THE RECIPROCITY GAP METHOD FOR A CAVITY IN AN INHOMOGENEOUS MEDIUM

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Abstract. We consider an interior inverse medium problem of reconstructing the shape of a cavity. Both the measurement locations and point sources are inside the cavity. Due to the lack of a priori knowledge of physical prosperities of the medium inside the cavity and to avoid the computation of background Green's functions, the reciprocity gap method is employed. We prove the related theory and present some numerical examples for validation.

1. **Introduction.** Inverse scattering problems have wide applications such as radar, medical imaging, geophysical explorations, etc. In contrast to the typical exterior scattering problem, we consider the interior inverse scattering problem of determining the shape of a cavity. Both measurements and point sources are distributed inside the cavity. The study of such problems is motivated by non-destructive testing in industrial applications such as monitoring the structural integrity of the fusion reactor [12]. Interior inverse scattering problems have attracted many researchers recently. In [12], Jakubik and Potthast used the solutions of the Cauchy problem by potential methods and the range test to study the integrity of the boundary of some cavity by acoustics. Later, the linear sampling method [21, 22, 25, 11, 3, 23], the nonlinear integral equation method [19], the decomposition method [26], the factorization method [15, 18], and the near-field imaging method [14] were applied.

In this paper, we study the interior scattering problem of identifying an inhomogeneous cavity, which maybe anisotropic, embedded in a known background medium using the reciprocity gap method due to Colton and Haddar [5], which has been applied to many exterior inverse scattering problems (cf. [9, 16, 17]). For the interior sound-soft obstacle scattering problem, we have shown in [27] that this method is well suited to the case when there is a lack of information of the physical properties of the medium inside the cavity. Furthermore, there is no need to consider the background Green's functions which may penalize the efficiency or are not even known. Here we extend the reciprocity gap method to penetrable obstacle under appropriate transmission boundary condition. The extension of the result to three dimension cases are straightforward and the reconstruction quality should be similar.

Note that inverse scattering problems for anisotropic media are challenging and many of them are not well-understood mathematically [24]. For example, it is not possible to uniquely determine the constitutive parameters of an anisotropic medium from the scattering data [4].

The rest of our paper is organized as the following. In Section 2, we consider the direct and inverse medium problems of an penetrable cavity with inhomogeneous medium. In Section 3, a reciprocity gap method based on a linear integral equation is introduced and the related theory is studied. We provide some preliminary numerical examples to show the viability of the method in Section 4.

2. **Direct and inverse problems.** Let D be a simply connected bounded Lipschitz domain in $\mathbb{R}^2$ and B be a region inside D which is a piecewise inhomogeneous medium with index of refraction $n_1(x)$. The medium in $D\backslash\overline{B}$ is homogeneous with the index of refraction 1. We denote by k the wave number. The medium outside D is assumed to be inhomogeneous and possibly anisotropic such that outside a large ball B_R it is homogenous with the same wave number as the medium in $D\backslash\overline{B}$.

As in [3], the physical properties of the medium in $\mathbb{R}^2\backslash\overline{D}$ are described by the 2×2 symmetric matrix valued function A with $L^\infty(\mathbb{R}^2\backslash\overline{D})$ entries such that

$$\overline{\xi} \cdot \mathrm{Re}(A)\xi \geq \alpha\|\xi\|^2 \quad \text{and} \quad \overline{\xi} \cdot \mathrm{Im(A)}\xi \leq 0$$

for all $\xi \in \mathbb{C}$ and some $\alpha > 0$, and the bounded function $n_2 \in L^\infty(\mathbb{R}^2\backslash\overline{D})$ such that

$$\mathrm{Re}(n_2) \geq n_0 > 0 \quad \text{and} \quad \mathrm{Im(n_2)} \geq 0$$

in $B_R\backslash\overline{D}$. Furthermore, we assume that $A \equiv I$ and $n_2 \equiv 1$ in $\mathbb{R}^2\backslash\overline{B_R}$ where B_R is a large ball containing D.

If u^i is the Green's function $\mathbb{G}(x, x_0)$ due to a point source x_0 on a smooth curve C contained in $D\backslash\overline{B}$, we can formulate the direct scattering problem of finding the total fields $(u, w) \in H^1(D\backslash\{x_0\}) \times H^1_{loc}(\mathbb{R}^2\backslash\overline{D})$ such that

(1a) $$\triangle u + k^2 n_1(x)u = 0 \quad \text{in} \quad D\backslash\{x_0\},$$

(1b) $$\nabla \cdot A\nabla w + k^2 n_2(x)w = 0 \quad \text{in} \quad \mathbb{R}^2\backslash\overline{D},$$

(1c) $$u = w \quad \text{and} \quad \partial_\nu u = \partial_{\nu_A} w \quad \text{on} \quad \partial D,$$

(1d) $$\lim_{r \to \infty} r^{1/2}(\frac{\partial w}{\partial r} - ikw) = 0,$$

where $n_1(x), n_2(x)$ are piecewise continuous, ν is the unit outward normal to the indicated curve, and $\partial_{\nu_A} w := A\nabla w \cdot \nu$. Furthermore, $n_1(x) \equiv 1$ for $x \in D\backslash\overline{B}$ and

$n_2(x) \equiv 1$ for $x \in \mathbb{R}^2 \backslash \overline{B}_R$. We assume that B has finitely many components and the curves across which $n_1(x)$ is discontinuous are piecewise smooth. The total field $u = u^s + u^i$, where u^s is the scattered field. In a similar way as Theorem 5.1 in [3], it is not difficult to show that u^i and u^s satisfy the reciprocity relations

$$(2) \qquad u^s(x, x_0) = u^s(x_0, x) \quad \text{and} \quad u^i(x, x_0) = u^i(x_0, x),$$

for x and x_0 in $D \backslash \overline{B}$. We note that u^i can be written in the form

$$u^i(x, x_0) = \Phi(x, x_0) + \Phi^s(x, x_0) = \mathbb{G}(x, x_0) \quad \text{in} \quad \mathbb{R}^2$$

for $x \neq x_0$, where

$$\Phi(x, x_0) = \frac{i}{4} H_0^{(1)}(k|x - x_0|).$$

Here $H_0^{(1)}$ is the Hankel function of the first kind of order zero. Using a variational approach (see e.g. [2]), it can be shown that the direct problem (1) has a unique solution.

In particular, the direct problem (1) can be formulated as the following general form. Given $f \in H^{1/2}(\partial D)$, $h \in H^{-1/2}(\partial D)$, find $(u, w) \in H^1(D) \times H^1_{loc}(\mathbb{R}^2 \backslash \overline{D})$ such that

$$(3a) \qquad \triangle u + k^2 n_1(x) u = 0 \quad \text{in} \quad D,$$

$$(3b) \qquad \nabla \cdot A \nabla w + k^2 n_2(x) w = 0 \quad \text{in} \quad \mathbb{R}^2 \backslash \overline{D},$$

$$(3c) \qquad u - w = f \quad \text{and} \quad \partial_\nu u - \partial_{\nu_A} w = h \quad \text{on} \quad \partial D,$$

$$(3d) \qquad \lim_{r \to \infty} r^{1/2} \left(\frac{\partial w}{\partial r} - ikw \right) = 0.$$

It is obvious that the scattered field u^s and the exterior field w satisfying (1) solve the general problem (3) with $f = -\mathbb{G}(\cdot, x_0)$ and $h = -\partial_\nu \mathbb{G}(\cdot, x_0)$.

Note that the exterior transmission problem plays an important role in the inverse problem of (3). Given $f \in H^{1/2}(\partial D)$, $h \in H^{-1/2}(\partial D)$, find $(u, w) \in H^1(D) \times H^1_{loc}(\mathbb{R}^2 \backslash \overline{D})$ such that

$$(4a) \qquad \triangle v + k^2 v = 0 \quad \text{in} \quad \mathbb{R}^2 \backslash \overline{D},$$

$$(4b) \qquad \nabla \cdot A \nabla w + k^2 n_2(x) w = 0 \quad \text{in} \quad \mathbb{R}^2 \backslash \overline{D},$$

$$(4c) \qquad v - w = f \quad \text{and} \quad \partial_\nu v - \partial_{\nu_A} w = h \quad \text{on} \quad \partial D,$$

$$(4d) \qquad \lim_{r \to \infty} r^{1/2} \left(\frac{\partial w}{\partial r} - ikw \right) = 0 \quad \text{and} \quad \lim_{r \to \infty} r^{1/2} \left(\frac{\partial v}{\partial r} - ikv \right) = 0.$$

We remark here that in [3] it was shown that problems (3) and (4) are well-posed. Now let Ω be a bounded Lipschitz domain in D such that $D_c \subset \Omega \subset D$ (see Figure 1), where D_c is the interior of C. The inverse problem we are interested in is to determine the shape of the scattering object from the knowledge of Cauchy data of the total field u on $\partial \Omega$ for fixed (but not necessarily known) n_1, n_2 and A satisfying the above assumptions. In [3], it is proved that the boundary of a homogeneous cavity is uniquely determined from a knowledge of the scattered field $u^s(x, y)$ for all $x, y \in C$. The argument can be carried over in a straightforward manner to our case. In some special cases, the shape of the targets can be determined even using a single point source [10].
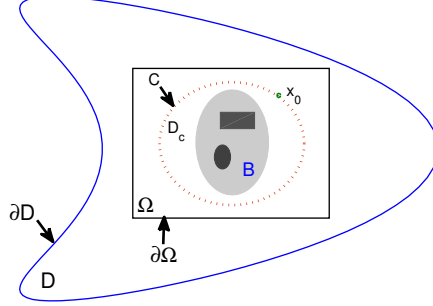
FIGURE 1. Explicative picture. The cavity is denoted by D. The domains D_c and Ω are contained in D and $D_c \subset \Omega$. Point sources and measurements are distributed on the boundaries C and $\partial\Omega$, respectively.

Definition 2.1. A non-zero value $k^2 \in \mathbb{C}$ is called a generalized Dirichlet eigenvalue of $-\triangle$ in D if there exists a non-trivial solution $u \in H^1(D)$ satisfying

$$\triangle u + k^2 n_1(x)u = 0 \quad \text{in} \quad D,$$
$$u = 0 \quad \text{on} \quad \partial D.$$

Definition 2.2. A value of $k \in \mathbb{C}$ with $\mathfrak{R}(k) > 0$ is called an exterior transmission eigenvalue if the homogeneous exterior transmission problem, i.e. (4) with $f = h = 0$, admits a nontrivial solution.

In this paper, we always assume that k^2 is not a generalized Dirichlet eigenvalue in D_c and k is not an exterior transmission eigenvalue.

3. **The reciprocity gap method.** In this section, we apply the reciprocity gap method to reconstruct the shape of the target. We first define two spaces [5].

- For an unbounded open domain $\mathbb{R}^2 \backslash \overline{\Omega}$, we denote

$$\mathbb{H}(\mathbb{R}^2\backslash\overline{\Omega}) := \left\{ v \in H^1_{loc}(\mathbb{R}^2\backslash\overline{\Omega}) : \Delta v + k^2 v = 0 \text{ in } \mathbb{R}^2\backslash\overline{\Omega}, \ \lim_{r\to\infty} r^{1/2}(\frac{\partial v}{\partial r} - ikv) = 0 \right\}.$$

- We denote

$$U := \{u : (u, w) \text{ solves (1) with } u^i = \mathbb{G}(\cdot, x_0), \ x_0 \in C\}.$$

For $v \in \mathbb{H}(\mathbb{R}^2\backslash\overline{\Omega})$ and $u \in U$ we define the reciprocity gap functional by

$$(5) \qquad \qquad \mathcal{R}(u, v) = \int_{\partial\Omega} (v\partial_\nu u - u\partial_\nu v)\,\mathrm{d}s,$$

where ν is the unit outward normal to $\partial\Omega$. The functional $\mathcal{R}(u, v)$ can be viewed as an operator $R : \mathbb{H}(\mathbb{R}^2\backslash\overline{\Omega}) \to L^2(C)$ given by

$$(6) \qquad \qquad R(v)(x_0) = \mathcal{R}(u, v)$$

for all point sources $x_0 \in C$ since u depends on x_0.

For the following discussion, we introduce the single layer potential v_g

$$(7) \qquad \qquad v_g(x) = \int_C \Phi(x, y)g(y)\mathrm{d}s(y), \quad g \in L^2(C).$$

We also define the exterior DtN map $\Lambda_D : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$ as $f \mapsto \partial_\nu v|_{\partial D}$, where v satisfies

$$\triangle v + k^2 v = 0 \quad \text{in} \quad \mathbb{R}^2 \backslash \overline{D}, \tag{8a}$$

$$v = f \quad \text{on} \quad \partial D, \tag{8b}$$

$$\lim_{r \to \infty} r^{1/2}(\frac{\partial v}{\partial r} - ikv) = 0. \tag{8c}$$

And the exterior NtD map is given by $\Lambda_N : H^{-1/2}(\partial D) \to H^{1/2}(\partial D)$ as $h \mapsto v|_{\partial D}$, where v is a solution of

$$\triangle v + k^2 v = 0 \quad \text{in} \quad \mathbb{R}^2 \backslash \overline{D}, \tag{9a}$$

$$\partial_\nu v = h \quad \text{on} \quad \partial D, \tag{9b}$$

$$\lim_{r \to \infty} r^{1/2}(\frac{\partial v}{\partial r} - ikv) = 0. \tag{9c}$$

From Rellich's Lemma, the operators Λ_D and Λ_N are well defined and bounded. Clearly, $\Lambda_D \Lambda_N = I$ in $H^{-1/2}(\partial D)$ and $\Lambda_N \Lambda_D = I$ in $H^{1/2}(\partial D)$. Moreover, Green's formula implies that $\Lambda_D = \Lambda_D^*$ and $\Lambda_N = \Lambda_N^*$ in the sense that $\langle \Lambda_D f_1, f_2 \rangle = \langle f_1, \Lambda_D f_2 \rangle$ for all $f_1, f_2 \in H^{1/2}(\partial D)$ and $\langle \Lambda_N h_1, h_2 \rangle = \langle h_1, \Lambda_N h_2 \rangle$ for all $h_1, h_2 \in H^{-1/2}(\partial D)$.

Lemma 3.1. *If k^2 is not a Dirichlet eigenvalue in D_c, then $(v_g|_{\partial D}, \partial_\nu v_g|_{\partial D})$ is complete in $H^{1/2}(\partial D) \times H^{-1/2}(\partial D)$.*

Proof. Define operator $H : L^2(C) \to H^{1/2}(\partial D) \times H^{-1/2}(\partial D)$ as $Hg = (H_1 g, H_2 g)$, where

$$H_1 g = v_g \quad \text{and} \quad H_2 g = \partial_\nu v_g.$$

Note that $H_1 g = \Lambda_N H_2 g$ and $H_2 g = \Lambda_D H_1 g$.

Now we just need to show that H has dense range. To this end it suffices to show that the corresponding dual operator $H^* : H^{-1/2}(\partial D) \times H^{1/2}(\partial D) \to L^2(C)$ defined by

$$\langle H_1 g, \varphi \rangle + \langle H_2 g, \psi \rangle = \langle g, H^*(\varphi, \psi) \rangle \tag{10}$$

for all $g \in L^2(C)$, $\varphi \in H^{-1/2}(\partial D)$, $\psi \in H^{1/2}(\partial D)$ is injective. By interchanging the order of integration, one can show that

$$H^*(\varphi, \psi)(x) = \int_{\partial D} \overline{\Phi(x, y)} \varphi(y) \mathrm{d}s(y) + \int_{\partial D} \overline{\frac{\partial \Phi(x, y)}{\partial \nu}} \psi(y) \mathrm{d}s(y). \tag{11}$$

Using the properties of operators Λ_D and Λ_N, by Green's formula, we can deduce that H^* have the forms of

$$H^*(\varphi, \psi)(x) = \int_{\partial D} \overline{\Phi(x, y)}(\varphi(y) + \Lambda_D \psi(y)) \mathrm{d}s(y), \tag{12}$$

$$H^*(\varphi, \psi)(x) = \int_{\partial D} \overline{\frac{\partial \Phi(x, y)}{\partial \nu}}(\Lambda_N \varphi(y) + \psi(y)) \mathrm{d}s(y). \tag{13}$$

Now assume that $H^*(\varphi, \psi) = 0$ on C. Since $H^*(\varphi, \psi)$ satisfies the Helmholtz equation in D_c, k^2 is not a Dirichlet eigenvalue in D_c yields that $H^*(\varphi, \psi) = 0$ in D_c. From the unique continuation principle, we have $H^*(\varphi, \psi) = 0$ in D which indicates that $H_-^* = 0$ and $\frac{\partial H_-^*}{\partial \nu} = 0$. Using the jump relations [7], we conclude from (12) that

$$H_+^* = H_-^* = 0, \tag{14}$$

and from (13) that

$$\frac{\partial H_+^*}{\partial \nu} = \frac{\partial H_-^*}{\partial \nu} = 0. \tag{15}$$

Finally from (11) and the results in (14) and (15), we get

$$\psi(y) = H_+^* - H_-^* = 0, \quad \text{and} \quad \varphi(y) = \frac{\partial H_-^*}{\partial \nu} - \frac{\partial H_+^*}{\partial \nu}.$$

Thus H^* is injective, i.e., H has dense range. The proof is complete. $\qquad\square$

Theorem 3.2. *The operator $R : \mathbb{H}(\mathbb{R}^2 \backslash \overline{\Omega}) \to L^2(C)$ defined by (6) is injective.*

Proof. Let $v \in \mathbb{H}(\mathbb{R}^2 \backslash \overline{\Omega})$ be such that $Rv = 0$, i.e.,

$$\mathcal{R}(u, v) = 0 \quad \text{for all } u \in U.$$

Let $(\widetilde{u}^s, \widetilde{w})$ be the solution of the following transmission problem

$$\triangle \widetilde{u}^s + k^2 n_1 \widetilde{u}^s = 0 \quad \text{in} \quad D, \tag{16a}$$

$$\nabla \cdot A \nabla \widetilde{w} + k^2 n_2(x) \widetilde{w} = 0 \quad \text{in} \quad \mathbb{R}^2 \backslash \overline{D}, \tag{16b}$$

$$\widetilde{w} - \widetilde{u}^s = v \quad \text{and} \quad \partial_{\nu_A} \widetilde{w} - \partial_\nu \widetilde{u}^s = \partial_\nu v \quad \text{on} \quad \partial D, \tag{16c}$$

$$\lim_{r \to \infty} r^{1/2} \left(\frac{\partial \widetilde{w}}{\partial r} - ik\widetilde{w} \right) = 0. \tag{16d}$$

Recalling that $u = u^s + \mathbb{G}(\cdot, x_0)$, integrating by parts, using the transmission conditions for u and v, together with that facts that $\widetilde{u}^s$ and u^s satisfy the same equation in D and that w and $\widetilde{w}$ are radiating solutions to the same equation outside D, we have that

$$
\begin{aligned}
\mathcal{R}(u, v) &= \int_{\partial \Omega} \{v \partial_\nu u - u \partial_\nu v\} \mathrm{d}s = \int_{\partial D} \{v \partial_\nu u - u \partial_\nu v\} \mathrm{d}s \\
&= \int_{\partial D} \{(\widetilde{w} - \widetilde{u}^s) \partial_\nu u - u(\partial_{\nu_A} \widetilde{w} - \partial_\nu \widetilde{u}^s)\} \mathrm{d}s \\
&= \int_{\partial D} \{\widetilde{w} \partial_{\nu_A} w - w \partial_{\nu_A} \widetilde{w}\} \mathrm{d}s \\
&\qquad - \int_{\partial D} \{\widetilde{u}^s \partial_\nu (u^s + \mathbb{G}(\cdot, x_0)) - (u^s + \mathbb{G}(\cdot, x_0)) \partial_\nu \widetilde{u}^s\} \mathrm{d}s \\
&= \int_{\partial D} \{\widetilde{w} \partial_{\nu_A} w - w \partial_{\nu_A} \widetilde{w}\} \mathrm{d}s - \int_{\partial D} \{\widetilde{u}^s \partial_\nu u^s - u^s \partial_\nu \widetilde{u}^s\} \mathrm{d}s \\
&\qquad - \int_{\partial D} \{\widetilde{u}^s \mathbb{G}(\cdot, x_0) - \mathbb{G}(\cdot, x_0) \partial_\nu \widetilde{u}^s\} \mathrm{d}s \\
&= \widetilde{u}^s(x_0).
\end{aligned}
$$

Thus $\widetilde{u}^s(x_0) = 0$ for all $x_0 \in C$. Since k is not a generalized Dirichlet eigenvalue in D_c, it deduce that $\widetilde{u}^s(x_0) = 0$ in D_c. Then $\widetilde{u}^s(x_0) = 0$ in D by the unique continuation principle. And from the trace theorem we have that

$$\widetilde{u}^s(x_0)|_{\partial D} = 0 \quad \text{and} \quad \partial_\nu \widetilde{u}^s(x_0)|_{\partial D} = 0.$$

Now since $v \in \mathbb{H}(\mathbb{R}^2 \backslash \overline{\Omega})$, we have that $\triangle v + k^2 v = 0$ and v satisfies the Sommerfeld radiation condition. Thus we can conclude that $(v, \widetilde{w})$ solves the following

exterior transmission problem

(17a) $\qquad \triangle v + k^2 v = 0 \quad \text{in} \quad \mathbb{R}^2 \backslash \overline{D},$

(17b) $\qquad \nabla \cdot A \nabla \widetilde{w} + k^2 n_2(x)\widetilde{w} = 0 \quad \text{in} \quad \mathbb{R}^2 \backslash \overline{D},$

(17c) $\qquad \widetilde{w} - v = 0 \quad \text{and} \quad \partial_{\nu_A}\widetilde{w} - \partial_\nu v = 0 \quad \text{on} \quad \partial D,$

(17d) $\qquad \lim\limits_{r \to \infty} r^{1/2}(\dfrac{\partial \widetilde{w}}{\partial r} - ik\widetilde{w}) = 0 \quad \text{and} \quad \lim\limits_{r \to \infty} r^{1/2}(\dfrac{\partial v}{\partial r} - ikv) = 0.$

Since k is not a transmission eigenvalue, we conclude that $v = 0$ in $\mathbb{R}^2 \backslash \overline{D}$. Thus $v = 0$ in $\mathbb{R}^2 \backslash \overline{\Omega}$ from the unique continuation principle. $\qquad \square$

Theorem 3.3. *The operator $R : \mathbb{H}(\mathbb{R}^2 \backslash \overline{\Omega}) \to L^2(C)$ defined by (6) has dense range.*

Proof. Let $\varphi \in L^2(C)$ be such that

$$(Rv, \varphi) = 0 \quad \text{for all } v \in \mathbb{H}(\mathbb{R}^2 \backslash \overline{\Omega}).$$

Then from (5) and the bi-linearity of $\mathcal{R}$ we have

$$(Rv, \varphi) = \int_C \mathcal{R}(u, v)\overline{\varphi(x_0)}\mathrm{d}s(x_0) = \mathcal{R}(h, v),$$

where

$$h(x) = \int_C u(x, x_0)\overline{\varphi(x_0)}\mathrm{d}s(x_0) \quad \text{for} \quad x \in D \backslash C.$$

Since $u = w$ and $\partial_\nu u = \partial_{\nu_A} w$ on ∂D, h can be continuously extended to $\mathbb{R}^2 \backslash C$ as

$$\widetilde{h}(x) = \begin{cases} \int_C u(x, x_0)\overline{\varphi(x_0)}\mathrm{d}s(x_0) & \text{for } x \in D \backslash C, \\ \int_C w(x, x_0)\overline{\varphi(x_0)}\mathrm{d}s(x_0) & \text{for } x \in \mathbb{R}^2 \backslash D. \end{cases}$$

Therefore, we have

(18) $\qquad \mathcal{R}(\widetilde{h}, v) = \int_{\partial D} v \partial_\nu \widetilde{h} - \widetilde{h} \partial_\nu v \mathrm{d}x = 0$

for all $v \in \mathbb{H}(\mathbb{R}^2 \backslash \overline{\Omega})$. And from Lemma 3.1, we get that

$$\widetilde{h}(x) = 0 \quad \text{and} \quad \partial_\nu \widetilde{h} = 0 \quad \text{on } \partial D.$$

At the same time, we have

$$\nabla \cdot A \nabla \widetilde{h} + k^2 n_2(x)\widetilde{h} = 0 \quad \text{in } \mathbb{R}^2 \backslash \overline{D},$$

$$\lim\limits_{r \to \infty} \sqrt{r}(\partial_r \widetilde{h} - ik\widetilde{h}) = 0.$$

If $\widetilde{h} \neq 0$ in $\mathbb{R}^2 \backslash \overline{D}$, then $(\widetilde{h}, 0)$ is a non-trivial solution of the homogeneous problem (4), which contradicts the fact that k is not an exterior transmission eigenvalue. Thus $\widetilde{h} \equiv 0$ in $\mathbb{R}^2 \backslash \overline{D}$. From analytic continuation argument we get that $\widetilde{h}(x) = 0$ in $\mathbb{R}^2 \backslash \overline{D}_c$, and so that $\widetilde{h}(x)|_c = 0$. Since $\triangle \widetilde{h} + k^2 n_1(x)\widetilde{h} = 0$ in D_c and k^2 is not a generalized Dirichlet eigenvalue, $\widetilde{h}(x) = 0$ in D_c. Thus the jump relation shows that

$$\overline{\varphi} = \frac{\partial \widetilde{h}^-}{\partial \nu} - \frac{\partial \widetilde{h}^+}{\partial \nu} = 0,$$

and the proof is complete. $\qquad \square$

For the rest of the paper, we set v to be the single layer potential v_g defined by (7). It is clear that $v_g \in \mathbb{H}(\mathbb{R}\backslash\overline{\Omega})$. The reciprocity gap method is to find an approximate solution $g \in L^2(C)$ to

$$(20) \qquad \mathcal{R}(u, v_g) = \mathcal{R}(u, \Phi_z) \quad \text{for all } u \in U,$$

where $\Phi_z := \Phi(\cdot, z)$ for z in the exterior of Ω. In particular, we will show how such a function g can be used to characterize ∂D. The advantage of the reciprocity gap method is that, for inhomogeneous cavities, only fundamental solutions are needed. In contrast, other qualitative methods, e.g., the linear sampling method, need the background Green's functions, which are either unknown or difficult to compute.

In general, the integral equation (20) does not have solution. Fortunately, it is possible to prove the existence of an approximate solution, which can be used to characterize the cavity D.

Remark 1. The linear sampling method can be viewed as a regularization strategy [1]. Unfortunately, to the authors' knowledge, similar result is not available to date for the reciprocity gap method.

Theorem 3.4. *Assume that k^2 is not a Dirichlet eigenvalue in D_c. Then we have*

(a): *If $z \in \mathbb{R}^2\backslash\overline{D}$, then there exists a sequence $\{g_n\}$, $g_n \in L^2(C)$, such that*

$$\lim_{n\to\infty} \mathcal{R}(u, v_{g_n}) = \mathcal{R}(u, \phi_z) \quad \text{for} \quad \text{all} \quad u \in U.$$

Furthermore, v_{g_n} converges in $H^1_{loc}(\mathbb{R}^2\backslash\overline{D})$ and $v_{g_n} \to \Phi_z$ in $H^{\frac{1}{2}}(\partial D)$.

(b): *If $z \in D\backslash\overline{\Omega}$, then for every sequence $\{g_n\}$, $g_n \in L^2(C)$, such that*

$$\lim_{n\to\infty} \mathcal{R}(u, v_{g_n}) = \mathcal{R}(u, \phi_z) \quad \text{for} \quad \text{all} \quad u \in U,$$

we have that $\quad \lim_{n\to\infty} \|v_{g_n}\|_{H^1_{loc}(\mathbb{R}^2\backslash\overline{D})} = \infty$.

Proof. **(a)** Suppose $z \in \mathbb{R}^2\backslash\overline{D}$. Let v be a radiating solution of Helmholtz equation in $\mathbb{R}^2\backslash\overline{D}$ with $v = \Phi_z \in H^{1/2}(\partial D)$. Then from [3], we see that there exists a sequence $\{v_{g_n}\}$ given by (7) such that $v_{g_n} \to v$ in $H^1_{loc}(\mathbb{R}^2\backslash\overline{D})$. Furthermore, the trace theorem and Lemma 3.1 show us that $(v_{g_n}, \partial_\nu v_{g_n}) \to (\Phi_z, \partial_\nu \Phi_z)$ in $H^{\frac{1}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D)$ which indicates

$$\begin{aligned}
\lim_{n\to\infty} \mathcal{R}(u, v_{g_n}) &= \lim_{n\to\infty} \int_{\partial D} \{v_{g_n}\partial_\nu u - u\partial_\nu v_{g_n}\}\mathrm{d}s \\
&= \int_{\partial D} \{\Phi_z\partial_\nu u - u\partial_\nu \Phi_z\}\mathrm{d}s \\
&= \mathcal{R}(u, \Phi_z) \quad \text{for} \quad \text{all} \quad u \in U.
\end{aligned}$$

(b) Now suppose that $z \in D\backslash\overline{\Omega}$. For $u(\cdot, x_0) \in U$, setting $\widetilde{u}^s(x, x_0) = u^s(x, x_0) + \Phi^s(x, x_0)$, one has

$$(21a) \qquad \mathcal{R}(u(\cdot, x_0), \Phi_z) = \int_{\partial\Omega} \Phi(x, z)\partial_\nu u(x, x_0) - u(x, x_0)\partial_\nu\Phi(x, z)\mathrm{d}s(x)$$

$$= \int_{\partial\Omega} \Phi(x, z)\partial_\nu \widetilde{u}^s(x, x_0) - \widetilde{u}^s(x, x_0)\partial_\nu\Phi(x, z)\mathrm{d}s(x)$$

$$(21b) \qquad + \int_{\partial\Omega} \Phi(x, z)\partial_\nu\Phi(x, x_0) - \Phi(x, x_0)\partial_\nu\Phi(x, z)\mathrm{d}s(x).$$

Since both u^s and Φ^s satisfy the Helmholtz equation and the reciprocity relation in

$D\backslash\overline{B}$, it is also true for $\widetilde{u}^s(x, x_0)$. Therefore, by reciprocity, $\widetilde{u}^s(x_0, x)$ and $\partial_\nu \widetilde{u}^s(x_0, x)$ are solutions of the Helmholtz equation with respect to x_0 for $x_0 \in D\backslash\overline{B}$. Denote by $v(x_0)$ that

$$v(x_0) = \int_{\partial\Omega} \Phi(x, z)\partial_\nu \widetilde{u}^s(x, x_0) - \widetilde{u}^s(x, x_0)\partial_\nu \Phi(x, z)\mathrm{d}s(x).$$

From above argument, one has that $v(x_0)$ is a solution of the Helmholtz equation in $D\backslash\overline{B}$. From (21b) we can conclude that, for every $x_0 \in C$, $\mathcal{R}(u(\cdot, x_0), \Phi_z)$ is of the form

(22) $$\mathcal{R}(u(\cdot, x_0), \Phi_z) = v(x_0) - \Phi(z, x_0)$$

and in view of (21a), $v(x_0) - \Phi(z, x_0)$ can be continued as a solution of $\triangle u + k^2 n_1 u = 0$ in D_c.

On the other hand,

(23) $$\mathcal{R}(u(\cdot, x_0), v_g) = \int_{\partial D} (v_g\partial_\nu u(\cdot, x_0) - u(\cdot, x_0)\partial_\nu v_g)\mathrm{d}s.$$

Assume that there exists a sequence $\{g_n\}$, $g_n \in L^2(C)$, such that for all $u \in U$,

(24) $$\mathcal{R}(u(\cdot, x_0), v_{g_n}) \to \mathcal{R}(u(\cdot, x_0), \Phi_z) \quad \text{as} \quad n \to \infty.$$

Suppose on the contrary that $\|v_{g_n}\|_{H^1_{loc}(\mathbb{R}^2\backslash\overline{D})}$ is bounded. Then there exists a weakly convergent subsequence $\{v_{g_n}\}$ in $H^1_{loc}(\mathbb{R}^2\backslash\overline{D})$ converging to $f \in H^1_{loc}(\mathbb{R}^2\backslash\overline{D})$. By the trace theorem, $(v_{g_n}, \partial_\nu v_{g_n})|_{\partial D} \to (f, \partial_\nu f)|_{\partial D}$ as $n \to \infty$, which shows

(25) $$\mathcal{R}(u(\cdot, x_0), v_{g_n}) \to V(x_0) \quad \text{as} \quad n \to \infty,$$

for all $x_0 \in C$ with

$$V(x_0) = \int_{\partial D} (f\partial_\nu u(\cdot, x_0) - u(\cdot, x_0)\partial_\nu f)\mathrm{d}s \quad \text{for } x_0 \in D\backslash\overline{B},$$

which can also be continued as a solution of $\triangle u + k^2 n_1 u = 0$ in D_c.

From the above argument, we have $V(x_0)$ coincides with $v(x_0) - \Phi(z, x_0)$ for $x_0 \in C$. In particular, $v_0 = V(x_0) - [v(x_0) - \Phi(z, x_0)]$ satisfies

$$\triangle v_0 + k^2 n_1 v_0 = 0 \quad \text{in} \quad D_c,$$
$$v_0 = 0 \quad \text{on} \quad C.$$

Since k^2 is not a generalized Dirichlet eigenvalue in D_c, $v_0 = 0$ in $\overline{D}_c$. By the unique continuation principle, $v_0 = 0$ in $D\backslash\{z\}$. Thus we conclude that $V(x_0)$ coincides with $v(x_0) - \Phi(z, x_0)$ for $x_0 \in D\backslash\{z\}$. However, the right-hand side is singular when $x_0 = z$ due to the term $\Phi(z, x_0)$. We arrive at a contradiction by letting $x_0 \to z$. Hence $\|v_{g_n}\|_{H^1_{loc}(\mathbb{R}^2\backslash\overline{D})}$ is unbounded. $\square$

4. **Numerical examples.** We present some numerical examples to verify the theory developed above. We choose the cavity D to be one of the following:

1. a square given by $(-2, 2) \times (-2, 2)$.
2. a triangle whose vertices are

$$(3, -\sqrt{3}), (0, 2\sqrt{3}), (-3, \sqrt{3}).$$

3. an ellipse given by

$$\frac{x^2}{2.5^2} + \frac{y^2}{1.5^2} = 1.$$

4. a kite given by

$$x = 2\cos\theta + 1.3\cos 2\theta - 0.8, \quad y = 3\sin\theta, \quad 0 \le \theta < 2\pi.$$

Inside D, there is an inhomogeneous medium occupying the disc B with radius 0.5 and the index of refraction $n(x) = n_1(x)$. There are 40 point sources distributed uniformly on a circle C with radius 0.6 such that B is contained in D_c. We choose Ω to be the disc with radius 1.0 and the measurements located uniformly on $\partial\Omega$. Outside D, the inhomogeneous medium has physical properties $A(x)$ and $n_2(x)$.

To obtain Cauchy data on $\partial\Omega$, we use a finite element solver to compute the direct scattering problem. Then we record the total fields u and approximate normal derivatives on $\partial\Omega$. We use a very fine mesh for the finite element solver such that the numerical error can be ignored. Then we add 3% noises to the data.

We discretize (20) to obtain the following ill-posed system

$$(26) \qquad\qquad\qquad \mathcal{A}\vec{g}(\cdot, z) = \vec{b}(z),$$

where $\mathcal{A}$ represents the matrix from the left hand side of (20) and $\vec{b}(z)$ is the discrete form for $\mathcal{R}(u, \Phi(\cdot, z))$. Here z is a sampling point. The boundary of the cavity is outside the measurement locations $\partial\Omega$. Hence we choose a sampling region to be a domain outside the circle with radius 1.1. In all examples, we choose the sampling region to be

$$S = \left\{ (x, y) \in \mathbb{R}^2 \,\middle|\, \sqrt{x^2 + y^2} > 1.1, -4 < x, y < 4 \right\}.$$

For each sampling point, we employ the Tikhonov regularization to solve (26). For simplicity, we use the L_2 norm of g instead of H^1_{loc} norm of v_g, which does not make significant difference according to our experience. The regularization parameter is 10^{-3} obtained by test and error.

4.1. **Homogeneous cavities.** We consider the case when the index of refraction inside B is 1, i.e., the cavity D is homogeneous. Note that linear sampling method can be used to reconstruct the shape of the cavity as well. For simplicity, we first set $A = I$ for the medium outside D. The index of refraction is $n(x) = 4 + i$. We show the results in Fig. 2 obtained by the reciprocity gap method. To get a better visualization, we take the indicator function as $I(z) = 1/\|g(\cdot, z)\|_{L^2(C)}$ at z in the sampling region.

Remark 2. Due to Theorem 3.4, $\|v_{g_n}\|_{H^1_{loc}(\mathbb{R}^2 \setminus \overline{D})}$ cannot be bounded for sampling points outside the cavity. For single layer potential, under suitable conditions on D, $\|v_{g_n}\|_{H^1_{loc}(\mathbb{R}^2 \setminus \overline{D})}$ is bounded by $C\|g(\cdot, z)\|_{L^2(C)}$ (see, for example, Ch. 3 of [7]).

Remark 3. In general, for sampling type methods, a cut-off value needs to be chosen such that the corresponding level curve can be taken as the reconstruction of the cavity. However, it is difficult to choose the cut-off value. Most existing works use test and error. Li, Liu, and Zou [13] proposed a strengthened linear sampling method with a reference ball.

Next we choose the medium outside the cavity D with the following properties

$$A = \text{diag}(2/3, 4/5), \quad \text{n}(x) = 4 + \text{i}.$$
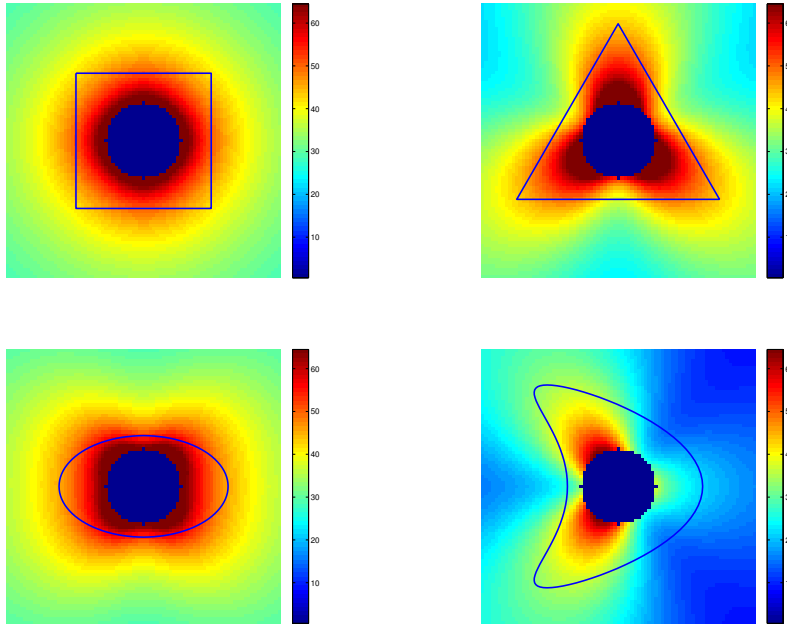
The construction is shown in Fig. 3.

FIGURE 2. The reconstructions for homogeneous cavities. The exact boundaries are the solid lines.
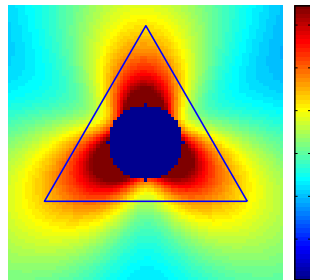

FIGURE 3. Reconstruction for the homogeneous triangle with $A = \text{diag}(2/3, 4/5)$ and $n(x) = 4 + i$ outside the cavity. The solid line is the exact boundary.

4.2. **Inhomogeneous cavities.** Now we consider the case when the cavity is inhomogeneous, i.e., the index of refraction of B is not 1. Note that for this case, other qualitative methods such as the linear sampling method cannot be applied directly. We first consider the case when $A = I$ and the media inside B and outside D have index of refraction $n(x) = 4$. The reconstruction of the four objects are shown in Fig. 4. The solid lines are the exact boundaries.
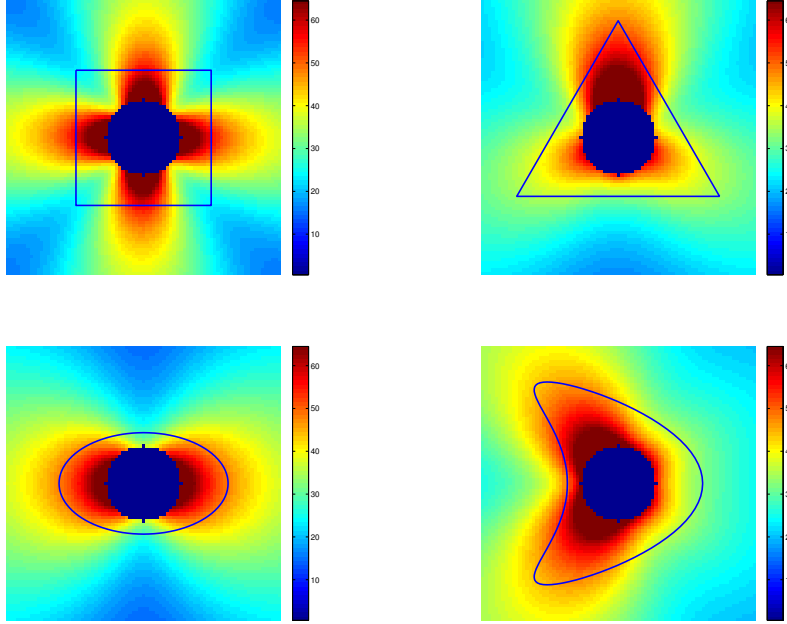
FIGURE 4. The reconstructions for non-absorbing media. The exact boundaries are the solid lines.

Next we consider the absorbing medium. The media inside B and outside D have index of refraction $n(x) = 4 + i$. We show the reconstructions in Fig. 5.

For the last example, we choose the

$$A = \text{diag}(2/3, 4/5), \quad \text{n(x)} = 4 + \text{i}$$

outside the cavity D and $n(x) = 3$ for the disc B. The construction is shown in Fig. 6.

For the exterior inverse scattering problems, the reciprocity gap method performs well over an interval of wave numbers. In contrast, for the interior scattering problems, numerical examples indicates that this admissible interval for wave numbers is significantly smaller. This could be caused by the "trapped" scattered fields inside the cavity, which bring extra difficulty for inverse problems.

Remark 4. The linear sampling type methods for interior inverse scattering problems do not perform as well as for exterior inverse scattering problems [21, 22, 25]. In some ways the interior inverse scattering problem is physically more complicated since the scattered waves are "trapped" inside the cavity [25]. The inhomogeneous background makes the scattering even more complicate. It is not clear at this point if this is intrinsic to the method or due to the inhomogeneous background. It would be interesting to develop other methods and compare the reconstructions with those for exterior problems.
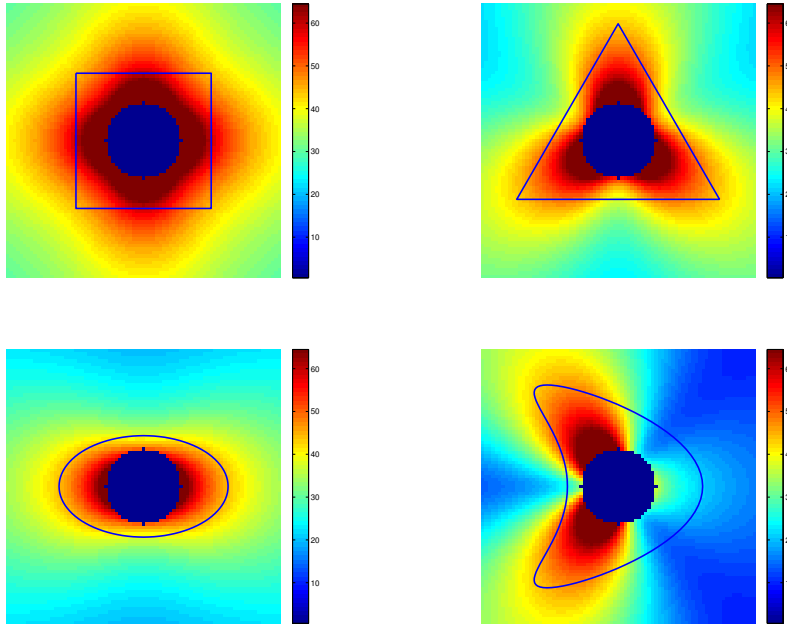
Figure 5. The reconstructions for absorbing media. The exact boundaries are the solid lines.
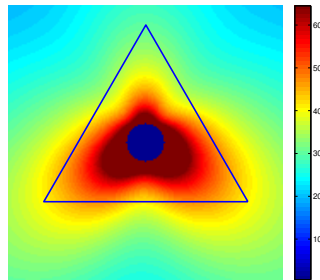
Figure 6. Reconstruction for the last example. The solid line is the exact boundary.

REFERENCES

[1] T. Arens, *Why linear sampling works*, Inverse Problems 20 (2004), no. 1, 163-173.

[2] F. Cakoni and D. Colton, Qualitative Approach to Inverse Scattering Theory, Springer, Berlin, 2014.

[3] F. Cakoni, D. Colton and S. Meng, *The inverse scattering problem for a penetrable cavity with internal measurements*, AMS Contemporary Mathematics, 615, 71-88, 2014.

[4] F. Cakoni, D. Colton and H. Haddar, *The linear sampling method for anisotropic media*, J. Comput. Appl. Math. 146 (2002), no. 2, 285?299.

[5] D. Colton and H. Haddar, *An application of the reciprocity gap functional to inverse scattering theory*, Inverse Problems 21, 383-398, 2005.

[6] D. Colton and R. Kress, Integral Equation Methods in Scattering Theory, Pure and Applied Mathematics, John Wiley & Sons Inc., New York, 1983.

[7] D. Colton and R. Kress, Inverse Acoustic and Electromagnetic Scattering Theory, 2nd Edition, Springer-Verlag, 1998.

[8] D. Colton and B. D. Sleeman, *An approximation property of importance in inverse scattering theory*, Proc. Edinburgh Math. Soc. 44, 449-454, 2001.

[9] M. D. Cristo and J. Sun, *An inverse scattering problem for a partially coated buried obstacle*, Inverse Problems 22, 2331-2350, 2006.

[10] G. Hu and X. Liu, *Unique determination of balls and polyhedral scatters with a single point source wave*, Inverse Problems 30 (2014), No. 6, 065010.

[11] Y. Hu, F. Cakoni and J. Liu, *The inverse scattering problem for a partially coated cavity with interior measurements*, Applicable Analysis, 93 (2013), No. 5, 936-956.

[12] P. Jakubik and R. Potthast, *Testing the integrity of some cavity – the Cauchy problem and the range test*, Appl. Numer. Math. Vol. 58 (2008), 1221-1232.

[13] J. Li, H. Liu, and J. Zou, *Strengthened linear sampling method with a reference ball*, SIAM J. Sci. Comput. 31 (2009/10), no. 6, 4013-4040.

[14] P. Li and Y. Wang, *Near-field imaging of interior cavities*, Commun. Comput. Phys., 17 (2015), 189-210.

[15] X. Liu, *The factorization method for cavities*, Inverse Problems 30 (2014), No. 1, 015006.

[16] P. Monk and J. Sun, *Inverse scattering using finite elements and gap reciprocity*, Inverse Problems and Imaging, Vol. 1(2007), No. 4, 643-660.

[17] P. Monk and V. Selgas, *Sampling type methods for an inverse waveguide problem*, Inverse Problems and Imaging 6 (2012), No. 4, 709-747.

[18] S. Meng, H Haddar and F. Cakoni, *The factorization method for a cavity in an inhomogeneous medium*, Inverse Problems 30, 045008, 2014.

[19] H. Qin and F. Cakoni, *Nonlinear integral equations for shape reconstruction in the inverse interior scattering problem*, Inverse Problems 27 (2011), No. 3, 035005.

[20] M. Powell, Approximation Theory and Methods, Cambridge University Press, Cambridge, 1981.

[21] H. Qin and D. Colton, *The inverse scattering problem for cavities*, Applied Numerical Mathematics 62, 699-708, 2012.

[22] H. Qin and D. Colton, *The inverse scattering problem for cavities with impedance boundary condition*, Advances in Computational Mathematics, Vol. 36 (2012), No. 2, 157-174.

[23] H. Qin and X. Liu, *The interior inverse scattering problem for cavities with an artificial obstacle*, Applied Numerical Mathematics 88, 18-30, 2015.

[24] G. Uhlmann, *Inverse scattering in anisotropic media*. in Surveys on solution methods for inverse problems, 235-251, Springer, Vienna, 2000.

[25] F. Zeng, F. Cakoni and J. Sun, *An inverse electromagnetic scattering problem for a cavity*, Inverse Problems 27 (2011), No. 12, 125002.

[26] F. Zeng, P. Suarez and J. Sun, *A decomposition method for an interior inverse scattering problem*, Inverse Problems and Imaging, Vol. 7(2013), Iss. 1, 291-303.

[27] F. Zeng, X. Liu, J. Sun and L. Xu, *Reciprocity gap method for an interior inverse scattering problem*, Journal of Inverse and Ill-posed Problems, online, Jan. 2016..