Notes on the Schwartz Alternating Method for Partition of Unity FEM

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Abstract. We consider discretization on overlapping non-matching grids for elliptic equations by using the Schwartz alternating (SA) method. We investigate the dependence between the angle of partition of unity (PU) subspaces, the condition number of the stiffness matrix, and the rate of convergence. The aim of the paper is to find strategies to choose optimal or quasi-optimal partition of unity set of functions for PU discretizaions for elliptic problems on overlapping non-matching grids.

Keywords. Partition of Unity, Finite Element Method, Overlapping Grids, Schwartz Alternating Method.

AMS (MOS) subject classification: 65N30.

1 Introduction

The partition of unity finite element method (PUFEM) was first proposed by Melenk and Babuska [8]. As one of the meshless method, the PUFEM has the ability to include a prior knowledge and to construct finite element spaces of any desired regularity. Since then, it has been applied to treat various problems including the elastically supported beam, bimaterial interface cracks, linear diffusion and convection problems, etc. e.g., see [2, 10, 9]. Extension and analysis of the PUFEM can be found, but not restrict to, in [5, 7]

In this paper we study the application of the Schwartz alternating method to the PUFEM function spaces. In particular, we present some preliminary study of the strengthened Cauchy-Schwartz (SCS) constant between different PUFEM spaces, the condition number of the global stiffness matrix and the convergence of the Schwartz alternating method.

2 Preliminary

Definition 2.1 *Let $\Omega \in R^n$ be an open set, $\{\Omega_i\}$ be an open cover of Ω satisfying a pointwise overlap condition:*

$$\exists M \in N \quad \forall x \in \Omega \quad card\{i | x \in \Omega_i\} \leq M.$$

Let $\{\phi_i\}$ be a Lipschitz partition of unity subordinate to the cover $\{\Omega_i\}$ satisfying

$$\begin{aligned} &supp\phi_i \subset closure(\Omega_i) \quad \forall i, \\ &\sum_i \phi_i \equiv 1 \quad on \quad \Omega, \\ &\|\phi_i\|_{L^\infty(R^n)} \leq C_\infty, \\ &\|\nabla\phi_i\|_{L^\infty(R^n)} \leq \frac{C_G}{diam\Omega_i}, \end{aligned}$$

where C_∞, C_G are two constants. Then $\{\phi_i\}$ is called a (M, C_∞, C_G) partition of unity subordinate to the cover $\{\Omega_i\}$. The partition of unity $\{\phi_i\}$ is said to be of degree $m \in N_0$ if $\{\phi_i\} \subset C^m(R^n)$. The covering sets $\{\Omega_i\}$ are called patches.

Definition 2.2 *Let $\{\Omega_i\}$ be an open cover of $\Omega \subset R^n$ and let $\{\phi_i\}$ be a (M, C_∞, C_G) partition of unity subordinate to $\{\Omega_i\}$. Let $V_i \subset H^1(\Omega_i \cap \Omega)$ be given. Then the space*

$$V := \sum_i \phi_i V_i = \{\sum_i \phi_i v_i | v_i \in V_i\} \subset H^1(\Omega)$$

is called the PUFEM space. The PUFEM space V is said to be of degree $m \in N_0$ if $V \subset C^m(\Omega)$. The space V_i are referred to as the local approximation spaces.

In [7], The authors consider the thin overlapping regions. Let each Ω_i is partitioned by quasi-uniform triangulation τ^{h_i} of maximal mesh size h_i. With each triangulation τ^{h_i}, associate a finite element spaces $V_i \subset H^r(\Omega_i)$. Let $u \in H^r(\Omega)$, and let $m_i \geq 1$ denote the additional degree of smoothness of u on Ω_i. Assume optimal approximation property on subdomains: For any $u \in H^{m_i+r}(\Omega_i)$, there exist a $v_h \in V_i$ such that

$$\sum_{k=0}^{r} h_i^k |u - v_h|_{k,\Omega_i} \leq c h_i^{m_i+r} \|u\|_{m_i+r,\Omega_i}.$$

Also assume that

$$|\nabla^k \phi_i| \leq c d_i^{-k}, 1 \leq k \leq r$$

where d_i is the minimal overlapping size of Ω_i with its neighboring subdomains.

Theorem 2.1 *(Huang-Xu, 2002) If the overlapping size $d_i \geq ch_i$, then for $0 \leq k \leq r$,*

$$\inf_{v_h \in V} \|u - v_h\|_{k,\Omega} \leq C \sum_{i=1}^{p} h_i^{m_i+r-k} \|u\|_{m_i+r,\Omega_i},$$

for any $u \in H^r(\Omega) \cap_{i=1}^p H^{m_i+r}(\Omega_i)$.

For $u \in H^2(\Omega)$ and H^1 conforming finite element space, we have

$$\begin{aligned} \inf_{v_h \in V} \|u - v_h\|_{0,\Omega} &\leq C \sum_{i=1}^{p} h_i^2 \|u\|_{2,\Omega_i}, \\ \inf_{v_h \in V} \|u - v_h\|_{1,\Omega} &\leq C \sum_{i=1}^{p} h_i^1 \|u\|_{2,\Omega_i}, \end{aligned}$$

where we set $k = 0$ and $k = 1$ in the above theorem and $m_i = 1$, $r = 1$.

The Schwartz's alternating method can be found in most domain decomposition books or finite element books (e.g., [6]). Consider the variational problem

$$a(u, v) =< f, v > \quad \forall v \in H. \tag{2.1}$$

Here $a(\cdot, \cdot)$ is the inner product of the Hilbert space H and $\|\cdot\|$ is the corresponding norm. Let H be the direct sum of two subspaces

$$H = V \oplus W, \tag{2.2}$$

and the solving (2.1) on either V or W is assumed to be easy. Let $u_0 \in H$. When u_{2i} is already determined, find $v_{2i} \in V$ such that

$$a(u_{2i} + v_{2i}, v_{2i}) =< f, v > \quad \forall v \in V. \tag{2.3}$$

Set $u_{2i+1} = u_{2i} + v_{2i}$. When u_{2i+1} is already determined, find $w_{2i+1} \in W$ such that

$$a(u_{2i+1}, w_{2i+1}) =< f, w > \quad \forall w \in W. \tag{2.4}$$

Then set $u_{2i+2} = u_{2i+1} + w_{2i+1}$.

The so-called strengthened Cauchy inequality is crucial in the analysis.

Theorem 2.2 Convergence Theorem. *Assume that there is a constant $\gamma \leq 1$ such that for the inner product in H*

$$|a(v, w)| \leq \gamma \|v\| \|w\| \quad \text{for } v \in V, w \in W. \tag{2.5}$$

Then, for the Schwartz alternating iteration, we have that the error reduction is given by

$$\|u_{k+1} - u\| \leq \gamma \|u_k - u\| \quad \text{for } k \geq 1. \tag{2.6}$$

A proof of the theorem can be found in [6]. The constant γ is often called the strengthened Cauchy-Schwartz (SCS) constant (in the energy inner product).

3 The SCS Constant

In this section, we consider the SCS constant (in L^2- inner product) for two PU finite element spaces and investigate its relation with the condition number of the global stiffness matrix for the model elliptic problem. For simplicity, we consider two overlapping subdomains Ω_1 and Ω_2. To make our presentation more precise, we define the following PUFEM subspace.

Definition 3.1 *Let $\{\Omega_i\}, i = 1, 2$ be an open cover of $\Omega \subset R^n$ and let $\{\phi_i\}, i = 1, 2$ be a (M, C_∞, C_G) partition of unity subordinate to $\{\Omega_i\}, i = 1, 2$. Let $V_i \subset H(\Omega_i \cap \Omega), i = 1, 2$ be given. Then the space*

$$V_i := \{\phi_i v_j | v_j \in V_i\} \subset H(\Omega), \quad i = 1, 2$$

is called the PUFEM subspace. Note $H(\Omega)$ is a Hilbert space of functions defined on Ω.

Remark 3.1 *From the computational point of of view, it is desirable to have the linearly independence of the collection of all PU base functions associated with the subdomains. It is easy to justify that, for thin overlapping case when the boundary of one mesh lies inside the other mesh (it is not aligned with the other mesh), that $\phi_1 V_1 \cap \phi_2 V_2 = \{\mathbf{0}\}$, which gives the linearly independence of the entire collection of all PU base functions.*

When the meshes are more regular on the overlapping region, the linear independence property was studied in [3]. Let Ω_1, Ω_2 be an overlapping covering of a two dimensional polygonal domain (see Figure 1). We assume that Ω_1 and Ω_2 are partitioned by quasi-uniform finite element triangulations of maximal mesh size h_1 and h_2, and $\Omega_0 = \Omega_1 \cap \Omega_2$ is a strip-type domain of width $d = O(h_1)$. Let $\{\phi_1, \phi_2\}$ be a PU subordinated to the

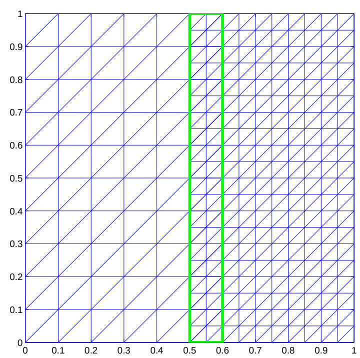

Figure 1: Overlapping Meshes in Ω.

covering $\{\Omega_1, \Omega_2\}$ of Ω, and assume that

$$\phi_1 + \phi_2 = 1, \quad 0 \leq \phi_1, \phi_2 \leq 1, \quad \|\nabla \phi_i\|_{L^\infty} = 1/d \tag{3.7}$$

$$V := \sum_{i=1,2} \phi_i V_i = \{ \sum_{i=1,2} \phi_i v_i, v_i \in V_i(\Omega_i)\}.$$

The following result, proved in [3], justifies the construction of a global stiffness matrix for the PUFEM space.

Theorem 3.2 *Suppose that $V_k, k = 1, 2$ are piecewise linear finite element spaces of functions which are zero on the boundary of Ω and ϕ_1, ϕ_2 are also piecewise linear functions. Then,*

$$\phi_1 V_1 \cap \phi_2 V_2 = \{\mathbf{0}\}$$

If $\{\psi_i\}$ and $\{\eta_j\}$ are bases for V_1 and V_2 respectively, then $\{\phi_1 \psi_i, \phi_2 \eta_j\}$ is a basis for $V = \phi_1 V_1 + \phi_2 V_2$.

If the spaces V_k do not have zero boundary conditions, it is enough to take ϕ_1, ϕ_2 to be piecewise quadratic. To estimate the condition number of the PUFEM space described in the above theorem we need to introduce the SCS angle γ in the L^2 inner product of $\phi_1 V_1$ and $\phi_2 V_2$. Let $\gamma \in (0, 1)$ be the SCS constant (the cosine of the angle between the subspaces $\phi_1 V_1$ and $\phi_2 V_2$) in the L^2 inner product, i.e.,

$$\gamma := \sup_{u_i \in \phi_i V_i} \frac{|(u_1, u_2)|}{\|u_1\| \|u_2\|} \leq 1.$$

Theorem 3.3 *Suppose that $V_k, k = 1,2$ are finite element spaces of continuous piecewise linear functions which are zero on the boundary of $\Omega \cap \Omega_k$ and ϕ_1, ϕ_2 are piecewise linear partition of unity functions subordinated to the covering $\Omega_1 \cup \Omega_2 = \Omega$, and satisfy (3.7). Let $V = \phi_1 V_1 + \phi_2 V_2$ and consider the problem: Find $u \in V$ such that*

$$a(u, v) = (f, v) \quad \text{for all} \quad v \in V, \tag{3.8}$$

where

$$a(u, v) := \int_\Omega \nabla u \cdot \nabla v \; dx,$$
$$(f, v) := \int_\Omega f \; v \; dx.$$

Let A be the stiffness matrix associated with form $a(\cdot,\cdot)$ and the PU nodal basis functions of Theorem 3.2. Assume that h_1, h_2 are the mesh sizes for $\mathcal{T}_1$ on Ω_1 and $\mathcal{T}_2$ on Ω_2, respectively, and that $h_1 \geq h_2$. Then,

$$c_1 \; \frac{1}{h_1^2} \leq \mathcal{K}(A) \leq c_2 \; \frac{1}{1-\gamma} \frac{1}{h_2^2}, \tag{3.9}$$

where $\mathcal{K}(A)$ is the condition number of the matrix A, and c_1, c_2 are two constants independent of γ, h_1, h_2.

Proof First we notice that, by using the CS and the SCS inequalities, we have

$$(1-\gamma)(\|u_1\|^2 + \|u_2\|^2) \leq \|u_1 + u_2\|^2 \leq 2\; (\|u_1\|^2 + \|u_2\|^2), \tag{3.10}$$

for all $u_k \in \phi_k V_k$, $k = 1,2$, where $\|\cdot\|$ denotes the standard L^2 inner product. To estimate the condition number we follow the same ideas presented in [4], Chapter 2. Let $\{\phi_1 \psi_i\}_{i=1,\ldots,n_1}$ and $\{\phi_2 \eta_j\}_{j=1,\ldots,n_2}$ be bases for $\phi_1 V_1$ and $\phi_2 V_2$, respectively. Then, according to Theorem 3.2, we have that $\{\varphi_k\}_{k=1,\ldots,n_1+n_2} := \{\phi_1 \psi_i\}_{i=1,\ldots,n_1} \cup \{\phi_2 \eta_j\}_{j=1,\ldots,n_2}$ is a basis for $V = \phi_1 V_1 + \phi_2 V_2$ and the entries of the stiffness matrix A are $A_{i,j} = a(\varphi_i, \varphi_j), i, j = 1, \ldots, n$, where $n = n_1 + n_2$, $\varphi_i = \phi_1 \psi_i$ for $i = 1, \ldots, n_1$ and $\varphi_{n_1+j} = \phi_2 \eta_j$ for $j = 1, \ldots, n_2$. Thus, if $u = \sum_{k=1}^n c_k \varphi_k \in V$, and $c := (c_1, c_2, \ldots, c_n)^T \in \mathbf{R}^n$, then $\mathcal{K}(A) = \frac{\lambda_{max}(A)}{\lambda_{min}(A)}$, where

$$\lambda_{min}(A) = \inf_{c \in \mathbf{R}^n} \frac{(Ac, c)}{(c, c)} = \inf_{u = \sum_{k=1}^n c_k \varphi_k \in V} \frac{a(u, u)}{(c, c)} \tag{3.11}$$

and

$$\lambda_{max}(A) = \sup_{c \in \mathbf{R}^n} \frac{(Ac, c)}{(c, c)} = \sup_{u = \sum_{k=1}^n c_k \varphi_k \in V} \frac{a(u, u)}{(c, c)}. \tag{3.12}$$

Here $(\cdot,\cdot)$ is the standard Euclidian inner product and the inf and the sup are taken over non-zero vectors. In what follows, C, C_1, C_2 are constants independent of γ, h_1 and h_2 and might be different at different occurrences. Using that ϕ_1 can be taken identical 1 on $\Omega_1 \cap \Omega_2^c$, and that on the overlapping region above any triangle $T_1 \in \mathcal{T}_1$ the function ϕ_1 is a linear combination of the nodal functions associated with the vertices of T_1 (with coefficients "independent" of T_1) we have that there are constants C, C_1, C_2 such that

$$C_1 \; \|u_1\|^2 \leq h_1^2 \sum_{i=1}^{n_1} \alpha_i^2 \leq C_2 \; \|u_1\|^2 \tag{3.13}$$

for all $u_1 = \phi_1 \sum_{i=1}^{n_1} \alpha_i \psi_i \in \phi_1 V_1$ and

$$a(\phi_1 \psi_i, \phi_1 \psi_i) \leq C, \quad i = 1, \ldots, n_1. \tag{3.14}$$

Similar arguments for the subspace V_2 lead to

$$C_1 \; \|u_2\|^2 \leq h_2^2 \sum_{j=1}^{n_2} \beta_j^2 \leq C_2 \; \|u_2\|^2 \tag{3.15}$$

for all $u_2 = \phi_2 \sum_{j=1}^{n_2} \beta_j \eta_j \in \phi_2 V_2$ and

$$a(\phi_2 \eta_j, \phi_2 \eta_j) \leq C, \quad j = 1, \ldots, n_2. \tag{3.16}$$

Let $u = u_1 + u_2 \in V$, where $u_1 = \phi_1 \sum_{i=1}^{n_1} \alpha_i \psi_i \in \phi_1 V_1$ and $u_2 = \phi_2 \sum_{j=1}^{n_2} \beta_j \eta_j \in \phi_2 V_2$. Then, using (3.14), (3.16) and the finite interaction of the nodal functions, we get

$$\begin{aligned} a(u, u) &\leq 2(a(u_1, u_1) + a(u_2, u_2)) \\ &\leq C(\sum_{i=1}^{n_1} \alpha_i^2 \; a(\phi_1 \psi_i, \phi_1 \psi_i) + \sum_{j=1}^{n_2} \beta_j^2 \; a(\phi_2 \eta_j, \phi_2 \eta_j) \\ &\leq C(\sum_{i=1}^{n_1} \alpha_i^2 + \sum_{j=1}^{n_2} \beta_j^2). \end{aligned}$$

From the above estimate and (3.12) we get that

$$\lambda_{max}(A) \leq C.$$

On the other hand, from the Poincare's inequality, (3.13), and (3.15) we have

$$\begin{aligned} a(u, u) &\geq C \; \|u\|^2 \geq C \; (1-\gamma)\|u_1\|^2 + \|u_2\|^2) \\ &\geq C \; (1-\gamma)(h_1^2 \sum_{i=1}^{n_1} \alpha_i^2 + h_2^2 \sum_{j=1}^{n_2} \beta_j^2) \\ &\geq C \; (1-\gamma) h_2^2 (\sum_{i=1}^{n_1} \alpha_i^2 + \sum_{j=1}^{n_2} \beta_j^2). \end{aligned}$$

From (3.11) we get that

$$\lambda_{min}(A) \geq C \; (1-\gamma) h_2^2.$$

Hence, we have proved the left side of (3.9). Next, we prove the right side of (3.9). Let $u = \phi_1 \psi_1$. The coefficient vector associated with u is $e_1 = (1, 0...., 0)^T \in \mathbf{R}^n$, and using the locality of ψ_1 and the properties of ϕ_1 we get

$$\frac{a(u, u)}{(e_1, e_1)} = C.$$

From (3.12) we get that

$$\lambda_{max}(A) \geq C. \tag{3.17}$$

To find an upper bound for $\lambda_{min}(A)$ we fix a non-zero function $u \in H_0^1$. According with Theorem 2.1 there exists $u_h \in V$ such that

$$\|u - u_h\| \le C(h_1|u|_{H^1(\Omega_1)} + h_2|u|_{H^1(\Omega_2)}), \qquad (3.18)$$

and

$$a(u_h, u_h) \le C\ a(u, u). \qquad (3.19)$$

From (3.18) we get

$$\|u_h\| \ge \|u\| - \|u - u_h\| \ge 1/2\|u\| = C_1, \qquad (3.20)$$

if $h = h_1$ is small enough. Assume that $u_h = u_1 + u_2$, where $u_1 = \phi_1 \sum_{i=1}^{n_1} \alpha_i \psi_i \in \phi_1 V_1$ and $u_2 = \phi_2 \sum_{j=1}^{n_2} \beta_j \eta_j \in \phi_2 V_2$. Then we have

$$\begin{aligned} C_1^2 &\le \|u_h\|^2 \le 2(\|u_1\|^2 + \|u_2\|^2) \\ &\le C\ (h_1^2 \sum_{i=1}^{n_1} \alpha_i^2 + h_2^2 \sum_{j=1}^{n_2} \beta_j^2) \\ &\le C\ h_1^2 (\sum_{i=1}^{n_1} \alpha_i^2 + \sum_{j=1}^{n_2} \beta_j^2). \end{aligned}$$

Combining (3.19) and the above estimate we have

$$\begin{aligned} \lambda_{min}(A) &\le \frac{a(u_h, u_h)}{\sum_{i=1}^{n_1} \alpha_i^2 + \sum_{j=1}^{n_2} \beta_j^2} \\ &\le \frac{C\ a(u, u)}{\sum_{i=1}^{n_1} \alpha_i^2 + \sum_{j=1}^{n_2} \beta_j^2} \le C h_1^2. \end{aligned} \qquad (3.21)$$

The right side of (3.9) follows now from (3.17) and (3.21).

Remark 3.4 *The SCS constant γ in the above theorem depends on $r = \frac{h_2}{h_1}$. We have that $\gamma \nearrow 1$ for $r \to 0$. A justification for this statement is that for $r \to 0$, the function in $\phi_1 V_1$ supported on $\Omega_1 \cap \Omega_2$ can be well approximated by functions in $\phi_2 V_2$ supported on $\Omega_1 \cap \Omega_2$. The above theorem still holds if the PU functions ϕ_1 and ϕ_2 are chosen to be piecewise polynomials of degree n. Numerical computations show that γ increases (to one) as h decreases, and (on the good side) γ decreases as the degree of PU functions n increases.*

Remark 3.5 *It is straightforward to apply the Schwartz' alternating method to the variational problem (3.8). To obtain the solution in each step in the Schwartz' alternating method, it is necessary to evaluate the following terms in the overlapping region*

$$a(\phi_1 v_i, \phi_2 w_j) \quad v_i \in V_1, w_j \in V_2$$

which is not trivial in general case, because the topology of the intersection of the two meshes on the overlapping region has to be considered for computations.

4 Numerical Results

In this section, we carry out some numerical tests in one dimension. Consider the following Poisson equation

$$u'' = 1 \quad \text{on} \quad (0, 1), \qquad (4.22)$$

$$u(0) = u(1) = 0. \qquad (4.23)$$

We consider two kinds of the overlapping region. The first kind is called thin overlapping because the width of the overlapping region is of the same order of a grid. To be precise, let $\Omega = (0, 1)$. Let $\Omega_1 = (0, 1/2)$ and divide it into N intervals uniformly. The width of the interval is simply $h = \frac{1}{2N}$. The second region is given by $\Omega_2 = (1/2 - h, 1)$ which is divided into $N+1$ intervals uniformly (see Fig. 2). Hence the overlapping region is $(1/2 - h, 1/2)$. This is an example of thin overlapping region. The second kind is called fixed overlapping because the width of the overlapping is fixed and does not change when the grid is refined. To be precise, $\Omega_1 = (0, 1/2)$, $\Omega_2 = (1/4, 1)$ and the overlapping region is always $\Omega_o = (1/4, 1/2)$. We use linear finite element basis functions and choose nonlinear polynomials for PU functions. Note that a simple choice of $\phi_1 = 1 - x$ and $\phi_2 = x$ (scaled to $[0, 1]$) does not work since $\{\phi_1 v, \phi_2 w\}$ are linearly dependent.

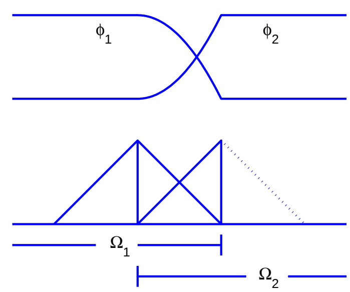

Figure 2: Explicative Mesh and PU Functions.

We choose four pairs of PU functions.

$$\begin{aligned} &\phi_1^a = 1 - x^2, \quad \phi_2^a = x^2, \\ &\phi_1^b = 1 - x^3, \quad \phi_2^b = x^3, \\ &\phi_1^c = 1 + 2x^3 - 3x^2, \phi_2^c = -2x^3 + 3x^2, \\ &\phi_1^d = 6x^2 - 7x + 1, \quad \phi_2^d = -4x^2 + 5x. \end{aligned}$$

4.1 Error in L^2 norm and H^1 semi-norm for thin overlapping

For convergence of the SA method for PUFEM, it is reasonable to look at the error in L^2 norm and H^1 semi-norm. We fix the PU functions ϕ_1^a and ϕ_2^a in this subsection. Note that the overlapping region is one element wide, i.e., if the interval $[0, 1]$ is divided into $M = 2N$ subinterval uniformly,

$$\begin{aligned} &0 = x_0 < x_1 < \ldots < x_M = 1, \\ &x_j = jh, h = 1/M, j = 0, \ldots, M, \end{aligned}$$

the overlapping region is $I_o = [x_{N-1}, x_N]$

Table 1: The error of the PUFEM.

N	$\|e_N\|_{L^2}$	Ratio	$\|e_N\|_{H^1}$	Ratio
8	1.334E-3	-	3.375E-2	-
16	3.453E-4	3.864	1.746E-2	1.932
32	8.774E-5	3.935	8.879E-3	1.967
64	2.211E-5	3.968	4.475E-3	1.984

Table 2: The error of the SA method for PUFEM.

N	$\|e_N\|_{L^2}$	$\|e_N\|_{H^1}$
8	1.340E-3	3.375E-2
16	4.064E-4	1.747E-2
32	1.171E-3	9.893E-3
64	7.601E-3	2.781E-2

The errors are shown in Table 1. We obtain $O(h^2)$ in L_2 norm and $O(h)$ in H^1 semi-norm. The results are consistent with the theory in [8] (c.f. Definition 2.1, 2.2 and Theorem 2.1 there). The estimate in [8] predicts the $O(h^2)$ convergence L_2 norm. Since the overlapping region I_o changes (same as h) as N changes, our case is slightly different. Instead, for the PU function we choose, we have the following estimate

$$\|\nabla\phi_i\|_{L^\infty(\mathbb{R}^n)} \leq \frac{C_G}{\text{diam}\Omega_i}\frac{1}{h^2}$$

where C_G is a constant. For the thin overlapping region we consider, taking into account $\text{diam}\Omega_i = 1/h$, we will still have the $O(h)$ convergence in H^1 semi-norm.

Remark 4.1 *The overlapping region we consider is different than that considered in [8]. Theorem 2.1 in [8] is not optimal for our case. Thus, our numerical results show an improvement for the approximation result of Theorem 2.1 for the special case of thin overlapping.*

Next we shall look at the convergence of the SA method. We fix the number of iteration with 50 and record the error for different N in Table 2. Note that when N increases, the width of the overlapping region I_o decreases. When N is small, the errors in Table 2 match those in Table 1 well. This indicates that the convergence of the SA method is better. As we increase N, the width of the overlapping region decreases and the convergence is worse. This is an indicator that the SCS constant is getting larger as the overlapping region is smaller.

4.2 Error in L^2 norm and H^1 semi-norm for fixed overlapping

In this section, we will consider the case when the overlapping region is fixed. For the same problem above, we have $I_o = [1/4, 1/2]$ fixed. In Table 3, we give the error in L_2 norm and H^1 semi-norm. The order of convergence is consistent with the results in [8].

Note that when $N = 4$, the case we consider here is the same as the one we consider in the previous subsection.

Table 3: The error of the PUFEM for fixed overlapping region.

N	$\|e_N\|_{L^2}$	Ratio	$\|e_N\|_{H^1}$	Ratio
8	1.235E-3	-	3.125E-2	-
16	3.088E-4	4.000	1.563E-2	2.000
32	7.720E-5	4.000	7.813E-3	2.000
64	1.930E-5	4.000	3.906E-3	2.000

Table 4: The error of the SA method for PUFEM for fixed overlapping region.

N	$\|e_N\|_{L^2}$	$\|e_N\|_{H^1}$
8	1.318E-3	3.140E-2
16	5.032E-4	1.855E-2
32	1.611E-4	1.109E-2
64	7.008E-5	6.081E-3

The errors are identical which are not shown. Next we look at the convergence of the SA method. Again, we fix the number of iteration at 50 and record the error for different N in Table 4. The errors do not become worse as we increase the number of elements as we expected since the overlapping region is fixed.

4.3 The SCS constant

Since the convergence rate of the SA method is decided by the the SCS constant, it would be plausible to look at the constants when the meshes are refined. Following [1], Let $u \in V = X \oplus Y$ with X and Y being PUFEM subspaces. If we partition the stiffness matrix into blocks

$$B = \begin{bmatrix} \mathbf{B}_{XX} & \mathbf{B}_{XY} \\ \mathbf{B}_{YX} & \mathbf{B}_{YY} \end{bmatrix},$$

then γ^2 is the maximum eigenvalue of the problem

$$\mathbf{B}_{XY}\mathbf{B}_{YY}^{-1}\mathbf{B}_{YX}\mathbf{x} = \mu\mathbf{B}_{XX}\mathbf{x}.$$

Table 5 shows the results for thin overlapping region and fixed overlapping region. Note that the error is in H^1 semi-norm.

One observation is that from Theorem 2.2 the SCS constant $\gamma \approx \|u - u_k\|/\|u - u_{k-1}\|$ for k sufficiently large. Fig. 3 shows the relative errors of SA method for both

Table 5: The CSC constants γ^2.

N	$fixedOverlapping$	$ThinOverlapping$
4	0.8571	0.8571
8	0.9489	0.8658
16	0.9870	0.8869
32	0.9969	0.9239
64	0.9993	0.9572

Table 6: The SCS constants γ^2 v.s. the partition of unity functions.

$N = 32$	$ThinOverlapping$	$fixedOverlapping$
ϕ^a	0.9239	0.9969
ϕ^b	0.9388	0.9990
ϕ^c	0.9005	0.9764
ϕ^d	0.9883	0.9999

Table 7: The SCS constants γ^2 for fixed overlapping.

N	ϕ^a	ϕ^b	ϕ^c	ϕ^d
4	0.8571	0.7995	0.5530	0.8813
8	0.9489	0.9504	0.7669	0.9888
16	0.9870	0.9924	0.9122	0.9986
32	0.9969	0.9990	0.9764	0.9998
64	0.9990	0.9999	0.9941	0.9999

thin and fixed overlapping regions. As the iteration number getting large, the relative errors approach the corresponding γ in both cases. We also look at the relation

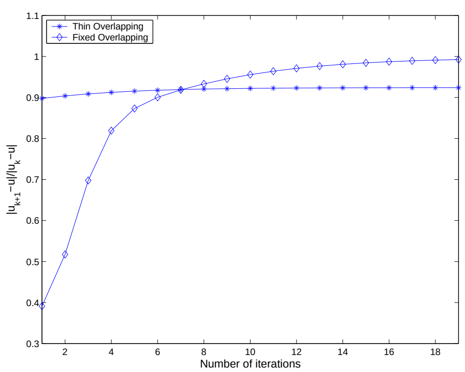

Figure 3: Relative erorrs of the SA method ($N = 32$).

between γ and the PU functions. In Table 6, we calculate γ for thin overlapping and fixed overlapping. We fix $N = 32$. For both cases, ϕ^c's give the smallest γ and ϕ^d's give the largest γ. Finally we look the effect of the mesh refinement on γ. Each time we cut the mesh size by half and calculate the SCS constant. For both thin and fixed overlapping, the constant is getting larger (closer to 1) as the mesh size decreases (see Table 7 and Table 8). Again, the PU functions ϕ_c's are the best in the sense that they give the smallest γ among all the test PU functions at the same mesh refinement level.

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Table 8: The SCS constants γ^2 for thin overlapping.

N	ϕ^a	ϕ^b	ϕ^c	ϕ^d
4	0.8571	0.7995	0.5530	0.8813
8	0.8657	0.8398	0.6488	0.9502
16	0.8868	0.8936	0.8096	0.9763
32	0.9239	0.9388	0.9005	0.9883
64	0.9572	0.9674	0.9491	0.9942

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