Ch^{s-1} \left(\|\mathbf{u}\|_{H^s(D)^3} + \|\operatorname{curl} \mathbf{u}\|_{H^s(D)^3}\right) \\
&\qquad + Ch^s \left(\|\operatorname{curl}^2 \mathbf{u}\|_{H^s(D)^3} + \|\operatorname{curl}^3 \mathbf{u}\|_{H^s(D)^3}\right) \\
&\leq Ch^{s-1} \left(\|\mathbf{u}\|_{H^s(D)^3} + \|\operatorname{curl} \mathbf{u}\|_{H^s(D)^3}\right).
\end{aligned}$$

□

We now use the theory from Section 2.3 to prove an L^2-norm convergence result for $\mathbf{u} - \mathbf{u}_h$. Of course, since we are using edge elements of the first kind [14], the convergence rate in L^2 cannot be better than the convergence rate in $H(\operatorname{curl})$. So nothing would be gained from a duality argument.

Theorem 6 *Under the conditions of Theorem 5, there exists a constant C independent of* $\mathbf{u}$, $\mathbf{u}_h$ *and* h *such that*

$$\|\mathbf{u} - \mathbf{u}_h\| \leq Ch^{s-1} \left(\|\mathbf{u}\|_{H^s(D)^3} + \|\operatorname{curl} \mathbf{u}\|_{H^s(D)^3}\right).$$

Proof Let $\mathbf{v}_h \in Y_h$ be the first component of the solution of (7) with $\mathbf{f} = \operatorname{curl}\operatorname{curl}\mathbf{u}$ so that $\mathbf{u}$ is the exact solution. By Lemma 6 and Lemma 3, we have that

$$\|\mathbf{u} - \mathbf{v}_h\|_{H(\operatorname{curl};D)} \leq Ch^s \left(\|\mathbf{u}\|_{H^s(D)^3} + \|\operatorname{curl} \mathbf{u}\|_{H^s(D)^3}\right). \tag{28}$$

Then, using the triangle inequality and the discrete Friedrichs inequality in Lemma 5, we have that

$$\begin{aligned}
\|\mathbf{u} - \mathbf{u}_h\| &\leq \|\mathbf{u} - \mathbf{v}_h\| + \|\mathbf{v}_h - \mathbf{u}_h\| \\
&\leq \|\mathbf{u} - \mathbf{v}_h\| + C\|\operatorname{curl}(\mathbf{v}_h - \mathbf{u}_h)\| \\
&\leq C(\|\mathbf{u} - \mathbf{v}_h\|_{H(\operatorname{curl};D)} + \|\operatorname{curl}(\mathbf{u} - \mathbf{u}_h)\|).
\end{aligned}$$

Combination of Theorem 5 and (28) completes the proof. □

4 The Quad-curl Eigenvalue Problem

The quad-curl eigenvalue problem is to find λ and $\mathbf{u}$ such that

$$(\mathrm{curl})^4\,\mathbf{u} = \lambda\mathbf{u} \quad \text{in } D, \tag{29a}$$

$$\mathrm{div}\mathbf{u} = 0 \quad \text{in } D, \tag{29b}$$

$$\mathbf{u} \times \boldsymbol{\nu} = 0 \quad \text{on } \partial D, \tag{29c}$$

$$(\mathrm{curl}\,\mathbf{u}) \times \boldsymbol{\nu} = 0 \quad \text{on } \partial D. \tag{29d}$$

We call λ a quad-curl eigenvalue and $\mathbf{u}$ the associated eigenfunction. Due to the well-posedness of the quad-curl problem, we can define an operator $T : L^2(D)^3 \to L^2(D)^3$ such that $T\mathbf{f} = \mathbf{u}$ for (12). It is obvious that T is self-adjoint. Furthermore, because of the compact imbedding of V into $L^2(D)^3$, T is a compact operator.

The weak formulation is to find $(\lambda, \mathbf{u}) \in \mathbb{R} \times V$ such that

$$\mathcal{C}(\mathbf{u}, \mathbf{q}) = \lambda(\mathbf{u}, \mathbf{q}) \quad \text{for all } \mathbf{q} \in V. \tag{30}$$

It is clear that λ is an eigenvalue satisfying (30) if and only if $\mu = 1/\lambda$ is an eigenvalue of T.

Lemma 8 *There is an infinite discrete set of quad-curl eigenvalues $\lambda_j > 0, j = 1, 2, \ldots$ and corresponding eigenfunctions $\mathbf{u}_j \in V$, $\mathbf{u}_j \neq \mathbf{0}$ such that* (30) *is satisfied and $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ Furthermore*

$$\lim_{j\to\infty} \lambda_j = \infty.$$

The eigenfunctions satisfy $(\mathbf{u}_j, \mathbf{u}_l)_{L^2(D)^3} = 0$ if $j \neq l$.

Proof Applying the Hilbert-Schmidt theory (see, for example, Theorem 2.36 of [12]), we immediately have the above theorem. □

Using the Helmholtz decomposition, we can easily obtain the following result. Thus we omit its proof.

Lemma 9 *The quad-curl eigenvalues coincide with the non-zero eigenvalues of the following problem. Find $(\lambda, \mathbf{u}) \in \mathbb{R} \times H_0^2(curl; D)$ such that*

$$\mathcal{C}(\mathbf{u}, \mathbf{q}) = \lambda(\mathbf{u}, \mathbf{q}) \quad \textit{for all } \mathbf{q} \in H_0^2(curl; D). \tag{31}$$

Then the quad-curl eigenvalue problem in mixed form can be written as: Find $\lambda \in \mathbb{R}$, $(\mathbf{0}, \mathbf{0}) \neq (\mathbf{u}, \boldsymbol{\phi}) \in W \times X$ satisfying

$$a(\boldsymbol{\phi}, \mathbf{v}) + b(\mathbf{v}, \mathbf{u}) = 0, \quad \text{for all } \mathbf{v} \in X, \tag{32a}$$

$$b(\boldsymbol{\phi}, \mathbf{q}) = -\lambda(\mathbf{u}, \mathbf{q}), \quad \text{for all } \mathbf{q} \in Y. \tag{32b}$$

It is easy to see that if $(\lambda, (\mathbf{u}, \boldsymbol{\phi}))$ is an eigenpair of (32), then $\lambda B\mathbf{u} = \mathbf{u}, \mathbf{u} \neq \mathbf{0}$, i.e., $(\lambda, \mathbf{u})$ is a quad-curl eigenpair. If $\lambda B\mathbf{u} = \mathbf{u}, \mathbf{u} \neq \mathbf{0}$, then there exists $\boldsymbol{\phi} \in X$ such that $(\lambda, (\mathbf{u}, \boldsymbol{\phi}))$ is an eigenpair of (32).

The mixed finite element method for the quad-curl problem can be stated as follows. For $\mathbf{f} \in H(\mathrm{div}^0; D)$, find $A_h\mathbf{f} \in Y_h$, $B_h\mathbf{f} \in X_h$ such that

$$a(A_h\mathbf{f}, \mathbf{v}_h) + b(\mathbf{v}_h, B_h\mathbf{f}) = 0, \qquad \text{for all } \mathbf{v}_h \in X_h, \tag{33a}$$
$$b(A_h\mathbf{f}, \mathbf{q}_h) = -(\mathbf{f}, \mathbf{q}_h), \qquad \text{for all } \mathbf{q}_h \in Y_h. \tag{33b}$$

From Theorems 5 and 6, we have that

$$\|(B - B_h)\mathbf{f}\| \le Ch^{s-1}\left(\|B\mathbf{f}\|_s + \|\mathrm{curl} B\mathbf{f}\|_s\right), \tag{34}$$
$$\|(A - A_h)\mathbf{f}\| \le Ch^{s-1}\left(\|B\mathbf{f}\|_s + \|\mathrm{curl} B\mathbf{f}\|_s\right). \tag{35}$$

In the following, we assume that $\|\mathbf{u}\|_s \le C\|\mathbf{f}\|$ and $\|\mathrm{curl}\,\mathbf{u}\|_s \le C\|\mathbf{f}\|$ holds for the quad-curl problem and some constant C. Note that when $s = 2$, the above regularity results are a consequence of Lemma 1 and the fact that $\mathbf{u}$ is the solution of the quad-curl problem. Thus we have the norm convergence

$$\lim_{h\to 0} \|B - B_h\| = 0 \quad \text{and} \quad \lim_{h\to 0} \|A - A_h\| = 0.$$

The discrete eigenvalue problem is to find $\lambda_h \in \mathbb{R}$, $(\mathbf{u}_h, \boldsymbol{\phi}_h) \in Y_h \times X_h$ such that

$$a(\boldsymbol{\phi}_h, \mathbf{v}_h) + b(\mathbf{v}_h, \mathbf{u}_h) = 0, \qquad \text{for all } \mathbf{v}_h \in X_h, \tag{36a}$$
$$b(\boldsymbol{\phi}_h, \mathbf{q}_h) = -\lambda_h(\mathbf{u}_h, \mathbf{q}_h), \qquad \text{for all } \mathbf{q}_h \in Y_h. \tag{36b}$$

Theorem 7 *The discrete quad-curl eigenvalues of* (36) *coincide with the non-zero eigenvalues of the following problem. Find* $\lambda_h \in \mathbb{R}$ *and* $\mathbf{u}_h \in U_{0,h}, \boldsymbol{\phi}_h \in U_h$ *such that*

$$(\boldsymbol{\phi}_h, \mathbf{v}_h) - (\mathit{curl}\,\mathbf{v}_h, \mathit{curl}\,\mathbf{u}_h) = 0, \qquad \textit{for all } \mathbf{v}_h \in U_h, \tag{37a}$$
$$(\mathit{curl}\,\boldsymbol{\phi}_h, \mathit{curl}\,\mathbf{q}_h) = -\lambda_h(\mathbf{u}_h, \mathbf{q}_h), \qquad \textit{for all } \mathbf{q}_h \in U_{0,h}. \tag{37b}$$

Proof We write

$$\mathbf{u}_h = \mathbf{u}_h^0 + \nabla\varphi_h, \quad \mathbf{u}_h^0 \in Y_h, \varphi_h \in S_h.$$

Letting $\mathbf{q}_h = \nabla\xi_h$ in (37b), we have that

$$0 = (\boldsymbol{\phi}_h, \mathrm{curl}\,\nabla\xi_h) = -\lambda_h(\mathbf{u}_h, \nabla\xi_h) = \lambda_h(\nabla\varphi_h, \nabla\xi_h) \quad \text{for all } \xi_h \in S_h.$$

Then either $\lambda_h = 0$ or $(\nabla\varphi_h, \nabla\xi_h) = 0$ for all $\xi_h \in S_h$. It is clear that if $\lambda_h \neq 0$, we have $(\nabla\varphi_h, \nabla\xi_h) = 0$ for all $\xi_h \in S_h$, which implies $\nabla\varphi_h = 0$. Thus $\mathbf{u}_h = \mathbf{u}_h^0$, which is discrete divergence-free. $\square$

Let μ be a non-zero eigenvalue of B. The ascent α of $\mu - B$ is defined as the smallest integer such that $N((\mu - B)^\alpha) = N((\mu - B)^{\alpha+1})$, where N denotes the null space. Let $m = \dim N((\mu - B)^\alpha)$ be the algebraic multiplicity of μ. The geometric multiplicity of μ is $\dim N(\mu - B)$. Note that since B is self-adjoint, the two multiplicities are same. Then there are m eigenvalues of B_h, $\mu_1(h), \ldots, \mu_m(h)$ such that

$$\lim_{h\to 0} \mu_j(h) = \mu, \quad \text{for } j = 1, \ldots, m. \tag{38}$$

Theorem 8 *Let $\lambda = 1/\mu$ be an exact quad-curl eigenvalue with multiplicity m and $\lambda_{j,h}, j = 1, \ldots, m$ be the corresponding computed eigenvalues. Then we have that*

$$|\lambda - \lambda_{j,h}| \le Ch^{2s-2} \tag{39}$$

for some constant C.

Proof From (34) and (35), we have that

$$\|(B - B_h)\| \le Ch^{s-1} \quad \text{and} \quad \|(A - A_h)\| \le Ch^{s-1}.$$

Then the theorem is verified using Theorem 11.1 of [1]. □

Acknowledgements The research was supported in part by NSF grant DMS-1016092. The author would like to thank Prof. Peter Monk for helpful discussions.

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