A new family of high regularity elements

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Abstract

In this paper, we propose a new family of high regularity finite element spaces. The global approximation spaces are obtained in two steps. We first build an open cover of the computational domain and local approximation spaces on each patch of the cover. Then we construct partition of unity functions subordinate to the open cover depending on the regularity requirement. The basis functions of the global space is given by the products of the local basis functions and the corresponding partition of unity functions. The method can be used to construct finite element spaces of any desired regularity. Approximation properties and implementation details are discussed. Numerical examples for the biharmonic equation are presented to show the effectiveness of the proposed method.

Keywords: high regularity finite element, partition of unity, biharmonic equation

1 Introduction

High regularity finite element spaces are of central importance for the approximation of partial differential equations of higher order, for example, the Argyris triangle [1] and the Bogner-Fox-Schmit rectangle [10] for the biharmonic equation. However, classical conforming finite element spaces are rarely used in practice because they are difficult to construct in general. We refer the readers to [1, 16, 10, 20, 24, 25] for some efforts to construct high regularity finite element spaces, in particular, C^1 elements. Alternative methods, such as nonconforming finite element methods [10, 8, 17, 18, 11] and mixed finite element methods [7, 15, 9], are also proposed to treat higher order problems.

In this paper, we propose a new family of high regularity finite element spaces. The main components are local approximation spaces on each patch of an open cover of the computational domain and the partition of unity functions. The global finite element spaces inherit the approximation properties of the local spaces and the smoothness of the partition of unity functions. The technique we used is the so-called partition of unity finite element method (PUFEM) [14, 4]. We refer the readers to [21, 19, 2, 3] and references therein for recent developments and applications of the PUFEM. Although pointed out in the fundamental paper of Melenk and Babuška [14], the ability of the PUFEM to construct finite element spaces of high regularity has not been fully explored to date. This is the motivation of this paper. To be precise, we apply a thin overlapping version of the PUFEM (see [12, 5, 6, 7]) to construct high regularity finite element

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spaces. The major advantages of the proposed method include (i) the simple choices of the local basis functions, for example, biquadratic polynomials in two dimension, (ii) the ability to construct higher regularity finite element spaces by choosing adequate partition of unity functions, and (iii) the easy extension to higher dimensions, for example, three dimensional C^1 elements.

The paper is organized as follows. In Section 2, we first introduce fundamental concepts and theories of the partition of unity method [14, 4]. Then we make necessary extensions in order to construct H^2 conforming finite element spaces. In Section 3, we show examples of how to construct high regularity spaces in detail in one, two and three dimensions. Then we use these spaces to solve the biharmonic equation in Section 4. In Section 5, we make conclusions and mention some future work.

2 Partition of unity method

In this section, we introduce the concept of the partition of unity finite element method [14, 4] and make necessary extensions to facilitate the construction of H^2 conforming finite element spaces. We will also remark on higher regularity finite element spaces.

Definition 2.1. *Let $\Omega \in R^d$ be an open set, $\{\Omega_i\}$ be an open cover of Ω satisfying a pointwise overlap condition:*

$$\exists M \in \mathbb{N} \quad \forall x \in \Omega \quad \mathit{card}\{i | x \in \Omega_i\} \leq M.$$

Let $\{\phi_i\}$ be a partition of unity subordinate to the cover $\{\Omega_i\}$ satisfying

$$\begin{aligned}
&\mathit{supp}\,\phi_i \subset \mathit{closure}\,(\Omega_i), \quad \forall\, i,\\
&\sum_i \phi_i \equiv 1 \quad \mathit{on} \quad \Omega,\\
&\|\phi_i\|_{L^\infty(R^n)} \leq C_\infty,\\
&\|\nabla \phi_i\|_{L^\infty(R^n)} \leq \frac{C_1}{d_i}, \quad d_i = \mathit{diam}\,\Omega_i,\\
&\left\|\frac{\partial^\alpha \phi_i}{\partial x^\alpha}\right\|_{L^\infty(R^n)} \leq \frac{C_2}{d_i^2}, \quad |\alpha| = 2,
\end{aligned}$$

where C_∞, C_1, C_2 are constants. Then $\{\phi_i\}$ is called an (M, C_∞, C_1, C_2) partition of unity subordinate to the cover $\{\Omega_i\}$. The partition of unity $\{\phi_i\}$ is said to be of degree $m \in \mathbb{N}$ if $\{\phi_i\} \subset C^m(R^n)$. The covering sets Ω_i's are called patches.

Remark 2.1. *The above conditions on the partition of unity functions are sufficient to construct H^2 conforming finite element spaces. To obtain higher regularity finite element spaces, we will need additional conditions on $\|\partial^\alpha_{x^\alpha}\phi_i\|_{L^\infty(R^n)}$ for $|\alpha| > 2$.*

Definition 2.2. *Let $\{\Omega_i\}$ be an open cover of $\Omega \subset R^d$ and let $\{\phi_i\}$ be a (M, C_∞, C_1, C_2) partition of unity subordinate to $\{\Omega_i\}$. Let V_i be the approximation space on Ω_i. Then the space*

$$V := \sum_i \phi_i V_i = \left\{ \sum_i \phi_i v_i \big| v_i \in V_i \right\} \tag{2.1}$$

is called the global approximation space. The space V is said to be of degree m if $V \subset C^m(\Omega)$. The space V_i are referred to as the local approximation spaces.

The approximation properties of V depends on the local approximation spaces V_i and the partition of unity functions as we can see in the following theorem.

Theorem 2.2. *Let $\Omega \in R^d$ be given, and $\{\phi_i\}$, $\{\Omega_i\}$, $\{V_i\}$ be as in the definitions above. Let u be the function to be approximated. Assume that the local approximation spaces V_i have the following approximation properties: on each patch $\Omega_i \cap \Omega$, u can be approximated by a function $v_i \in V_i$ such that*

$$\begin{aligned} &\|u - v_i\|_{L^2(\Omega_i\cap\Omega)} \le \epsilon_1(i), \\ &\|\nabla(u - v_i)\|_{L^2(\Omega_i\cap\Omega)} \le \epsilon_2(i), \\ &\left\| \sum_{|\alpha|=2} D^\alpha(u - v_i) \right\|_{L^2(\Omega_i\cap\Omega)} \le \epsilon_3(i). \end{aligned}$$

Then the function

$$u_{ap} = \sum_i \phi_i v_i \tag{2.2}$$

satisfies

$$\|u - u_{ap}\|_{L^2(\Omega)} \le \sqrt{M} C_\infty \left(\sum_i \epsilon_1^2(i) \right)^{1/2} \tag{2.3}$$

$$\|\nabla(u - u_{ap})\|_{L^2(\Omega)} \le \sqrt{2M} \left(\sum_i \left[\frac{C_1}{d_i} \right]^2 \epsilon_1^2(i) + C_\infty^2 \epsilon_2^2(i) \right)^{1/2} \tag{2.4}$$

$$\begin{aligned} &\left\| \sum_{|\alpha|=2} D^\alpha(u - u_{ap}) \right\|_{L^2(\Omega)} \\ &\le \sqrt{M N_0} \left(\sum_i \left[\frac{C_2}{d_i^2} \right]^2 \epsilon_1^2(i) + \left[\frac{C_1}{d_i} \right]^2 \epsilon_2^2(i) + {C_\infty}^2 \epsilon_3^2(i) \right)^{1/2} \end{aligned} \tag{2.5}$$

where N_0 is a constant depending on the dimension d.

Proof. We will show the proof of (2.5). The proof of (2.3) and (2.4) can be found in [14]. Since the functions ϕ_i form a partition of unity, we have $u = \sum_i \phi_i u$. Let x_1 be the first component of $x \in R^d$. We have

$$\begin{aligned} \left\| \frac{\partial^2}{x_1^2}(u - v) \right\|^2_{L^2(\Omega)} &= \left\| \frac{\partial^2}{x_1^2} \sum_i \phi_i (u - v_i) \right\|^2_{L^2(\Omega)} \\ \le\ 3 \left\| \sum_i \frac{\partial^2 \phi_i}{\partial x_1^2}(u - v_i) \right\|^2_{L^2(\Omega)} &+ 12 \left\| \sum_i \frac{\partial \phi_i}{\partial x_1} \frac{\partial (u - v_i)}{\partial x_1} \right\|^2_{L^2(\Omega)} + 3 \left\| \sum_i \phi_i \frac{\partial^2 (u - v_i)}{\partial x_1^2} \right\|^2_{L^2(\Omega)}. \end{aligned}$$

For any given $x \in \Omega$, there are at most M patches covering it. Thus the sums

$$\sum_i \frac{\partial^2 \phi_i}{\partial x_1^2}(u - v_i), \quad \sum_i \frac{\partial \phi_i}{\partial x_1}\frac{\partial (u - v_i)}{\partial x_1}, \quad \text{and} \ \sum_i \phi_i \frac{\partial^2 (u - v_i)}{\partial x_1^2}$$

contain at most M terms for any fixed $x \in \Omega$. We obtain

$$\begin{aligned}
& 3\left\|\sum_i \frac{\partial^2 \phi_i}{\partial x_1^2}(u - v_i)\right\|_{L^2(\Omega)}^2 + 12\left\|\sum_i \frac{\partial \phi_i}{\partial x_1}\frac{\partial (u - v_i)}{\partial x_1}\right\|_{L^2(\Omega)}^2 + 3\left\|\sum_i \phi_i \frac{\partial^2 (u - v_i)}{\partial x_1^2}\right\|_{L^2(\Omega)}^2 \\
\leq \ & 3M\sum_i \left\|\frac{\partial^2 \phi_i}{\partial x_1^2}(u - v_i)\right\|_{L^2(\Omega)}^2 + 12M\sum_i \left\|\frac{\partial \phi_i}{\partial x_1}\frac{\partial (u - v_i)}{\partial x_1}\right\|_{L^2(\Omega)}^2 \\
& \qquad + 3M\sum_i \left\|\phi_i \frac{\partial^2 (u - v_i)}{\partial x_1^2}\right\|_{L^2(\Omega)}^2, \\
= \ & 3M\sum_i \left\|\frac{\partial^2 \phi_i}{\partial x_1^2}(u - v_i)\right\|_{L^2(\Omega_i \cap \Omega)}^2 + 12M\sum_i \left\|\frac{\partial \phi_i}{\partial x_1}\frac{\partial (u - v_i)}{\partial x_1}\right\|_{L^2(\Omega_i \cap \Omega)}^2 \\
& \qquad + 3M\sum_i \left\|\phi_i \frac{\partial^2 (u - v_i)}{\partial x_1^2}\right\|_{L^2(\Omega_i \cap \Omega)}^2, \\
\leq \ & M\sum_i \left(3\left[\frac{C_2}{d_i^2}\right]^2 \epsilon_1^2(i) + 12\left[\frac{C_1}{d_i}\right]^2 \epsilon_2^2(i) + 3{C_\infty}^2 \epsilon_3^2(i)\right).
\end{aligned}$$

Then there exist a constat N_0 depending on the dimension d and $|\alpha|(= 2)$ such that

$$\left\|\sum_{|\alpha|=2} D^\alpha (u - u_{ap})\right\|_{L^2(\Omega)} \leq \sqrt{MN_0}\left(\sum_i \left[\frac{C_2}{d_i^2}\right]^2 \epsilon_1^2(i) + \left[\frac{C_1}{d_i}\right]^2 \epsilon_2^2(i) + {C_\infty}^2 \epsilon_3^2(i)\right)^{1/2}.$$

□

Combining the results in the above theorem, we obtain the following estimate in H^2 norm. Let $u \in H^2(\Omega)$ and u_{ap} be defined as in (2.2) above. Then there exists a constant C depending on $M, C_\infty, C_1, C_2, N_0$ such that

(2.6)
$$\|u - u_{ap}\|_{H^2(\Omega)}^2 \leq C\left(\sum_i \left\{\frac{1}{d_i^4} + \frac{1}{d_i^2} + 1\right\}\epsilon_1^2(i) + \left\{\frac{1}{d_i^2} + 1\right\}\epsilon_2^2(i) + \epsilon_3^2(i)\right).$$

In order to obtain a conforming H^2 finite element space, we need more restrictions on the open cover $\{\Omega_i\}$ and the local approximation spaces V_i's. Let $u \in H^k(\Omega), k \geq 2$. We assume that

(H1) There exist two constants $C, c > 0$ such that

$$ch \leq \operatorname{diam}\Omega_i \leq Ch, \quad \text{for all } i,$$

where $h = \max_i \operatorname{diam}\Omega_i$.

(H2) Each V_i has the following approximation properties:

$$\epsilon_1(i) \le C d_i^{\mu+2} \|u\|_{H^k(\Omega\cap\Omega_i)},$$
$$\epsilon_2(i) \le C d_i^{\mu+1} \|u\|_{H^k(\Omega\cap\Omega_i)},$$
$$\epsilon_3(i) \le C d_i^{\mu} \|u\|_{H^k(\Omega\cap\Omega_i)},$$

for some $\mu > 0$, the local approximation order.

Using the previous theorem, it is straightforward to show the following theorem.

Theorem 2.3. *Assume the conditions in Theorem 2.2 and (H1-H2) hold. Then there exist constants C depending on M, C_∞, C_1 and C_2 such that*

$$\|u - u_{ap}\|_{L^2(\Omega)} \le C h^{\mu+2} \|u\|_{H^k(\Omega)}, \tag{2.7}$$
$$\|\nabla(u - u_{ap})\|_{L^2(\Omega)} \le C h^{\mu+1} \|u\|_{H^k(\Omega)}, \tag{2.8}$$
$$\left\| \sum_{|\alpha|=2} D^\alpha (u - u_{ap}) \right\|_{L^2(\Omega)} \le C h^{\mu} \|u\|_{H^k(\Omega)}. \tag{2.9}$$

3 Construction of global approximation spaces

Now we discuss the detail of how to construct the high regularity conforming spaces. In particular, we will construct H^2 conforming spaces. We will also remark on the higher regularity spaces.

3.1 One dimension case

Let $\Omega = (0,1)$, $n \in \mathbb{N}$, $h = 1/n$ and define $x_i = ih, i = -1, 0, \dots, n, n+1$. Let $\Omega_i = (x_{i-1}, x_{i+1}) \cap \Omega$. Obviously $\{\Omega_i\}$ is an open cover of Ω corresponding $M = 2$ in Definition 2.1.

On each patch Ω_i, we need to define a local space V_i. In view of Theorem 2.3, it is necessary to use at least quadratic basis functions for V_i if we try to solve fourth order problems, e.g., the biharmonic equation. A quadratic basis on the reference element $\hat{\Omega} = (0,1)$ is given by (see Fig. 1)

$$v_1 = 2x^2 - 3x + 1,$$
$$v_2 = -4x^2 + 4x,$$
$$v_3 = 2x^2 - x.$$

For better approximation, we can choose higher order polynomial basis, for example, the cubic basis or the Hermite basis on the reference element $\hat{\Omega}_0 = (0,1)$, i.e.,

$$v_1 = -\frac{9}{2}x^3 + 9x^2 - \frac{11}{2}x + 1,$$
$$v_2 = \frac{27}{2}x^3 - \frac{45}{2}x^2 + 9x,$$
$$v_3 = \frac{27}{2}x^3 + 18x^2 - \frac{9}{2}x,$$
$$v_4 = \frac{9}{2}x^3 - \frac{9}{2}x^2 + x,$$

or

$$\begin{aligned} v_1 &= 2x^3 - 3x^2 + 1, \\ v_2 &= -2x^3 + 3x^2, \\ v_3 &= x^3 - 2x^2 + x, \\ v_4 &= x^3 - x^2. \end{aligned}$$

Next we consider the partition of unity functions subordinate to $\{\Omega_i\}$. The simplest choice of partition of unity functions might be $\phi_i^1(x) = \phi^1(x - x_i)$ where

$$\phi^1(x) = \frac{1}{h^3} \begin{cases} (x+h)^2(h-2x), & x \in (-h, 0], \\ (h-x)^2(h+2x), & x \in (0, h), \\ 0, & \text{elsewhere}. \end{cases} \tag{3.10}$$

Other choices of partition of unity functions are possible, for example,

$$\phi^2(x) = \frac{1 + \cos(\pi x/h)}{2}, \quad x \in (-h, h) \tag{3.11}$$

and

$$\phi^3(x) = \frac{1}{h^5} \begin{cases} (h+x)^3(h^2 - 3hx + 6x^2), & x \in (-h, 0], \\ (h-x)^3(h^2 + 3hx + 6x^2), & x \in (0, h), \\ 0, & \text{elsewhere}. \end{cases} \tag{3.12}$$

Assume that the local approximation spaces V_i's are given by

$$\text{span}\,\{v_{i,j}, j = 1, \ldots, N_i\}.$$

Also assume that the partition of unity functions are given as above. Then the following theorem shows that the global approximation space V is $H^2(0,1)$ conforming.

Theorem 3.1. *Let V be the global approximation space constructed using the local approximation spaces and partition of unity functions given above. Then $V \subset H^2(0,1)$.*

Proof. Let $v \in V_i$. Then $\phi_i v$ can be viewed as a function defined on the whole domain Ω which is a piecewise polynomial and a $C^1(0,1)$ function. Hence any linear combination of these functions is also a $C^1(0,1)$ function which implies $V \subset C^1(0,1)$. By Theorem 2.1.2 in [10], we have that $V \subset H^2(0,1)$. □

Remark 3.2. *To obtain higher regularity global approximation spaces, we need to put more restrictions on the partition of unity functions and local approximation spaces. For example, to construct an H^3 conforming space, one could use the cubic basis or the Hermite basis for V_i (for convergence, we need polynomial basis of order 3 or higher) and partition of unity functions ϕ^3 defined in (3.12) (for regularity, we need that $\|\partial_{x^\alpha}^\alpha \phi_i\|_{L^\infty(R^n)} \le C_3/d_i^3$ for some constant C_3 and $|\alpha| = 3$. See Remark 2.1.).*

Figure 1: Left: Quadratic basis functions on Ω_0. Middle: The partition of unity function ϕ^2 on Ω_0. Right: The H^2 conforming basis functions on Ω_0.

3.2 Two dimensional rectangular meshes

The above construction can be extended to two and three dimension cases easily. For simplicity, let $\Omega = (0,1) \times (0,1)$. We consider a uniform overlapping rectangular mesh on Ω. Let $n \in \mathbb{N}$, $h = 1/n$ and define

$$x_i = ih, i = -1, 0, \ldots, n, n+1,$$
$$y_j = jh, j = -1, 0, \ldots, n, n+1.$$

Let

$$\Omega_{i,j} = (x_{i-1}, x_{i+1}) \times (y_{j-1}, y_{j+1}) \cap \Omega.$$

Obviously $\{\Omega_{i,j}\}$ is an open cover of Ω.

On each $\Omega_{i,j}$, we need a local approximation space $V_{i,j}$. The choice of the local space is quite flexible as long as it can provide necessary local approximation ability. For example, one may use the biquadratic basis functions for H^2 conforming spaces.

For the partition of unity functions in two dimension, one can use the product of the partition of unity functions in the one dimension case. For example, the partition of unity functions can be defined by $\phi_{i,j}(x,y) = \phi_i^1(x - x_i)\phi_j^1(y - y_j)$, i.e., (see Fig. 2)

$$\phi(x,y) = \frac{1}{h^6}\begin{cases} (x+h)^2(h-2x)(y+h)^2(h-2y), & (x,y) \in (-h,0] \times (-h,0], \\ (x+h)^2(h-2x)(h-y)^2(h+2y), & (x,y) \in (-h,0] \times (0,h), \\ (h-x)^2(h+2x)(y+h)^2(h-2y), & (x,y) \in (0,h) \times (-h,0], \\ (h-x)^2(h+2x)(h-y)^2(h+2y), & (x,y) \in (0,h) \times (0,h), \\ 0, & \text{elsewhere.} \end{cases}$$

As in one dimensional case, the high regularity of the space can be achieved by a proper choice of the partition unity functions while the approximation property relies on the local approximation spaces. For example, the global space V constructed from the bi-quadratic local basis and the partition of unity function $\phi_{i,j}$ given above is H^2 conforming. The proof is a straightforward generalization of the one dimensional case.

3.3 Three dimensional rectangular meshes

The construction of V for three dimension case is similar. Let $\Omega = (0,1) \times (0,1) \times (0,1)$. We consider a uniform rectangular mesh on Ω. Let $n \in \mathbb{N}$,

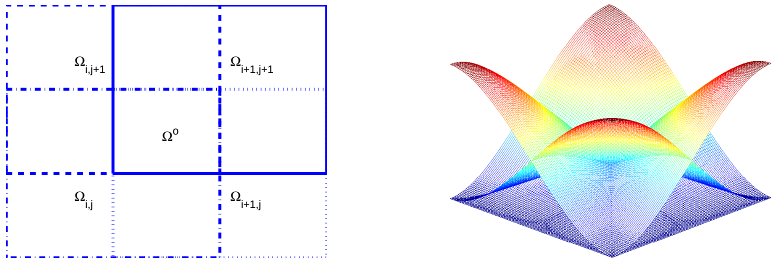

Figure 2: Left: Overlapping rectangular mesh in 2D. $\Omega^o = \Omega_{i,j} \cap \Omega_{i,j+1} \cap \Omega_{i+1,j} \cap \Omega_{i+1,j+1}$. Note that every point x in the domain is covered by 4 patches, i.e., $M = 4$ in Definition 2.1). Right: The partition of unity functions on Ω^o.

$h = 1/n$ and define

$$\begin{aligned} x_i &= ih, i = -1, 0, \ldots, n, n+1, \\ y_j &= jh, j = -1, 0, \ldots, n, n+1, \\ z_k &= kh, k = -1, 0, \ldots, n, n+1. \end{aligned}$$

Let

$$\Omega_{i,j,k} = (x_{i-1}, x_{i+1}) \times (y_{j-1}, y_{j+1}) \times (z_{k-1}, z_{k+1}) \cap \Omega.$$

Then $\{\Omega_{i,j,k}\}$ is an open cover of Ω. As above, one may use the simple tri-quadratic polynomials as local basis.

For the partition of unity functions, one can simply use

$$\phi_{i,j,k}(x, y, z) = \phi(x - x_i, y - y_j, z - z_k)$$

where

$$\phi = \frac{1}{h^9} \begin{cases} (x+h)^2(h-2x)(y+h)^2(h-2y)(z+h)^2(h-2z), & (-h,0] \times (-h,0] \times (-h,0], \\ (x+h)^2(h-2x)(y+h)^2(h-2y)(h-z)^2(h+2z), & (-h,0] \times (-h,0] \times (0,h), \\ (x+h)^2(h-2x)(h-y)^2(h+2y)(z+h)^2(h-2z), & (-h,0] \times (0,h) \times (-h,0], \\ (x+h)^2(h-2x)(h-y)^2(h+2y)(h-z)^2(h+2z), & (-h,0] \times (0,h) \times (0,h), \\ (h-x)^2(h+2x)(y+h)^2(h-2y)(z+h)^2(h-2z), & (0,h) \times (-h,0] \times (-h,0], \\ (h-x)^2(h+2x)(y+h)^2(h-2y)(h-z)^2(h+2z), & (0,h) \times (-h,0] \times (0,h), \\ (h-x)^2(h+2x)(h-y)^2(h+2y)(z+h)^2(h-2z), & (0,h) \times (0,h) \times (-h,0], \\ (h-x)^2(h+2x)(h-y)^2(h+2y)(h-z)^2(h+2z), & (0,h) \times (0,h) \times (0,h), \\ 0, & \text{elsewhere.} \end{cases}$$

As above, one can easily construct a H^2 conforming global space using the above partition of unity functions and the tri-quadratic polynomial local basis.

3.4 Two dimensional case on triangular meshes

Now we discuss how to construct H^2 conforming spaces based on given triangular meshes. Let $\mathcal{T}$ be a triangular mesh for Ω. We first construct an open cover $\{\Omega_i\}$. Our construction is associated to the nodes of $\mathcal{T}$. Let i be a node of

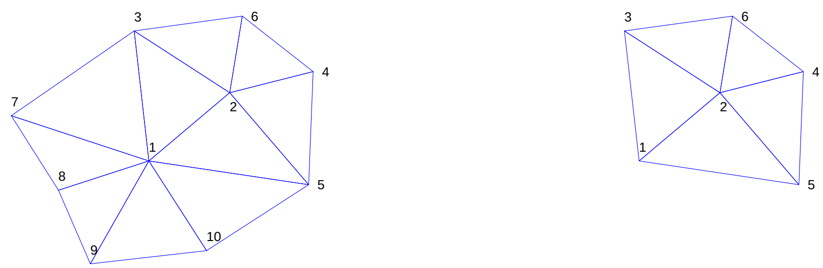

Figure 3: Left: A triangular mesh of Ω. Right: A polygon patch Ω_2 associated with vertex 2.

$\mathcal{T}$. The union of all triangles whose vertices include i gives a polygon, denoted by Ω_i. Let $\mathcal{N}$ be the index set of all nodes in $\mathcal{T}$. Then $\{\Omega_i\}, i \in \mathcal{N}$ is an open cover of Ω.

For example, Fig. 3 shows a triangular mesh. Then Ω_2 is the polygon which is the union of triangles $(2,1,5)$, $(2,5,4)$, $(2,4,6)$, $(2,6,3)$, $(2,3,1)$. It is obvious that $\{\Omega_i\}, i = 1, \ldots, 10$ is an open cover of Ω.

Remark 3.3. *Other types of construction are possible. For example, we can associate the open cover with triangles in the original mesh. The patch Ω_i is the union of a triangle and all other triangles surrounding it.*

Now we need to define a local approximation space V_i on Ω_i. Since Ω_i is the union of triangles, one may choose V_i as the space of continuous piecewise quadratic functions with respect to the triangulation of Ω_i.

Remark 3.4. *Polygonal finite element interpolants using rational polynomials for convex polygons was discussed by Wachspress [23]. We refer the readers to [22] and references therein for recent developments in the construction of finite element interpolants on polygonal domains.*

The partition of unity functions on Ω_i can be defined similarly. We first construct a "proper" function on each triangle and the union of all these functions will give the corresponding partition of unity functions. For example, on the reference triangle $\hat{K} = \{(0,0),(0,1),(1,0)\}$, the "proper" function for $\Omega_{(0,0)}$ can be defined as

$$\phi_{\hat{K}} = 1 - 10x^3 - 10y^3 + 15x^4 - 30x^2y^2 + 15y^4 - 6x^5 + 30x^3y^2 + 30x^2y^3 - 6y^5.$$

The node $(0,0)$ is the center of $\Omega_{(0,0)}$ and $\{(0,1),(1,0)\}$ is one edge of it. At $(0,0)$, $\phi_{\{(0,0),(0,1),(1,0)\}}$ is one and all its first order and second order partial derivatives vanish. At $(0,1)$ and $(0,1)$, the value and all first order and second order partial derivatives of $\phi_{\{(0,0),(0,1),(1,0)\}}$ vanish. Fig. 4 shows the partition of unity function on the patch Ω_2 shown in Fig. 3.

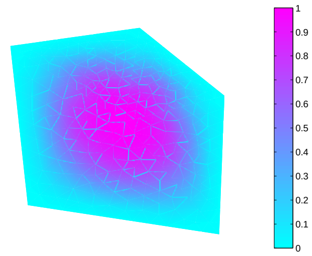

Figure 4: The partition of unity function on the polygonal patch Ω_2.

4 Application to the biharmonic equation

We consider the following homogeneous Dirichlet problem for the biharmonic equation

$$\Delta^2 u = f \quad \text{in } \Omega, \tag{4.13a}$$

$$u = \partial_\nu u = 0 \quad \text{on } \Gamma, \tag{4.13b}$$

where $\Gamma = \partial\Omega$ and ν is the unit outward normal to Γ. The weak formulation is to find $u \in H_0^2(\Omega)$ such that

$$a(u,v) := \int_\Omega \triangle u \triangle v \, dx = \int_\Omega f v \, dx =: (f,v), \quad \forall v \in H_0^2(\Omega). \tag{4.14}$$

Since the bilinear form $a(u,v)$ is $H_0^2(\Omega)$-elliptic, there exists a unique solution to (4.14) [10]. The corresponding discrete problem can be stated as to find $u_h \in V \subset H_0^2(\Omega)$ such that

$$\int_\Omega \triangle u_h \triangle v \, dx = \int_\Omega f v \, dx, \quad \forall v \in V \tag{4.15}$$

where V is the global approximation space constructed in the previous section.

Theorem 4.1. *Assume that V is the global approximation space obtained as in (2.1) and conditions in Theorem 2.2 hold. Let $u \in H_0^2(\Omega)$ and $u_h \in V$ be the solutions of (4.14) and (4.15), respectively. Then there exists a constant C such that*

$$\|u-u_h\|_{2,\Omega} \le C\left(\sum_i \left\{\frac{1}{d_i^4} + \frac{1}{d_i^2} + 1\right\}\epsilon_1^2(i) + \left\{\frac{1}{d_i^2}+1\right\}\epsilon_2^2(i) + \epsilon_3^2(i)\right)^{1/2} \tag{4.16}$$

Proof. Since the bilinear form a is elliptic, we have, by Céa's lemma,

$$\|u-u_h\|_{2,\Omega} \le C \inf_{v_h \in V} \|u-v_h\|_{H^2(\Omega)}.$$

Now $u_{ap} \in V$ and it is obvious that

$$\inf_{v_h \in V} \|u-v_h\|_{H^2(\Omega)} \le \|u-u_{ap}\|_{H^2(\Omega)}.$$

Thus (4.16) holds. □

Combining Theorem 2.3 with Theorem 4.1, we obtain the following theorem for one dimension case corresponding to the example we discuss above.

Theorem 4.2. *Assume that $u \in H_0^2(\Omega)$ and (H2) holds for $\mu \geq 1$ and $k \geq 3$. Then there exists a constant C such that*

$$\|u - u_h\|_{2,\Omega} \leq Ch^{\mu-1/2}\|u\|_{H^k(\Omega)}. \tag{4.17}$$

The following theorem gives the L^2 estimate.

Theorem 4.3. *Assume that $u \in H_0^2(\Omega)$ and (H2) holds for $\mu \geq 1$ and $k \geq 3$. Then there exists a constant C such that*

$$\|u - u_h\|_{0,\Omega} \leq Ch^{2\mu-1}\|u\|_{H^k(\Omega)}. \tag{4.18}$$

Proof. The theorem can be proved using Nitsche's trick which is similar to the L^2 estimate for Poisson equation with homogeneous Dirichlet boundary condition (for example, see [13]). Thus we omit the details. □

4.1 One dimensional examples

In the following we will show some numerical results for the one dimensional case. Let $\Omega = (0, 1)$, we have

$$u^{(4)} = f, \quad \text{on } (0, 1) \tag{4.19}$$

$$u(0) = u(1) = u'(0) = u'(1) = 0. \tag{4.20}$$

Let $u = \sin^2(\pi x)$ which satisfies the biharmonic equation (4.19) and (4.20) with $f(x) = -8\pi^4\cos(2\pi x)$. We use quadratic local basis and partition of unity functions given by (3.10), (3.11) and (3.12). The errors of the numerical solution are shown in Fig. 5, Fig. 6 and Fig. 7, respectively. All these three examples, the error convergence rate is $O(h)$ in L^2 norm and $O(h^{1/2})$ in the H^2 semi-norm.

Next we let $u(x) = x^6 - 3x^5 + 3x^4 - x^3$. It is easy to check that u satisfies the biharmonic equation (4.19) and (4.20) with $f(x) = 360x^2 - 360x + 72$. We first use the quadratic basis function on each Ω_i. The partition of unity function is given by (3.12). We show the error in the L^2 norm and the H^2 semi-norm of the numerical solution in Fig. 8. It can be seen that the convergence rate is $O(h^2)$ in the L^2 norm and $O(h)$ in the H^2 semi-norm. Then we use Hermite basis functions on each Ω_i and keep the same partition of unity functions. The numerical results are shown in Fig. 9. The error convergence rate is $O(h^4)$ in the L^2 norm and $O(h^2)$ in the H^2 semi-norm. For this particular example, we obtain better convergence rates.

4.2 A two dimensional example

Now we consider a two dimensional example on a rectangular mesh. We choose $u = \sin^2(\pi x)\sin^2(\pi y)$ such that

$$f(x, y) = 24\pi^4 - 40\pi^4\cos^2(\pi y) - 40\pi^4\cos^2(\pi x) + 64\pi^4\cos^2(\pi x)\cos^2(\pi y).$$

We use the H^2 conforming finite element space described in Section 3.2. In Fig. 10, we plot the numerical result. The error convergence rate is $O(h^2)$ in the L^2 norm and $O(h)$ in the H^2 semi-norm.

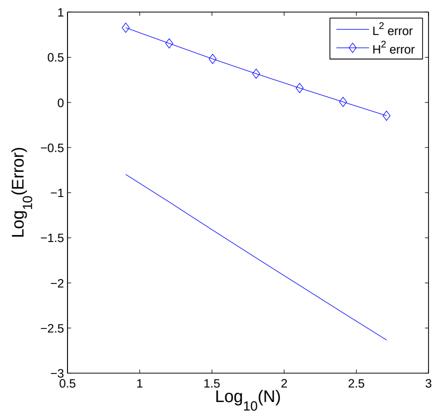

Figure 5: The error of the numerical solution in log scale. The exact solution is given by $u(x) = \sin^2(\pi x)$. Local basis are quadratic basis functions and the partition of unity functions are given by (3.10). The error convergence rate is $O(h)$ in the L^2 norm and $O(h^{1/2})$ in the H^2 semi-norm.

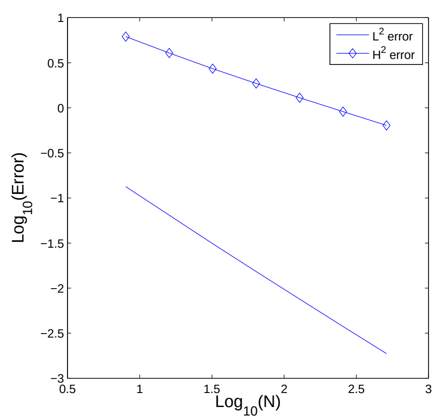

Figure 6: The error of the numerical solution of in log scale. The exact solution is given by $u(x) = \sin^2(\pi x)$. Local basis are quadratic basis functions and the partition of unity functions are given by (3.11). The error convergence rate is $O(h)$ in the L^2 norm and $O(h^{1/2})$ in the H^2 semi-norm.

5 Conclusions and future work

In this paper, we propose a new family of high regularity finite element spaces. Based on an overlapping mesh of the computation domain, the global approximation spaces are obtained by choosing the local approximations spaces and the partition of unity functions appropriately. The major advantage lies in the simplicity and efficiency of the method for higher dimensional problems. For

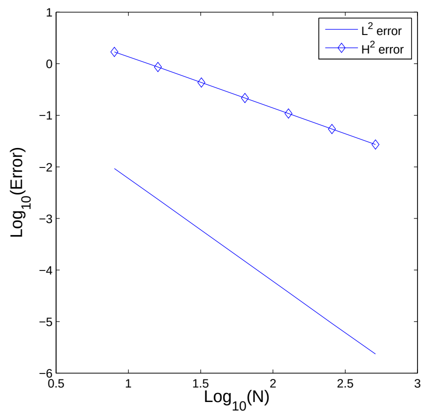

Figure 7: The error of the numerical solution of u_h in log scale. The exact solution is given by $u(x) = \sin^2(\pi x)$. Local basis are quadratic basis functions and the partition of unity functions are given by (3.12). The error convergence rate is $O(h)$ in the L^2 norm and $O(h^{1/2})$ in the H^2 semi-norm.

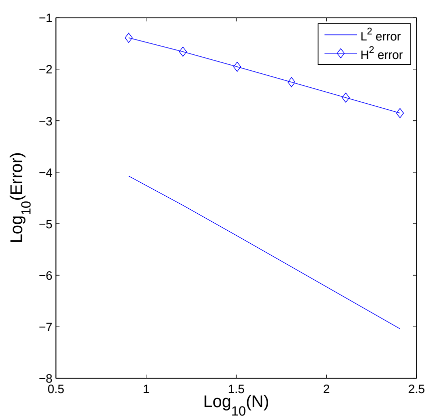

Figure 8: The error of the numerical solution of u_h in log scale. The exact solution is given by $u(x) = x^6 - 3x^5 + 3x^4 - x^3$. Local basis are quadratic functions and the partition of unity functions are given by (3.12). The error convergence rate is $O(h^2)$ in the L^2 norm and $O(h)$ in the H^2 semi-norm.

example, in the 2D case of rectangular meshes, the local degree of freedom is 9 for the proposed method comparing to 16 for the Bogner-Fox-Schmit rectangle. While this does not necessarily implies less degrees of freedom globally since we use an overlapping mesh, the proposed method lead simple implementation and potential savings for higher dimensional problems. For higher regularity elements ($H^k, k \geq 3$, conforming elements), the proposed method is simpler and will lead much less degrees of freedom since the possible use of normal vectors

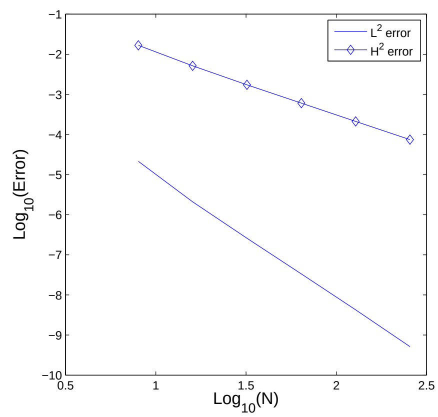

Figure 9: The error of the numerical solution of u_h in log scale. The exact solution is given by $u(x) = x^6 - 3x^5 + 3x^4 - x^3$. Local basis are Hermite basis functions and the partition of unity functions are given by (3.12). The error convergence rate is $O(h^4)$ in the L^2 norm and $O(h^2)$ in the H^2 semi-norm.

to define degrees of freedom is avoided. Note that the degrees of freedom for normal vectors are not respected by affine transformations in general. Furthermore, the proposed method inherited the major advantages of the PUFEM such as the ability to use different local basis functions to address local behavior of the solution which is known as an a priori.

Future work includes the application of the method on triangle and tetrahedron meshes in two and three dimension, and the implementation of essential boundary conditions.

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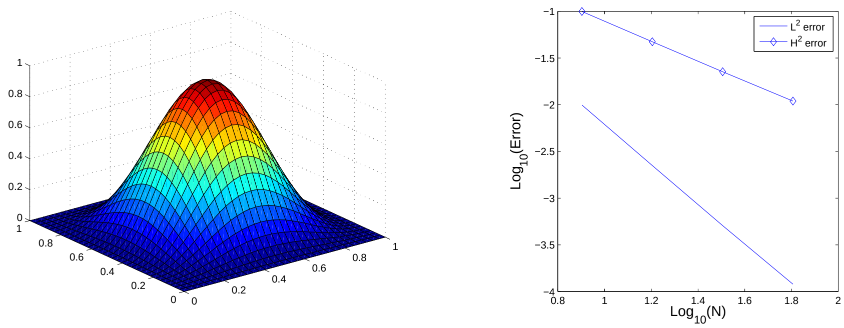

Figure 10: Left: The plot of the numerical solution ($h = 1/32$). Right: The error of the numerical solution in log scale. The error convergence rate is $O(h^2)$ in the L^2 norm and $O(h)$ in the H^2 semi-norm.

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