Estimation of transmission eigenvalues and the index of refraction from Cauchy data

Jiguang Sun *

Abstract

Recently the transmission eigenvalue problem has come to play an important role and received a lot of attention in inverse scattering theory. This is due to the fact that transmission eigenvalues can be determined from the far field data of the scattered wave and used to obtain estimates for the material properties of the scattering object. In this paper, we show that transmission eigenvalues can also be obtained from the near field Cauchy data. In particular, we use the gap reciprocity method to estimate the lowest transmission eigenvalue. To determine the index of refraction, we apply an optimization scheme based on a finite element method for transmission eigenvalues. Numerical examples show that the method is stable and effective.

1 Introduction

Recently a new qualitative method using transmission eigenvalues to estimate the index of refraction of the non-absorbing inhomogeneous medium emerged [11, 5, 4]. For the case of scattering of acoustic waves by an inhomogeneous medium $D \subset \mathbb{R}^2$, the transmission eigenvalue problem is to find $k \in \mathbb{C}$, $w, v \in L^2(D)$, $w - v \in H^2(D)$ such that

$$\Delta w + k^2 n(x) w = 0, \quad \text{in } D, \tag{1.1a}$$

$$\Delta v + k^2 v = 0, \quad \text{in } D, \tag{1.1b}$$

$$w - v = 0, \quad \text{on } \partial D, \tag{1.1c}$$

$$\frac{\partial w}{\partial \nu} - \frac{\partial v}{\partial \nu} = 0, \quad \text{on } \partial D, \tag{1.1d}$$

where ν is the unit outward normal to the boundary ∂D and the index of refraction $n(x)$ is positive. Values of $k \neq 0$ such that there exists a nontrivial solution to (1.1) are called transmission eigenvalues. The existence of transmission eigenvalues has been studied by many researchers [17, 15, 7, 14] and numerical methods have been developed as well [10, 18].

Since transmission eigenvalues can be determined from the far field data, they have been used to estimate the material properties of the scattering object

[1]Department of Mathematical Sciences, Delaware State University, Dover, DE 19901, U.S.A.
E-mail: `jsun@desu.edu`

[2, 3, 6, 5, 4]. The process can be divided into two steps. Transmission eigenvalues are first computed from the far field data by solving linear ill-posed integral equations. Then the estimation of index of refraction is computed from the transmission eigenvalues using inequalities such as Faber-Krahn type inequality [11, 6, 7, 4].

In this paper, we consider the estimation of the index of refraction of the inhomogeneity with Cauchy data using the reciprocity gap (RG) method. The RG method was proposed by Colton and Haddar [9] to obtain the support of the scattering object using Cauchy data. The method was further developed to treat other inverse problems by many researchers [12, 16, 13, 1]. Assuming that the support of the inhomogeneity is determined by the RG method, we show that the transmission eigenvalues can be computed from the near field data and apply an optimization method to estimate the index of refraction based on a continuous finite element method for transmission eigenvalues [10]. Numerical examples show that the method is stable and provides useful information in addition to the lower and/or upper bounds obtained from the inequalities such as Faber-Krahn type inequality.

The rest of the paper is organized as the following. In Section 2, we present the scattering problem of an inhomogeneous non-absorbing medium and the reciprocity gap method. Then we show that the RG method can be used to obtain transmission eigenvalues from near field Cauchy data in Section 3. In Section 4, base on the numerical scheme developed in [10], we apply an optimization scheme to estimate the index of refraction. Finally we draw some conclusions and discuss some future work in Section 5.

2 The reciprocity gap functional

We consider the scattering problem of an inhomogeneous non-absorbing medium due to a point source. The direct problem is to find a solution $u \in H^1_{loc}(\mathbb{R}^2 \setminus \{z\})$ such that

$$\Delta u + k^2 n(x) u = 0 \qquad \text{in } \mathbb{R}^2 \setminus \{z\}, \tag{2.2a}$$

$$u = u^i + u^s, \tag{2.2b}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0, \tag{2.2c}$$

where k is the wavenumber, $n(x)$ is the index of refraction, u^i is the incident field, and u^s is the scattered field. We assume that $D := \text{supp}(n(x) - 1)$ has finitely many components, each component having smooth boundary ∂D with unit outward normal ν, and that the curves across which $n(x)$ is discontinuous are piecewise smooth. Domain D is contained in the interior of a second bounded domain Ω whose boundary is denoted by $\Gamma = \partial\Omega$ (see Fig.1). Let u^i be the incident field due to a point source at z given by

$$u^i(x, z) = \Phi(x, z) \tag{2.3}$$

where $\Phi(x, z) = \frac{i}{4} H_0^{(0)}(k|x - z|)$. It can be shown that there exists a unique solution to the direct scattering problem (2.2) [9].

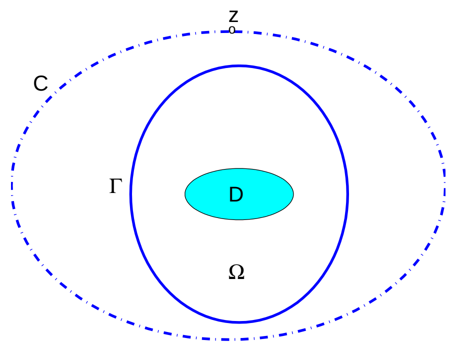

Figure 1: Explicative Example.

The inverse problem we are interested in is to estimate $n(x)$ from a knowledge of the Cauchy data of the total field u on Γ. We assume that both u and $\frac{\partial u}{\partial \nu}$ are known on Γ for each source point $z \in C$ where C is a simple closed curve containing Ω. Denote by $\mathbb{H}(\Omega)$ the set

$$\mathbb{H}(\Omega) = \{v \in H^1(\Omega) \,:\, \Delta v + k^2 n(x) v = 0 \text{ in } \Omega\}$$

and by U the set of solutions to (2.2a)-(2.2c) for all $z \in C$. For $v \in \mathbb{H}(\Omega)$ and $u \in U$ we define the reciprocity gap functional by

$$\mathcal{R}(u, v) = \int_\Gamma \left(u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} \right) ds. \tag{2.4}$$

Since u depends on the point $z \in C$, the reciprocity gap functional can be seen as an operator

$$\mathrm{R} : \mathbb{H}(\Omega) \to L^2(C),$$

defined by

$$\mathrm{R}(v)(z) = \mathcal{R}(u(\cdot, z), v). \tag{2.5}$$

The following theorems provide important properties of the reciprocity gap operator [2].

Theorem 2.1. *The reciprocity gap operator* $\mathrm{R} : \mathbb{H}(\Omega) \to L^2(C)$ *is injective if and only if* k *is not a transmission eigenvalue.*

Theorem 2.2. *If* k *is not a transmission eigenvalue, then the reciprocity gap operator* $\mathrm{R} : \mathbb{H}(\Omega) \to L^2(C)$ *has dense range.*

To reconstruct D, the RG method computes the regularized solutions $v \in \mathbb{H}(\Omega)$ to the integral equation

$$\mathcal{R}(u, v) = \mathcal{R}(u, \Phi_z) \qquad \text{for all} \quad u \in U \tag{2.6}$$

where $\Phi_z := \Phi(\cdot, z)$ for sampling points $z \in S$, a domain inside Ω containing D. When k is not a transmission eigenvalue, the reciprocity gap functional method characterizes D using the norm of v_z, the regularized solution of (2.6), for different sampling points $z \in \Omega$. The following theorem in [9] justifies the reconstruction of D using the RG method.

Theorem 2.3. *Assume that k is not a transmission eigenvalue for D.*

(a) If $z \in D$ then there exist a sequence $\{v_n\}$, such that

$$\lim_{n\to\infty} \mathcal{R}(u, v_n) = \mathcal{R}(u, \Phi_z) \qquad \text{for all } u \in U.$$

Furthermore, v_n converges in $L^2(D)$.

(b) If $z \in \Omega \setminus D$ then for every sequence $\{v_n\}$, such that

$$\lim_{n\to\infty} \mathcal{R}(u, v_n) = \mathcal{R}(u, \Phi_z) \qquad \text{for all } u \in U$$

we have that

$$\lim_{n\to\infty} \|v_n\|_{L^2(D)} = \infty.$$

We refer the readers to [9] for more details on the RG method. In the rest the paper we assume that D is obtained using the RG method and focus on the computation of transmission eigenvalues from Cauchy data and estimation of the index of refraction using transmission eigenvalues.

3 Computation of transmission eigenvalues from Cauchy data

Now we consider the problem of computing transmission eigenvalues from Cauchy data. The near field equation (2.6) can be written as

$$R(v)(z) = l(z) \tag{3.7}$$

where $l(z) := \mathcal{R}(u, \Phi(\cdot, z))$. To solve the above equation, we need to use a convenient family of solutions in $\mathbb{H}(\Omega)$ which satisfies appropriate density properties. In particular, we use Herglotz wave functions defined as

$$v = \mathcal{H}g := \int_\Omega e^{ikx\cdot d} g(d)\, \mathrm{d}s_d,\ g \in L^2(S^1),$$

where $S^1 = \{x \in \mathbb{R}^2, |x| = 1\}$.

Let R^δ be the reciprocity gap operator corresponding to the noisy measurement $(u^\delta, \frac{\partial u^\delta}{\partial \nu})$. The Tikhonov regularized solution $g^\delta_{z,\epsilon}$ of the near field equation is defined as the unique minimizer of the Tikhonov functional

$$\|R^\delta(\mathcal{H}g) - l(z)\|^2_{L^2(S^1)} + \epsilon\|g\|^2_{L^2(S^1)} \tag{3.8}$$

where ϵ is the regularization parameter. We denote $g^\delta_{z,\epsilon(\delta)}$ by $g_{z,\delta}$ when $\epsilon = \epsilon(\delta) \to 0$ as $\delta \to 0$. It can be shown that if k is not a transmission eigenvalue then $\mathcal{H}g_{z,\delta}$ converges in the $H^1(D)$ norm as $\delta \to 0$ [9]. Now we consider the case when k is a transmission eigenvalue. We assume that, for all points $z \in D$, the perturbed operator R^δ satisfies

$$\lim_{\delta \to 0} \|R^\delta \mathcal{H} g_{z,\delta} - \Phi(\cdot, z)\|_{L^2(S^1)} = 0. \tag{3.9}$$

If the operator R has dense range, the above assumption is true [3]. It is well-known that R has dense range except when k is a transmission eigenvalue associated with non-trivial solutions (w_0, v_0) of (1.1) such that v_0 can be represented as a Herglotz wave function.

The following theorem shows the behavior of the solution of (3.8) when k is a transmission eigenvalue. Its proof is similar to the Theorem 3.2 in [3] and thus is omitted here.

Theorem 3.1. *Let k be a transmission eigenvalue and assume that* (3.9) *is verified. Then for almost every $z \in D$, $\|g_{z,\delta}\|_{L^2(S^1)}$ cannot be bounded in D as $\delta \to 0$.*

Combining Theorems 3.1 and 2.3, if we choose a point z inside D and plot the norms of the kernels of the Tikhonov regularized solutions against k, we would expect the norms are relatively large when k is a transmission eigenvalue and relatively small when k is not a transmission eigenvalue.

Now we show numerically that transmission eigenvalues can be obtained from Cauchy data. We assume that D is reconstructed by the reciprocity gap method [9]. Note that if we only need to estimate transmission eigenvalues we do not need precise reconstruction of ∂D. A knowledge of a point z in D would suffice. For simplicity, we will use the exact shape of D in our numerical examples. We use a finite element method to solve the scattering problem on a mesh fine enough for all wavenumbers in the region $I = [1, 4]$. We put 40 point sources on the curve C which is the boundary of the circle with radius 4. We record the Cauchy data on $\partial\Omega$, the boundary of the circle with radius 3, and add 3% noise. Then we choose a point inside D and solve the ill-posed integral equation (2.6) using Tikhonov regularization with Morozov discrepancy. Finally we plot the norm of the Herglotz kernel g against the wavenumber k.

Let D be a disk with radius 1/2 centered at $(0, 0)$ and the index of refraction $n(x) = 16$. The exact lowest transmission eigenvalue is 1.99 given in [10]. We choose a point $(0.2, 0.2)$ inside D and solve (2.6) using Tikhonov regularization for all wavenumbers in I. In Fig. 2 we show the plot of $\|g_z\|_{L^2(S^1)}$ against k. It can be seen that roughly around 2.00 the norm of g is significantly larger. There are other locations where the norms are larger indicating the possibility of other transmission eigenvalues. Note that to improve the estimation, it is possible to choose a collection of points inside D and take the average of $\|g_z\|_{L^2(S^1)}$ for all these points [4]. Next, let D be the unit square and everything else keeps the same as the above example. We plot $\|g_z\|_{L^2(S^1)}$ against k in Fig. 3. This time we have the estimated lowest transmission eigenvalues as 1.76 comparing to the value 1.89 given in [10].

Now we consider the case when $n(x)$ is a function. Let D be a disk centered at $(0, 0)$ with radius 1/2. The index of refraction $n(x) = 8 + 4|x|$. The lowest transmission eigenvalue is 2.83. We repeat the above procedure and plot $\|g_z\|_{L^2(S^1)}$ against k (Fig. 4). The computation gives $k_1^\delta = 2.78$. Next let D be the unit square given by $(-1/2, 1/2) \times (-1/2, 1/2)$. The index of refraction $n(x) = 8 + x_1 - x_2$. The lowest transmission eigenvalue is 2.88. We repeat the

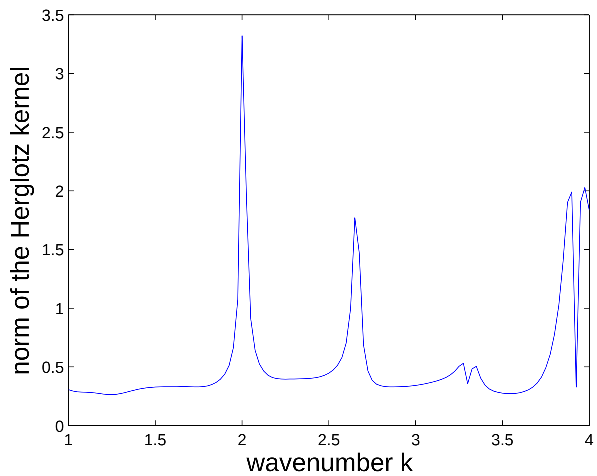

Figure 2: The plot of $\|g_z\|_{L^2(S^1)}$ against k for a point $(0.2, 0.2)$ inside the D. Here D is a disk with radius $1/2$ and the index of refraction $n(x) = 16$. The exact lowest transmission eigenvalue is 1.99.

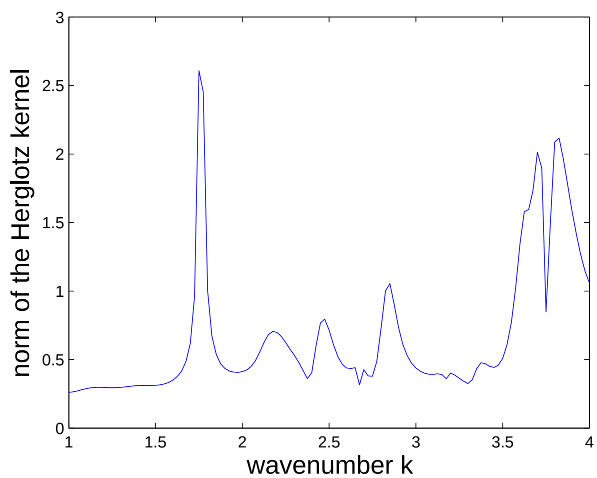

Figure 3: The plot of $\|g_z\|_{L^2(S^1)}$ against k for a point $(0.2, 0.2)$ inside the D. Here D is the unit square and the index of refraction $n(x) = 16$. The exact lowest transmission eigenvalue is 1.89.

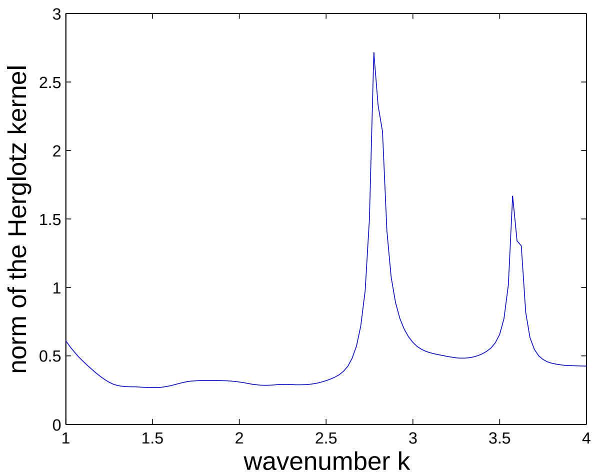

Figure 4: The plot of $\|g_z\|_{L^2(S^1)}$ against k. Here D is a disk with radius 1/2 centered at $(0,0)$ and the index of refraction $n(x) = 8 + 4|x|$. The exact lowest transmission eigenvalue is 2.83.

Table 1: The lowest transmission eigenvalues k_1 and the estimated lowest transmission eigenvalues k_1^δ using Cauchy data.

domain D	index of refraction n	k_1	k_1^δ
disk $r = 1/2$ centered at $(0,0)$	16	1.99	2.00
unit square $(-1/2,1/2) \times (-1/2,1/2)$	16	1.89	1.76
disk $r = 1/2$ centered at $(0,0)$	$8 + 4|x|$	2.83	2.78
unit square $(-1/2,1/2) \times (-1/2,1/2)$	$8 + x_1 - x_2$	2.88	2.90

above procedure and plot $\|g_z\|_{L^2(S^1)}$ against k (Fig. 5). The computation gives $k_1^\delta = 2.90$ for this case.

In Table 2, we summarize the results. Considering the noise and the ill-posedness of the problem, the RG method provide excellent estimations for the lowest transmission eigenvalue using Cauchy data.

4 Estimation of the index of refraction

In the previous section, we obtain an approximation $k_1^\delta(D)$ of the lowest transmission eigenvalue $k_1(D)$. Now we turn to the problem of estimating the index of refraction using $k_1^\delta(D)$. In [11] Colton et al. applied the Faber-Krahn type inequality to obtain a lower bound for $\sup_D n(x)$.

$$k_1^2(D) > \frac{\lambda_0(D)}{\sup_D n(x)} \tag{4.10}$$

where $\lambda_0(D)$ is the first Dirichlet eigenvalue.

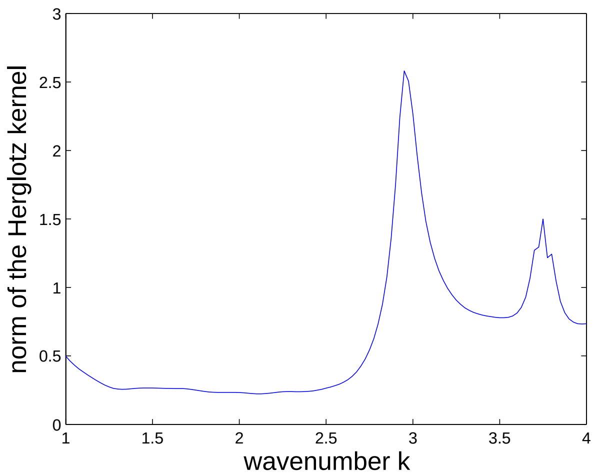

Figure 5: The plot of $\|g_z\|_{L^2(S^1)}$ against k. Here D is the unit square $(-1/2, 1/2) \times (-1/2, 1/2)$ and the index of refraction $n(x) = 8 + x_1 - x_2$. The exact lowest transmission eigenvalue is 2.88.

Based on the continuous finite element method for transmission eigenvalues presented in [10] and the result in [4], we apply an optimization method to estimate the index of refraction $n(x)$. The result will provide information of $n(x)$ in addition to its upper or lower bounds using methods such as the Faber-Krahn type inequality.

Let $\mu_D : L^\infty(D) \to \mathbb{R}$ which maps a given index of refraction n to the lowest transmission eigenvalue of D, i.e.,

$$\mu_D(n) = k_1(D). \tag{4.11}$$

Assuming $k_1^\delta(D)$ is obtained as in the previous section, we seek a constant n_0 minimizing the difference between $\mu_D(n)$ and $k_1^\delta(D)$, i.e.,

$$n_0 = \underset{n}{\operatorname{argmin}} |\mu_D(n) - k_1^\delta(D)|. \tag{4.12}$$

When the index of refraction is constant, the following lemma holds (see [4] for its proof).

Lemma 4.1. *The function μ_D is a differentiable function of n. Moreover, denoting $\tau := k^2$, if $f(\tau, n) := \mu_1(n\tau^2) - (n+1)\tau$, then $\frac{\partial f}{\partial \tau} < 0$ when $\tau < \frac{n+1}{2n}\lambda_0(D)$ where $\lambda_0(D)$ is the first Dirichlet eigenvalue of the negative Laplacian in D.*

We show a plot of the lowest transmission eigenvalue against the constant index of refraction in Fig. 6. Since $k_1(D)$ is a continuous function of n for $n > 1$, we can look for n_0 such that the computed lowest transmission eigenvalue $\mu(D)$ coincides with the value $k_1^\delta(D)$ obtained from the near field data using the following AlgorithmN. In each step, the transmission eigenvalues are computed using the continuous finite element method proposed in [10]. Note that the

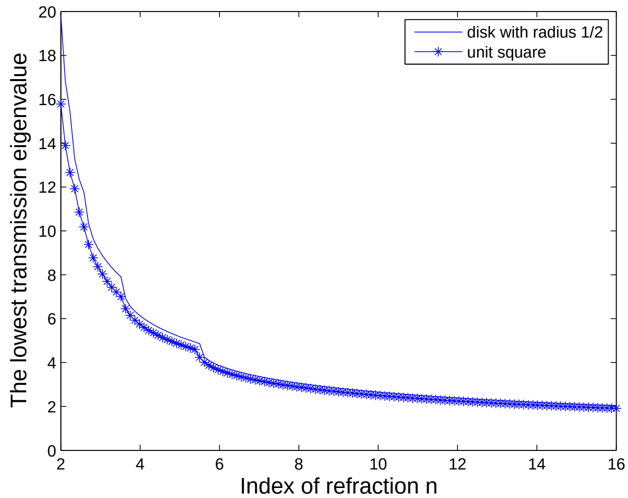

Figure 6: The lowest transmission eigenvalue v.s. the index of refraction for the disk with radius 1/2 and the unit square. For both cases, $\mu(D)$ is a decreasing function of n in the domain studied.

proposed method could also be used for the far field case since the algorithm only use the lowest transmission eigenvalue.

AlgorithmN $n_0 = algorithmN(D, k_1^\delta, tol)$

generate a regular triangular mesh for D

estimate an interval a and b based on the Fraba-Krahn type inequality

compute k_1^a and k_1^b using the continuous the finite element method

while abs$(a - b) > tol$

$c = (a + b)/2$ and compute k_1^c

if $|k_1^c - k_1^\delta| < |k_1^a - k_1^\delta|$

$a = c$

else

b=c

end

end

$n_0 = c$

Table 2: Estimation of the index of refraction. The last column is the lower bound for $\sum_D n$ computed using the Faber-Krahn type inequality.

domain D	exact n	n_0	$\sup_D n$
disk $r = 1/2$ centered at $(0,0)$	16	16.40	5.80
unit square $(-1/2,1/2) \times (-1/2,1/2)$	16	18.30	6.37
disk $r = 1/2$ centered at $(0,0)$	$8 + 4\|x\|$	9.33	3.00
unit square $(-1/2,1/2) \times (-1/2,1/2)$	$8 + x_1 - x_2$	7.87	2.35

We now show some numerical examples. We first choose D to be the circle with radius $1/2$ and $n = 16$. From last section using noisy Cauchy data, we have $k_1^\delta = 2.00$. Since $\lambda_0(D) = 23.21$, inequality (4.10) gives a lower bound for $\sup_D n(x)$ as 5.80. The above algorithm gives $n_0 = 16.40$. Next we let D be the unit square and thus $\lambda_0(D) = 2\pi^2$. Let $n(x) = 16$. From last section, we have $k_1^\delta = 1.76$. Using (4.10), we obtain a lower bound for $\sup_D n(x)$ given by 6.37. The above algorithm gives $n_0 = 18.30$.

When n is a function, we can still seek a constant estimation. Let D be a disk centered at $(0,0)$ with radius $1/2$. The index of refraction $n(x) = 8 + 4|x|$. The computed lowest transmission eigenvalue is $k_1^\delta = 2.78$. The algorithm gives $n_0 = 9.33$. Using (4.10), we obtain a lower bound for $\sup_D n(x)$ given by 3.00. Next let D be the unit square given by $(-1/2,1/2) \times (-1/2,1/2)$. The index of refraction $n(x) = 8 + x_1 - x_2$. The lowest transmission eigenvalue is $k_1^\delta = 2.90$. The algorithm gives $n_0 = 7.87$. Using (4.10), we obtain a lower bound for $\sup_D n(x)$ given by 2.35.

We summarize the results in Table 2 together with the estimation of the lower bound of $\sup_D(n)$ using the Faber-Krahn inequality (4.10). The algorithm gives a stable estimation of the index of refraction accurately. It is easy to see that the algorithm depends on how well we can estimate the lowest transmission eigenvalue from Cauchy data.

Finally we consider the case when the shape of the target is not known exactly. Again let D be the unit square. We first use the reciprocity gap method to obtain the reconstruction of ∂D. We choose the wavenumber $k = 3$ and compute the Cauchy data on $\partial\Omega$ for all point sources on the curve C. The sampling domain is the square given by $(-1,1) \times (-1,1)$. For each sampling point z, we solve the ill-posed integral equation (2.6) by Tikhonov regularization with Morozov discrepancy. For better visualization, we plot the contour of $1/\|g_z\|$ and choose a cut off value by "Calibration" [8] to obtain a reconstruction contour of ∂D (see Fig. 7). Then this contour is used as the boundary for D to compute the lowest transmission eigenvalue in AlgorithmN. When the index of refraction is $n = 16$, we obtain an estimation $n_0 = 18.90$. The same procedure is carried out for $n = 8 + x_1 - x_2$. The reconstruction of ∂D is shown in Fig. 8 and the estimation is $n_0 = 8.55$. It can be seen that when reconstruction is close to ∂D the estimation of the index of refraction is good. In fact, instead of the reciprocity gap method, any other method can be used as long as a good reconstruction of ∂D can be obtained.

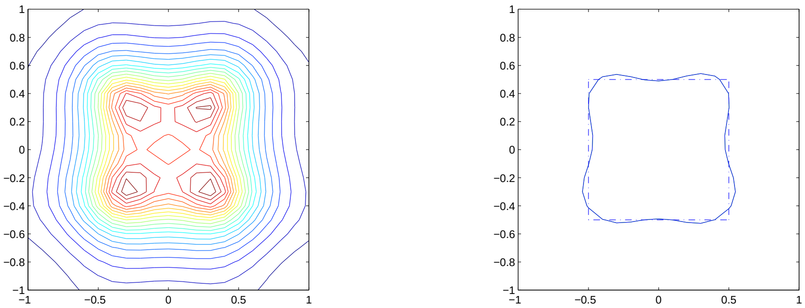

Figure 7: Reconstruction of the unit square when $n = 16$. Left: contour plot of $1/\|g_z\|_{L^2(S^1)}$ in the sampling domain. Right: the reconstruction (solid line) and the exact boundary (dashed line) of D.

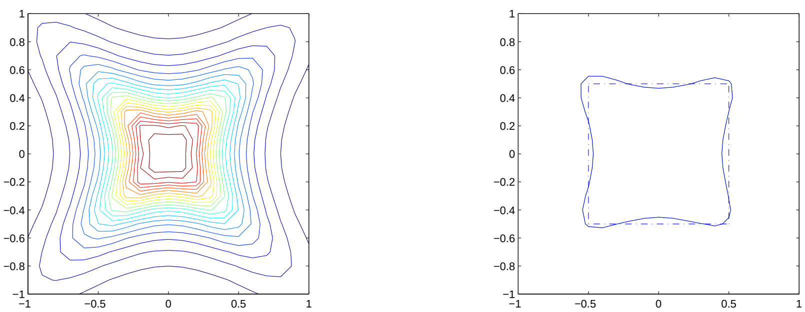

Figure 8: Reconstruction of the unit square when $n = 8 + x_1 - x_2$. Left: contour plot of $1/\|g_z\|_{L^2(S^1)}$ in the sampling domain. Right: the reconstruction (solid line) and the exact boundary (dashed line) of D.

5 Conclusion and future works

In this paper, we show that transmission eigenvalues can be computed from the near field Cauchy data. In addition, we apply an optimization method to estimate the index of refraction based on the lowest transmission eigenvalue. The numerical results validate the effectiveness of the method. The case for Maxwell's equations is under investigation.

Acknowledgement

The research of Jiguang Sun was supported in part by NSF grant DMS-1016092 and DEPSCoR grant W911NF-07-1-0422.

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