Partition of unity refinement for local approximation

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Abstract

In this paper, we propose a Partition of Unity Refinement (PUR) method to improve the local approximations of elliptic boundary value problems in regions of interest. The PUR method only needs to refine the local meshes and hanging nodes generate no difficulty. The mesh qualities such as uniformity or quasi-uniformity are kept. The advantages of the PUR include its effectiveness and relatively easy implementation. In this paper, we present the basic ideas and implementation of the PUR method on triangular meshes. Numerical results for elliptic Dirichlet boundary value problem on an L-shaped domain are shown. The extensions of the PUR method to multilevel and higher dimension are straightforward.

1 Introduction

In order to reduce the local approximation error of a finite element solution to PDEs, mesh refinement is widely used [1, 3, 18]. Most classical techniques need to refine beyond the local regions [1, 3] which leads to the mesh quality degeneracy to some extent [14]. For quadrilateral and hexahedral meshes, it is especially difficult to perform refinement which preserves compatibility and mesh quality [10, 13]. For three dimensional problem, mesh refinement is still a challenging topic [8, 18]. Moreover, the computational cost related to local mesh refinement can not be ignored [8].

In this paper, we propose a Partition of Unity Refinement (PUR) method to improve the local approximation of elliptic boundary value problems. Starting with some initial mesh, this method only (uniformly) refines elements in the local regions. Combining a subset of the initial mesh and the refined mesh, the PUR method leads to a nonmatching overlapping grid. Then it constructs a global conforming finite element space by using partition of unity finite element method (PUFEM) [4, 12] (see [2, 11, 17, 19] and references therein for recent developments and applications of the PUFEM). The issues of inter-element continuity and local approximability is separated. Since the method allows hanging nodes, there is no need to refine elements outside the local regions. Typical problems in classical local mesh refinement techniques such as refinement beyond the local

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regions and mesh quality degeneracy are avoided. Thus the related computation cost of mesh refinement is minimized.

The advantages of the PUR include its effectiveness and relatively easy implementation. It makes no difference for triangular, quadrilateral or hexahedral meshes. In addition, the PUR can be applied recursively to generate a multilevel version. Its extensions to higher dimensions are also straightforward. For simplicity, we will restrict our presentation in two dimensions. The paper is organized as follows. In Section 2, we describe the PUR method for local approximation in the context of finite element method for elliptic boundary value problem, in particular, Dirichlet problem. Our presentation is based on triangular meshes. In Section 3, we discuss various implementation issues of the PUR by using an L-shaped domain. In Section 4, we show some numerical examples. Finally we draw some conclusions in Section 5.

2 Partition of unity refinement

In this section, we give a presentation of the PUR method in the frame work of the finite element method for elliptic boundary value problems. The weak formulation is given by: Find $u \in U$ such that

$$a(u, v) = f(v), \quad \forall v \in V, \tag{2.1}$$

where $a(\cdot, \cdot)$ is a continuous bilinear form on $U \times V$ and $f(\cdot)$ is continuous linear functional on V. The corresponding discrete problem is: Find $u_h \in U_h$ such that

$$a(u_h, v) = f(v), \quad \forall v \in V_h, \tag{2.2}$$

where U_h and V_h are finite dimensional subspaces of Hilbert spaces U and V, respectively.

Assume that a computational domain Ω and an initial (triangular) mesh $\mathcal{T}$ are given. Based on some a priori information or a posteriori estimation, the solution on some disjoint local regions $\Omega_i \subsetneq \Omega, i = 1, \ldots, N$, needs to be improved. Let $\mathcal{T}_i \subset \mathcal{T}, i = 1, \ldots, N$ be the corresponding triangulation on Ω_i. Choose a "boundary layer" close to $\partial\Omega_i \setminus \partial\Omega$ which is contained in Ω_i and is denoted by Ω_i^o. More precisely, Ω_i^o is a strip type region made up from triangles of $\mathcal{T}_i$ which are close to $\partial\Omega_i \setminus \partial\Omega$.

Let

$$\Omega_0 = \Omega \setminus \cup_{i=1}^{N} \left(\Omega_i \setminus \Omega_i^o\right).$$

Then $\{\Omega_i\}, i = 0, \ldots, N$ is an open cover (patches) of Ω such that

$$\forall x \in \Omega, \quad \text{card}\{i | x \in \Omega_i\} \leq 2.$$

We define a Lipschitz partition of unity $\{\phi_i\}$ subordinate to $\{\Omega_i\}$:

$$\begin{aligned}
&\text{supp}\phi_i \subset \text{closure}(\Omega_i) \quad i = 0, \ldots, N, \\
&\phi_i(x) = \begin{cases} 0 & \text{on } \Omega \setminus \Omega_i, \\ 1 & \text{on } \Omega_i \setminus \Omega_i^o, \end{cases} \\
&\phi_0(x) + \phi_i(x) = 1 \text{ on } \Omega_i^o.
\end{aligned}$$

To improve the approximation in local regions $\Omega_i, i = 1, \ldots, N$, the corresponding mesh $\mathcal{T}_i$ is refined to obtain mesh $\mathcal{T}_i^f, i = 1, \ldots, N$. In general, $\mathcal{T}_i^f$ is chosen to be a uniform refinement of $\mathcal{T}_i$. Let $\mathcal{T}_0$ denote the set of triangles in $\mathcal{T}$ covering Ω_0. Thus $\{\mathcal{T}_0, \mathcal{T}_i^f, i = 1, \ldots, N\}$ gives an overlapping non-matching mesh on Ω. To minimize the computation effort, the size of overlapping region Ω_i^o, $i = 1, \ldots, N$, is the size of a few grids in the initial mesh. Thus we end up with thin overlapping and non-matching grids [6, 11]. A schematic is shown in Fig. 2 for two overlapping regions Ω_0 and Ω_1 for an L-shaped domain.

Remark 2.1. *The simplest choice of the partition of unity function is piecewise linear functions. Note that the choices of partition of unity functions have impact on the condition number of the stiffness matrix, for example, see [5] .*

Remark 2.2. *Since the PUR method involves uniform refinement in local regions, the refinement cost is minimized.*

Remark 2.3. *The refinement on each patch $\Omega_i, i = 1, \ldots, N$ needs not to be same. For instance, we can uniformly divide the triangles in $\mathcal{T}_1$ into 4 triangles and divide the triangles in $\mathcal{T}_2$ into 9 triangles.*

Remark 2.4. *Since hanging nodes are allowed, the PUR for quadrilateral and hexahedral meshes can be done similarly to triangular meshes. Note that classical techniques for refinement of quadrilateral and hexahedral meshes are more difficult [10, 13].*

On each patch $\Omega_i, i = 1, \ldots, N$, we build a local approximation space V_i corresponding to the mesh T_i. Note that the mesh on Ω_0, which occupies most portion of Ω, does not change. Thus the corresponding approximation space V_0 does not need to change. Now we can define the following global approximation space as in [12].

Definition 2.1. *Let $\{\Omega_i\}$ be an open cover of $\Omega \subset R^n$ constructed as above and let $\{\phi_i\}$ be a partition of unity subordinate to $\{\Omega_i\}$. Let $V_i \subset H^1(\Omega_i \cap \Omega)$ be given. Then the space*

$$V := \sum_i \phi_i V_i = \{\sum_i \phi_i v_i | v_i \in V_i\} \subset H^1(\Omega)$$

is called the global approximation space. The space V_i are referred to as the local approximation spaces.

In this paper, we will simply consider linear finite element for V_i's, $i = 0, \ldots, N$. The approximation property of V is stated in the following theorem which can be proved similarly as [12] (also see [11]).

Theorem 2.5. *Let $\Omega \in R^n$ be given. Let $\{\phi_i\}$, $\{\Omega_i\}$ and $\{V_i\}$ be as in the definitions above and h_i the maximal mesh size of the triangulation on Ω_i. Let $u \in H^2(\Omega)$ be the function to be approximated and $h = \max_i h_i$. Assume that the local approximation spaces V_i have the following approximation properties: On each patch $\Omega_i \cap \Omega$, u can be approximated by a function $v_i \in V_i$ such that*

$$\|u - v_i\|_{L^2(\Omega_i \cap \Omega)} \leq c h_i^2 |u|_2,$$
$$\|\nabla(u - v_i)\|_{L^2(\Omega_i \cap \Omega)} \leq c h_i |u|_2.$$

Then the function

$$v = \sum_i \phi_i v_i \in V \subset H^1(\Omega)$$

satisfies

$$\|u - v\|_{L^2(\Omega)} \leq ch^2 |u|_2, \tag{2.3}$$

$$\|\nabla(u - v)\|_{L^2(\Omega)} \leq ch|u|_2. \tag{2.4}$$

Proof. We will prove (2.4) and (2.3) can be shown similarly. Since the functions ϕ_i form a partition of unity, we have $u = \sum_{i=0}^N \phi_i u$ and thus

$$\begin{aligned} \|\nabla(u-v)\|^2_{L^2(\Omega)} &\leq \|\nabla \sum_{i=0}^N \phi_i (u - v_i)\|^2_{L^2(\Omega)} \\ &\leq 2\|\sum_{i=0}^N \nabla\phi_i (u - v_i)\|^2_{L^2(\Omega)} + 2\|\sum_{i-0}^N \phi_i \nabla(u - v_i)\|^2_{L^2(\Omega)}. \end{aligned}$$

For any given $x \in \Omega$, there are at most two patches cover it. Thus the sums $\sum_{i=0}^N \nabla\phi_i(u - v_i)$ and $\sum_{i-0}^N \phi_i \nabla(u - v_i)$ contains at most 2 terms for any fixed $x \in \Omega$. For any $x \in \Omega$,

$$\begin{aligned} |\sum_{i=0}^N \nabla\phi_i (u - v_i)|^2 &\leq 2\sum_{i=0}^N |\nabla\phi_i (u - v_i)|^2, \\ |\sum_{i=0}^N \phi_i \nabla(u - v_i)|^2 &\leq 2\sum_{i=0}^N |\phi_i \nabla(u - v_i)|^2. \end{aligned}$$

We obtain

$$\begin{aligned} & 2\|\sum_{i=0}^N \nabla\phi_i (u - v_i)\|^2_{L^2(\Omega)} + 2\|\sum_{i-0}^N \phi_i \nabla(u - v_i)\|^2_{L^2(\Omega)} \\ \leq\ & 4\sum_{i=0}^N \|\nabla\phi_i (u - v_i)\|^2_{L^2(\Omega)} + 4\sum_{i=0}^N \|\phi_i \nabla(u - v_i)\|^2_{L^2(\Omega)} \\ =\ & 4\sum_{i=0}^N \|\nabla\phi_i (u - v_i)\|^2_{L^2(\Omega_i \cap \Omega)} + 4\sum_{i=0}^N \|\phi_i \nabla(u - v_i)\|^2_{L^2(\Omega_i \cap \Omega)} \\ \leq\ & 4\sum_{i=0}^N \frac{c}{h_i^2} h_i^4 |u|_2^2 + 4\sum_{i=0}^N c h_i^2 |u|_2^2 \\ \leq\ & ch^2 |u|_2^2. \end{aligned}$$

Hence (2.4) follows immediately. □

A major advantage of the PUR method is that local mesh generation (refinement) only involves uniform refinement and thus the mesh quality is kept. No mesh optimization is needed. The mesh in Ω_0 is untouched and there is no need to reconstruct the stiffness matrix corresponding to this part. Note that in general, the region Ω_0 occupies a large portion of the whole domain. The actual work involved is to construct a partition of unity function.

The choice of the overlapping region and the partition of unity functions affect the properties of the global approximation space. If we keep the mesh ratio between neighboring subdomains constant, then the condition number of the global stiffness matrix is bounded by Ch^{-2} , where h is the mesh size of the finest meshed region (see [5] for some discussion).

The hanging nodes from local refinement are taken care of by the partition of unity functions. The local spaces are glued by PU functions to produce a global conforming approximation space [12]. This is our conforming space for the PUR method. It is obvious that the extension of the PUR method to higher dimension is straightforward. Moreover, the PUR method can be extended to multilevel by applying the PUR around the local region iteratively when necessary.

3 Implementation of the PUR

Now we illustrate the implementation of the PUR by using an L-shaped domain. Suppose we have an initial quasi-uniform mesh (see Fig. 1). We would like to improve the approximation in a local region around the reentrant angle (blue region in Fig. 1). We will refine the mesh in the local region uniformly and end

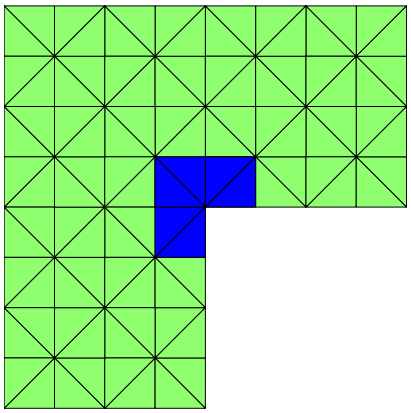

Figure 1: An L-shaped domain and the initial mesh. The region around $(0, 0)$ (the blue region around the re-entrant corner) is chosen for more accurate approximation.

up with overlapping regions covering the whole domain.

To get a better visualization, in Fig. 2, we make refinement region larger than necessary. Let Ω_a denote the (green) region outside the banded (light blue) region, Ω_1^o the banded (light blue) region, and Ω_b the region inside the banded region Ω_1^o. Then $\Omega_0 = \Omega_a \cup \Omega_1^o$ and $\Omega_1 = \Omega_b \cup \Omega_1^o$ give a cover of the L-shaped domain.

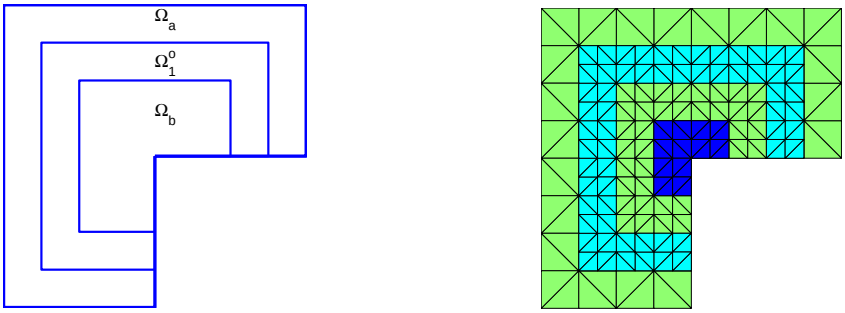

Figure 2: An example of the PUR method for the L-shaped domain. Left: A schematic figure for the PUR. $\Omega_0 = \Omega_a \cup \Omega_1^o$, $\Omega_1 = \Omega_b \cup \Omega_1^o$. Right: The overlapping mesh for the L-shaped domain. The coarse mesh in Ω_a outside the banded region (light blue) is kept. The coarse mesh in Ω_1 is refined. The banded region Ω_1^o (light blue) is the overlapping region where partition of unity functions needs to be defined. The region closely surrounding the re-entrant corner $(0, 0)$ is also highlighted (blue).

3.1 Data Structure

The data structure related to the PUR method is rather simple. Suppose the initial mesh is given by a series of triangles $\{K_i, i = 1, \ldots, M\}$. The local mesh on Ω_1 is given by $\cup_j K_j, j \in R$, where $R \subset \{1, \ldots, M\}$ is the index set. Then we choose a boundary region Ω_1^o of Ω_1. Assume the mesh on Ω_1^o is $\cup_j K_j, j \in O \subset R$ where O is the corresponding index set. Now the triangles in $\cup_j K_j, j \in R$, are refined to obtain a finer mesh $\{k_i\}$. The mesh $\{k_i\}$ and $\{K_i\}, i \in \{1, \ldots M\} \setminus R$, give an overlapping mesh for Ω.

What we need is the mapping between the triangles $K_j, j \in O$ and k_i. This can be done by constructing a simple $o \times n$ matrix $C2F$ where o is the number of triangles in Ω_1^o and n is the number of triangles each K_j is divided into. For example, if each K_j is divided uniformly into 4 congruent triangles, the jth row of the matrix is given by

$$p, \quad q, \quad r, \quad s$$

which indicates that triangle K_j is divided into four small triangles and their indices in the refined mesh $\{k_i\}$ is p, q, r, s. The data structure $C2F$ is used together with the partition of unity functions to construct the global stiffness matrix.

3.2 Partition of Unity function in the overlapping region

The choices of the partition of unity function in the overlapping region are flexible. However, it does have an impact on the condition number of the global stiffness matrix. In our implement, we choose piecewise linear functions, i.e., the partition of unity functions are linear on each triangle.

Partition of unity functions ϕ_0 and ϕ_1 associated with $\{\Omega_0, \Omega_1\}$ for the L-shaped domain are shown in Fig. 3.

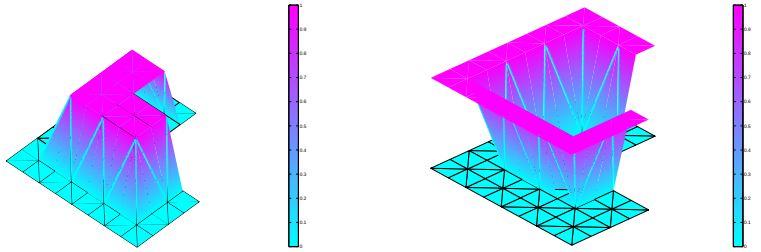

Figure 3: Piecewise linear partition of unity functions corresponding to Fig. 2. Left: ϕ_0. Right: ϕ_1. Note that ϕ_0 is 1 on $\Omega_0 \setminus \Omega^o$ and 0 on $\Omega_1 \setminus \Omega^o$ while ϕ_1 is 0 on $\Omega_0 \setminus \Omega^o$ and 1 on $\Omega_1 \setminus \Omega^o$. On the overlapping region Ω^o, the partition of unity functions are piecewise linear.

3.3 The global approximation space

The global approximation space V for the PUR method is constructed from the partition of unity function ϕ_i's, $i = 0, 1$, and the local approximation spaces V_i's. One expects that the functions in the form of

$$\{\phi_i v_{i,m}\} \text{ where } \{v_{i,m}\} \text{ are basis functions of } V_i$$

are the basis functions for V. Unfortunately, $\{\phi_i v_{i,m}\}$ might be linearly dependent. For example, if we choose linear basis function in each V_i and linear partition of unity functions in the overlapping region, the set $\{\phi_i v_{i,m}\}$ is linearly dependent (see the discussion on basis functions on the reference triangle in Section 3.4). However, by forcing the Dirichlet boundary condition, we can show that the set $\{\phi_i v_{i,m}\}$ is indeed linearly independent.

Theorem 3.1. *Assume that the boundary condition is Dirichlet and the overlapping meshes for Ω_0 and Ω_1 are obtained as above. If $V_k, k = 0, 1$ are linear finite element spaces with basis functions $\{v_i\}$ and $\{w_j\}$, then the basis functions of V are exactly*

$$\{\phi_1 v_i, \phi_2 w_j\}.$$

We refer the readers to [6] for the proof of the theorem.

3.4 Basis functions in the overlapping region

In the overlapping region, the basis functions from two patches lead to more degrees of freedom. Thus we have an enriched basis. We will use the reference triangle $\hat{K}$ to illustrate this.

Assume that we use linear finite element basis functions. On the reference triangle, the basis functions are given by

$$v_1 = 1 - x - y, \tag{3.5a}$$

$$v_2 = x, \tag{3.5b}$$

$$v_3 = y. \tag{3.5c}$$

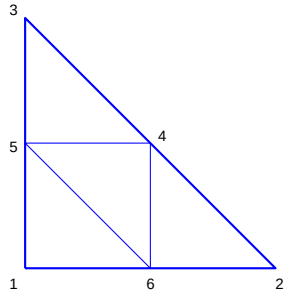

Figure 4: The reference triangle $\hat{K}$ are divided into four triangles in the overlapping region. Basis functions on the reference triangle and four small triangles are multiplied by partition of unity functions respectively to form an approximation space.

Suppose a uniform refinement leads to four congruent small triangles. On the small triangle $\hat{k}$ whose vertices are given by $(0,0)$, $(1/2,0)$ and $(0,1/2)$, the basis functions are given by

$$w_1 = 1 - \frac{x}{2} - \frac{y}{2}, \tag{3.6a}$$

$$w_2 = \frac{x}{2}, \tag{3.6b}$$

$$w_3 = \frac{y}{2}. \tag{3.6c}$$

Suppose that the partition of unity functions are given by

$$\phi_1 = 1 - x - y,$$
$$\phi_2 = x + y,$$

such that $\phi_1 + \phi_2 = 1$ on the reference triangle. The induced functions on the small triangles $\hat{k}$ are given by

$$U = \{\phi_1 v_i, i = 1,2,3\} \bigcup \{\phi_2 w_i, i = 1,2,3\}.$$

The approximation by U is better than linear. To be precise, U contains the following functions

$$(1 - x - y)(1 - x - y), \tag{3.7a}$$

$$x(1 - x - y), \tag{3.7b}$$

$$y(1 - x - y), \tag{3.7c}$$

$$(1 - \frac{x}{2} - \frac{y}{2})(x + y), \tag{3.7d}$$

$$\frac{x}{2}(x + y), \tag{3.7e}$$

$$\frac{y}{2}(x + y). \tag{3.7f}$$

The above functions are linearly dependent. After simple calculation we can show that a basis for span$U$ is given by

$$\{1, x, y, x^2 + xy, y^2 + xy\}.$$

This is better than linear approximation but not a complete quadratic finite element basis.

3.5 The PUR algorithm

Now we give the major steps of the PUR method.

1. Start with some initial mesh on the domain.
2. Identify the local regions where improvement of approximation are needed.
3. Uniformly refine the local regions identified in Step 2.
4. Construct adequate partition of unity functions and the finite element approximation space.
5. Construct the stiffness matrix and right hand side.
6. Solve the matrix problem.

If further improvement of local approximations is necessary, it is easy to define a multilevel PUR method similarly to the above algorithm. We will present a two level example in the next section.

4 Numerical Examples

Now we show some numerical examples which are implemented using Matlab. We use the following model problem:

$$-\triangle u = 0 \tag{4.8}$$

on an L-shaped domain $(-1,1)\times(-1,1)\setminus([0,1]\times[-1,0])$ with a reentrant corner at $(0,0)$. Dirichlet boundary condition is specified such that the exact solution reads

$$u(r,\theta) = r^{2/3}\sin\left(\frac{2}{3}\theta\right)$$

in polar coordinates.

4.1 Improvement of local approximation

For a linear finite element solution on uniform triangular mesh, a residual type a posteriori estimator indicates the local refinement is needed around the reentrant corner at $(0,0)$ (see Fig.5) .

Remark 4.1. *In Fig. 5, we apply a residual type a posteriori error estimator for illustration. Posteriori error estimates are of central importance for adaptive finite element and numerous paper have been devoted to them. We simply refer readers to [3, 15], and [16] which gives an excellent introduction on the derivation of an explicit residual-based a posteriori error estimates for Galerkin finite element discretizations of general linear elliptic operators.*

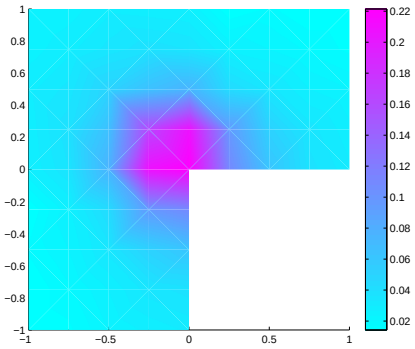

Figure 5: A residual type a posteriori error estimator shows that the error around the corner $(0, 0)$ is significantly larger than that away from the corner.

For comparision, we first use the linear finite element solver and the exact solution to calculate the local error (the blue region in Fig. 1) on a series of uniformly refined meshes. Fig.1 shows a rather coarse initial mesh and a region of interest around the origin. The mesh size is $h = 1/4$. The total degree of freedom (Dof) is 33. The L^2 error for a linear finite element method on the whole domain is 0.0213. While the L^2 error on the small shaded region, 1/32 of the total area, is 0.0180, contributing around 85% of the global error and indicating a local refinement is necessary. Similarly, the local H^1 error is 0.1215 while the global H^1 error is 0.1451. A global uniform refinement reduces the local L^2 error to 0.0066, the local H^1 error to 0.0849 ($h = 1/8$, DoF: 144). Another level of global uniform refinement reduces the local L^2 error to 0.0025, the local H^1 error to 0.0563 ($h = 1/16$, DoF: 705).

Now we apply the PUR method. We first refine local mesh in the shaded region. Away from the shaded region, the mesh keeps unchanged. In Fig. 2, we show a locally refined mesh. Hanging nodes can be seen easily. Note that the banded region is the overlapping region. It is where the partition of unity takes effect. In this case, the local L^2 error is 0.0066, which is close to the local L^2 error of the global uniform refinement. The local H^1 error is 0.0852. Note that the total degrees of freedom are 148 and $h_l = 1/8$ in the region of interest.

If we take the refinement as shown Fig.6, the local L^2 error is 0.0066, which is close to the local L^2 error of the global uniform refinement. The local H^1 error is 0.0851. Note that the total degrees of freedom are 89, which is much less than that of the uniform refinement (DoF: 144).

To obtain a finer local refinement, we might divide a triangle from the initial mesh into more $(9, 16, \ldots)$ triangles uniformly. In Fig. 7 we divide the initial triangle from the coarse mesh into 16 triangles uniformly. The L^2 error is 0.0040, the local L^2 error is 0.0026, the local H^1 error is 0.0564. The total degree of freedom is 241.

The results are summarized in Table. 1. It can be seen that using considerably small number of degree of freedoms, the PUR obtain local errors similar to those of uniform finite element method.

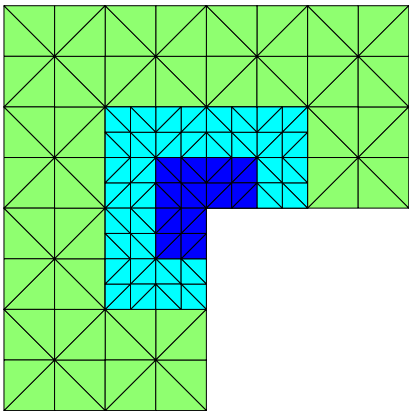

Figure 6: Another locally refined mesh. Note that the refined region is chosen closely surrounded the interested area around $(0, 0)$.

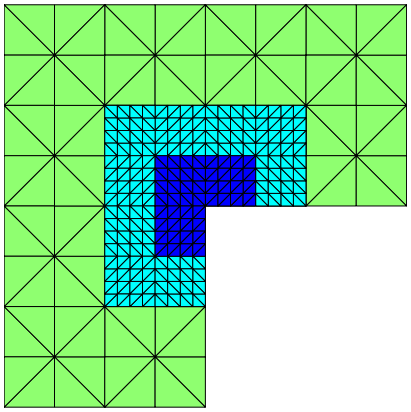

Figure 7: The finer local refinement. Each triangle in the initial mesh around the interested region is uniformly divided into 16 small triangles.

4.1.1 A two level example

The PUR method can be easily taken to multilevel. Suppose we have done one level PUR as in Fig.2. In the refinement region, we can carry another level of partition of unity finite element around $(0, 0)$. The sizes of triangles in each level are $h = 1/4$, $h = 1/8$ and $h = 1/16$, respectively. The local L^2 error is 0.0026 and the local H^1 error is 0.0563. The extension of the PUR method to multilevel is similar.

$h = 1/8$ locally	local L^2 error	local H^1 error	DoF
FEM on uniform mesh	0.0066	0.0851	144
PUR	0.0068	0.0852	89
$h = 1/16$ locally	local L^2 error	local H^1 error	DoF
FEM on uniform mesh	0.0025	0.0563	705
PUR	0.0026	0.0564	241

Table 1: Local L^2 and H^1 errors of the PUR and classical finite element method. It can be seen that using considerably small number of degree of freedoms, the PUR obtain local errors similar.

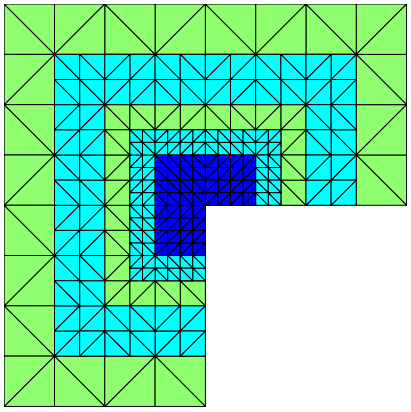

Figure 8: An example of two level PUR method. The first level PUR is the same as in Fig. 2. A second level refinement is performed to obtain finer mesh around $(0, 0)$.

4.2 Use the PUR method to obtain optimal convergence

Again we use model problem (4.8). Due the singularity around $(0, 0)$, the L_2 norm of the error for linear finite element method on uniform mesh is worse than $O(DoF^{-2})$. This is verified by the numerical results in Fig. 9 where we plot the L_2 norm of the error against the number of degree of freedom in log scale. Similarly, the H^1 norm of the error for linear finite element is worse than $O(DoF^{-1})$ (see Fig. 10).

Now we apply the PUR method. We choose a small region around the singularity $(0, 0)$ and uniformly refine the mesh in the selected region. We do this for several initial meshes.

In Fig. 9, we plot the L_2 norm of the error of the PUR method. We can see that the global convergence is almost $O(DoF^{-2})$. The H^1 norm of the error of the PUR method is almost $O(DoF^{-1})$.

Remark 4.2. *In our implementation, we divide each triangle into 16 small triangles in the local region. The ratio needed to achieve optimal convergence for a general elliptic problem depends on the singularities of the solution. An*

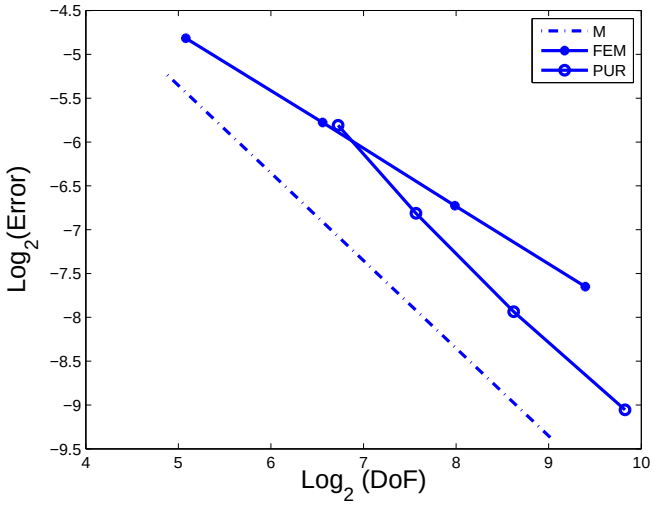

Figure 9: The L_2 error order of PUR method for linear element. It can be seen that the global convergence order of L_2 error is close to $M := O(DoF^{-2})$.

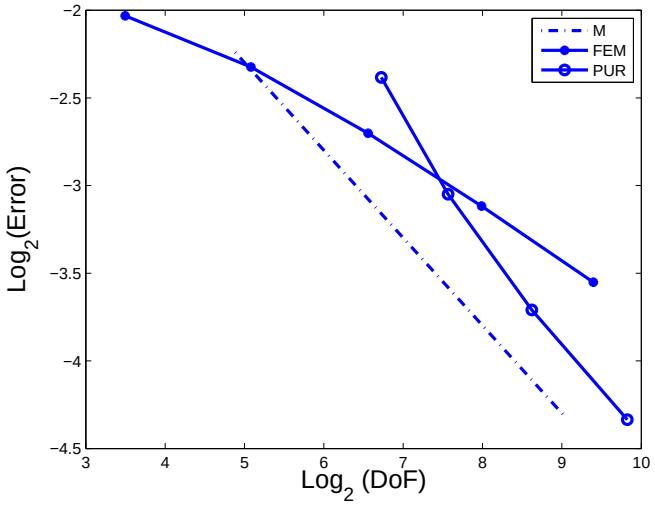

Figure 10: The global H_1 error order of the PUR method. The global convergence order is almost $M := O(DoF^{-1})$.

optimal strategy of refinement for elliptic problems on polygonal domains can be based on the equi-distribution principle presented in [9] or on the graded meshes strategy as presented e.g., in [3, 7, 15].

Remark 4.3. *Multilevel PUR can be applied to obtain optimal convergence. Based on the discussion in [9], it is possible to find out the number of levels*

needed, the thickness of each layer and the size of triangulation in each layer. It is currently under investigation.

5 Conclusions

In this paper, we propose a PUR method to improve local approximations for finite element method of PDEs. Since the method only (uniformly) refine the elements in the local regions, it minimizes the refinement cost. Moreover, the proposed method does not depend on the shape of the elements (triangular, quadrilateral or hexahedral meshes). Future works include application of the PUR method to three dimension large scale problems and the multilevel extension to obtain optimal convergence.

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References

[1] D.N. Arnold, A. Mukherjee and L. Pouly, Locally adapted tetrahedral meshes using besection, *SIAM J. Sci. Comput.*, **22** (2001), no. 2, pp. 431–448.

[2] I. Babuška, U. Banerjee and J.E. Osborn, Survey of meshless and generalized finite element methods: a unified approach. *Acta Numer.*, **12** (2003), pp. 1–125

[3] I. Babuška, R.B. Kellog, and J. Pitkäranta. Direct and inverse error estimates for finite elements with mesh refinements, *Numer. Math.*, **33** (1979), pp. 447–471.

[4] I. Babuška, J.M. Melenk, The partition of unity method, *Internat. J. Numer. Methods Eng.*, **40** (1997) pp. 727–758.

[5] C. Bacuta and J. Sun, Notes on the Schwarz alternating method for partition of unity FEM, *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.*, **16** (2009), Differential Equations and Dynamical Systems, suppl. S1, pp.15–21.

[6] C. Bacuta and J. Sun, Partition of Unity Finite Element Method Implementation for Poisson Equation, Advances in Applied and Computational Mathematics, Nova Publishers, 2006, pp.35–46

[7] C. Bacuta, V. Nistor, and L. Zikatanov. Improving the rate of convergence of high order finite elements on polygons and domains with cusps. *Numer. Math.*, **100** (2005), pp.165–184.

[8] E. Bellenger and P. Coorevits, Controlled cost of adaptive mesh refinement in practical 3D finite element analysis, *Adv. Eng. Softw.*, **38** (2007) 846–859.

[9] L. Chen and J. Xu, Convergence of Adaptive Finite Element Methods, Adaptive Computations: Theory and Algorithms, Mathematics Monograph Series, Tao Tang and Jinchao Xu ed., (2007), no. 6, pp 1–33.

[10] P.J. Frey and P.L. George, Mesh Generation, Application to Finite Elements, Hermes, 2000.

[11] Y. Huang and J. Xu, A conforming finite element method for overlapping and nonmatching grids, *Math. Comput.*, **72**, (2002), no. 243, pp.1057–1066.

[12] J.M. Melenk and I. Babuška, The partition of unity finite element method: Basic theory and applications, *Comput. Method Appl. Mech. Engrg.*, **139** (1996), pp. 289–314.

[13] G.P. Nikishkov, Finite element algorithm with adaptive quadtree-octree mesh refinement, *ANZIAM J.*, **46** (E) (2005), pp.C15–C28.

[14] M.A. Padron, J.P. Surez, A. Plaza and G.F. Carey, A comparative study between some bisection based partitions in 3D, *Appl. Numer. Math.*, **55**(2005) 357-367.

[15] G. Raugel, Résolution numérique par une méthode d'éléments finis du problème de Dirichlet pour le laplacien dans un polygone, *C. R. Acad. Sci. Paris Sér. A-B* **286** (1978), no. 18, A791–A794.

[16] J.R. Stewart and T. Hughes, A tutorial in elementary finite element error analysis: A systematic presentation of a prior and a posteriori error estimates, *Comput. Methods Appl Mech. Engry.*, **158** (1998) pp.1–22.

[17] T. Strouboulis, I. Babuška, R. Hidajat, The generalized finite element method for Helmholtz equation: theory, computation, and open problems, *Comput. Methods Appl. Mech. Eng.*, **195** (2006), pp. 4711– 4731.

[18] J.P. Suarez, P. Abad, A. Plaza and M.A. Padron, Computational aspects of the refiment of 3D tetrahedral meshes, *J. Comput. Methods Sci. Eng.*, **5** (2005), 1–10.

[19] T. Strouboulis, I. Babuška and R. Hidajat, The generalized finite element method for Helmholtz equation: theory, computation, and open problems. *Comput. Methods Appl. Mech. Engrg.* **195** (2006), no. 37-40, pp. 4711–4731.