Error Analysis for the Finite Element Approximation of Transmission Eigenvalues

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Abstract

In this paper we consider the transmission eigenvalue problem corresponding to acoustic scattering by a bounded isotropic inhomogeneous object in two dimensions. This is a non self-adjoint eigenvalue problem for a quadratic pencil of operators. In particular we are concerned with theoretical error analysis of a finite element method for computing the eigenvalues and corresponding eigenfunctions. Our analysis of convergence makes use of Osborn's perturbation theory for eigenvalues of non self-adjoint compact operators. Some numerical examples are presented to confirm our theoretical error analysis.

1 Introduction

The transmission eigenvalue problem arises in inverse scattering theory for an inhomogeneous object of bounded support [9]. The transmission eigenvalue problem is a quadratic and non self-adjoint eigenvalue problem that is not covered by the standard theory of eigenvalue problems for elliptic equations. Despite this, existence of a countable number of eigenvalues has been proved [5]. Furthermore it has been shown theoretically and tested numerically that transmission eigenvalues can be determined from typical scattering data [3, 16, 4]. In addition, it has been suggested that such measured transmission eigenvalues can be used to determine properties of the the scatterer [4]. These applications have lead to the need to compute transmission eigenvalues, and it is the numerical computation of these eigenvalues by finite elements that we shall analyze in this paper.

We start by describing the transmission eigenvalue problem. Given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ and a real valued function $n \in L^\infty(\Omega)$ such that $n-1$ is strictly positive (or strictly negative) almost everywhere in Ω we seek a scalar $k \in \mathbb{C}$ and a non trivial pair

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of functions $(v,w) \in L^2(\Omega) \times L^2(\Omega)$ such that $w - v \in H^2(\Omega)$ satisfying

$$\Delta w + k^2 n(x) w = 0 \qquad \text{in } \Omega, \tag{1}$$

$$\Delta v + k^2 v = 0 \qquad \text{in } \Omega, \tag{2}$$

$$w = v \qquad \text{on } \partial\Omega, \tag{3}$$

$$\frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \qquad \text{on } \partial\Omega. \tag{4}$$

Under the above assumption that $n-1$ is strictly of one sign, it is possible to write (1)-(4) as an equivalent eigenvalue problem for $u = w - v \in H_0^2(\Omega)$ [21]. In particular

$$(\Delta + k^2)u = \Delta w + k^2 w = k^2(1-n)w.$$

Dividing by $(n-1)$ and applying the operator $(\Delta + k^2 n)$ to the resulting equality we obtain the problem of finding $k \in \mathbb{C}$ and $u \in H_0^2(\Omega)$ that satisfies the following fourth order equation [21] (see also [5, 6]):

$$\left(\Delta + k^2 n\right) \frac{1}{n-1} \left(\Delta + k^2\right) u = 0 \text{ in } \Omega. \tag{5}$$

In variational form this becomes the problem of finding a nontrivial transmission eigenfunction $u \in H_0^2(\Omega)$ and corresponding eigenvalue $k \in \mathbb{C}$ such that

$$\int_\Omega \frac{1}{n-1}(\Delta u + k^2 u)(\Delta \overline{v} + k^2 n \overline{v})\, dA = 0 \qquad \text{for all } v \in H_0^2(\Omega), \tag{6}$$

where $\overline{v}$ denotes the complex conjugate of v. In the remainder of this paper we shall assume $n > n_0 > 1$ almost everywhere where n_0 is constant, although, with obvious changes, the theory also holds for n strictly less than 1.

The transmission eigenvalue problem first appeared in the analysis of inverse problems in Kirsch [14] and in more generality in Colton and Monk [10]. The main goal at that time was to show that transmission eigenvalues can be easily avoided. In particular Rynne and Sleeman [21] showed that there is at most a countable set of real transmission eigenvalues with the only possible accumulation point being infinity. More recently Païvarinta and Sylvester [20] proved the existence of at least one eigenvalue, and soon thereafter Cakoni, Gintides and Haddar [5] proved the existence of infinitely many real transmission eigenvalues, together with estimates that started the program of research on using transmission eigenvalues to infer properties of the scatterer.

Practically Cakoni, Colton and Haddar [3] and later Kirsch and Lechleiter [16] have shown that transmission eigenvalues can be recovered from measurements of the scattered field. In order to asses the reliability of this procedure Cakoni, Colton, Monk and Sun [4] used finite element methods to compute transmission eigenvalues, but without a theoretical error analysis. This paper addresses the theoretical justification of methods like those in [22] (the method here is not the same as the one in [22]). Other techniques for computing transmission eigenvalues have been proposed, including for example [13, 18, 1, 17]. Error estimates for another finite element method for solving the interior transmission boundary value problem (away from eigenvalues) are given in [23].

The outline of the paper is as follows. In the next section we propose a modified version of Kirsch's formulation of the transmission eigenvalue problem [15] that is well adapted to implementation by finite elements. We then apply Osborn's perturbation theory for eigenvalues of non self-adjoint compact operators to prove convergence [19]. Some numerical results are presented in Sections 3 and some conclusions are drawn in the final section.

2 The Numerical Eigenvalue Problem

Mathematically, one proof of the discreteness of eigenvalues of (6) uses fractional powers of certain compact operators [6] (see also [7]) to convert the problem to an eigenvalue problem for a system of compact operators. These operators are not convenient for numerical computation since computing fractional powers of inverses of solution operators is time consuming, and instead we first introduce related operators involving just the Laplacian that are easy to implement.

Expanding (6) we obtain the problem of finding non-trivial $u \in H_0^2(\Omega)$ and $k \in \mathbb{C}$ such that

$$(\Delta u, \Delta v)_{n-1} + k^2(u, \Delta v)_{n-1} + k^2(\Delta u, nv)_{n-1} + k^4(nu, v)_{n-1} = 0$$

where

$$(u, v)_{n-1} = \int_\Omega \frac{1}{n-1} u\overline{v}\, dA.$$

Obviously $k = 0$ is not an eigenvalue of this problem since the sesquilinear form $(\Delta u, \Delta v)_{n-1}$ is coercive on $H_0^2(\Omega)$. Now define $\tau = k^2$ and let $w \in H_0^1(\Omega)$ satisfy

$$\Delta w = \tau \frac{n}{n-1} u \text{ in } \Omega.$$

Then we may rewrite the above transmission eigenvalue problem as the problem of finding a non trivial pair of functions $(u, w) \in H_0^2(\Omega) \times H_0^1(\Omega)$, and constant $\tau \in \mathbb{C}$ such that

$$\begin{aligned}
(\Delta u, \Delta v)_{n-1} &= -\tau\left((u, \Delta v)_{n-1} + (\Delta u, nv)_{n-1} - (\nabla w, \nabla v)\right) \text{ for all } v \in H_0^2(\Omega),\\
(\nabla w, \nabla z) &= -\tau (nu, z)_{n-1} \text{ for all } z \in H_0^1(\Omega).
\end{aligned}$$

This is our new non self-adjoint eigenvalue problem.

To analyze the problem, let us define the following sesquilinear forms where $u, v \in H_0^2(\Omega)$ and $z, w \in H_0^1(\Omega)$:

$$\begin{aligned}
a(u, v) &= (\Delta u, \Delta v)_{n-1},\\
b_1(u, v) &= (u, \Delta v)_{n-1} + (\Delta u, nv)_{n-1},\\
b_2(w, v) &= -(\nabla w, \nabla v),\\
c(u, z) &= (nu, z)_{n-1},\\
d(w, z) &= (\nabla w, \nabla z).
\end{aligned}$$

Then define the sesquilinear form A on $(H_0^2(\Omega) \times H_0^1(\Omega)) \times (H_0^2(\Omega) \times H_0^1(\Omega))$ by

$$A((u, w), (v, z)) = a(u, v) + d(w, z)$$

Note that A is an inner product on $H_0^2(\Omega) \times H_0^1(\Omega)$.

The eigenvalue problem is then to find non trivial $(u, w) \in H_0^2(\Omega) \times H_0^1(\Omega)$ and $\lambda \in \mathbb{C}$ such that

$$\lambda A((u, w), (v, z)) = b_1(u, v) + b_2(w, v) + c(u, z) \quad \text{for all } (v, z) \in H_0^2(\Omega) \times H_0^1(\Omega)$$

where $\lambda = -1/\tau$ (recall that $\tau = k^2 = 0$ is not a transmission eigenvalue).

Now define the operator $T : H_0^2(\Omega) \times H_0^1(\Omega) \to H_0^2(\Omega) \times H_0^1(\Omega)$ by

$$A(T(u, w), (v, z)) = b_1(u, v) + b_2(w, v) + c(w, z) \text{ for all } (v, z) \in H_0^2(\Omega) \times H_0^1(\Omega).$$

Then, in operator notation, we seek $\lambda \in \mathbb{C}$ and non-trivial $(u, w) \in H_0^2(\Omega) \times H_0^1(\Omega)$ such that

$$\lambda(u, w) = T(u, w).$$

Note that if $\lambda \neq 0$, $(0, w)$, $w \in H_0^1(\Omega)$, is not an eigenfunction of this system, so we have not introduced spurious eigenvalues into the problem.

Now suppose we use conforming subspaces $X_h \subset H_0^2(\Omega)$ and $Y_h \subset H_0^1(\Omega)$ to compute a finite dimensional eigenproblem. To fix ideas we suppose Ω is a Lipschitz polyhedron. Then we can cover Ω with a shape regular triangulation $\mathcal{T}_h$ consisting of triangles K with maximum diameter h. In this case an obvious choice is to use Argyris elements [8] to build X_h, and this is the choice we shall use later in our numerical tests. To build Y_h we could use simple continuous piecewise polynomials, and in our code we use piecewise linear or piecewise quadratic Lagrange elements.

The finite element problem is to seek non-trivial $(u_h, v_h) \in X_h \times Y_h$ and $\lambda_h \in \mathbb{C}$ such that

$$\lambda_h A((u_h, w_h), (v_h, z_h)) = b_1(u_h, v_h) + b_2(w_h, v_h) + c(w_h, z_h) \quad \text{for all } (v_h, z_h) \in X_h \times Y_h.$$

We next define an approximation to the operator T denoted $T_h : H_0^2(\Omega) \times H_0^1(\Omega) \to X_h \times Y_h$ such that for $(p, q) \in H_0^2(\Omega) \times H_0^1(\Omega)$, $T_h(p, q) \in X_h \times Y_h$ satisfies

$$A(T_h(p, q), (v_h, z_h)) = b_1(p, v_h) + b_2(q, v_h) + c(q, z_h) \text{ for all } (v_h, z_h) \in X_h \times Y_h.$$

We seek to prove that the discrete problem of finding approximate transmission eigenvalues $\lambda_h \in \mathbb{C}$ and non-trivial eigenfunctions $(u_h, w_h) \in X_h \times Y_h$ satisfying

$$\lambda_h(u_h, w_h) = T_h(u_h, w_h)$$

are close to the exact eigenvalues provided h is small enough.

To prove convergence we will apply a theorem due to Osborn [19, Theorem 3] (stated here in terms of Hilbert spaces rather than Banach spaces as in Osborn's paper). Let $\mathcal{X}$ denote a complex Hilbert space with $S : \mathcal{X} \to \mathcal{X}$ a compact operator. For λ an non-zero eigenvalue of S with algebraic multiplicity m and let Γ be a circle centered at λ containing no other eigenvalues. Then denote by E the spectral projection

$$E = \frac{1}{2\pi i} \int_\Gamma (z - S)^{-1} \, dz$$

and $R(E)$ the range of E (the dimension of $R(E)$ is m). Similarly, let $R(E^*)$ denote the range of the spectral projection E^* for the Hilbert adjoint S^* of S where now the eigenvalue is $\overline{\lambda}$.

Let $S_h : \mathcal{X} \to \mathcal{X}$ denote a sequence of compact operators for $h > 0$ (in fact constructed by finite elements). Osborn [19, Theorem 2] gives conditions under which the eigenvalues of S_h converge to those of S. If λ is an eigenvalue of S with multiplicity m, suppose $\lambda_{h,1}, \cdots, \lambda_{h,m}$ converge to λ then define

$$\hat{\lambda}_h = \frac{1}{m} \sum_{j=1}^{m} \lambda_{h,j}.$$

Theorem 1 (Theorem 3 from [19]). *Suppose $S_h \to S$ in norm and $S_h^* \to S^*$ in norm. Let $\phi_1, \cdots, \phi_m$ be a basis for $R(E)$ and let $\phi_1^*, \cdots, \phi_m^*$ be the dual basis. Then there is a constant C such that*

$$|\lambda - \hat{\lambda}_h| \leq \frac{1}{m} \sum_{j=1}^{m} |[(S - S_h)\phi_j, \phi_j^*]| + C \|(S - S_h)|_{R(E)}\| \, \|(S^* - S_h^*)|_{R(E^*)}\|,$$

where $[(S - S_h)\phi_j, \phi_j^*]$ denotes the Hilbert space duality pairing.

Our first goal is to prove the norm convergence of T_h to T and T_h^* to T^*:

Lemma 1. *Under the standing conditions on the domain and finite element spaces and provided n is smooth and $n - 1 > 0$ in Ω, $T_h \to T$ as $h \to 0$ in norm. In particular*

$$\|T - T_h\|_{\mathcal{L}(H^2(\Omega) \times H^1(\Omega), H^2(\Omega) \times H^1(\Omega))} \leq C h^{\min(\alpha, 2s)},$$

where $\min(\alpha, 2s) > 0$ and depends on the interior angles of the Lipschitz polyhedron as described in the proof. Similarly $T_h^* \to T^*$ in norm, and the same estimate holds for $\|T^* - T_h^*\|_{\mathcal{L}(H^2(\Omega) \times H^1(\Omega), H^2(\Omega) \times H^1(\Omega))}$.

Remark 1. *If the domain is convex, we have at least first order convergence.*

Proof. Of course we have Galerkin orthogonality:

$$A((T - T_h)(u, w), (v_h, z_h)) = 0 \text{ for all } (v_h, z_h) \in X_h \times Y_h.$$

Then as usual

$$A((T - T_h)(u, w), (T - T_h)(u, w)) \;=\; A((T - T_h)(u, w), T(u, w) - (v_h, z_h))$$

for any $v_h, z_h \in X_h \times Y_h$. Hence

$$\|(T - T_h)(u, w)\|_{H^2(\Omega) \times H^1(\Omega)} \leq \|T(u, w) - (v_h, z_h))\|_{H^2(\Omega) \times H^1(\Omega)}. \tag{7}$$

We can now complete the estimate using the regularity of u and v and standard finite element error estimates. First let $T(u, w) = (k_1, k_2) \in H_0^2(\Omega) \times H_0^1(\Omega)$. Then $k_2 \in H_0^1(\Omega)$ satisfies

$$(\nabla k_2, \nabla z) = (nu, z)_{n-1}.$$

Since $n/(n - 1) \in L^\infty(\Omega)$ and Ω is a Lipschitz polygon, there is an $\alpha_0 > 0$ such that

$$\|k_2\|_{H^{1+\alpha}(\Omega)} \leq C \|nu/(n - 1)\|_{H^{-1+\alpha}(\Omega)}$$

where $\alpha_0 > \alpha \geq 1/2$ and where α_0 depends on the interior angles of the polygon. In particular $\alpha_0 > 1/2$ and if the domain is convex $\alpha_0 = 1$ [12]. Choosing $z_h = P_{1,h}k_2$ where $P_{1,h}$ is the $H_0^1(\Omega)$ projection into Y_h we have

$$\|k_2 - z_h\|_{H^1(\Omega)} \leq Ch^\alpha \|k_2\|_{H^{1+\alpha}(\Omega)} \leq Ch^\alpha \|nu/(n-1)\|_{H^{-1+\alpha}(\Omega)} \leq Ch^\alpha \|u\|_{L^2(\Omega)} \tag{8}$$

for $1/2 < \alpha < \min(\alpha_0, 1)$, provided Y_h contains polynomials of degree at least one (which must hold since Y_h is H^1-conforming).

Now $k_1 \in H_0^2(\Omega)$ satisfies

$$(\Delta k_1, \Delta v)_{n-1} \;=\; (u, \Delta v)_{n-1} + (\Delta u, nv)_{n-1} - (\nabla w, \nabla v) \text{ for all } v \in H_0^2(\Omega).$$

The regularity of k_1 is a bit more difficult to determine. In strong form $k_1 \in H_0^2(\Omega)$ satisfies

$$\Delta\left(\frac{1}{n-1}\Delta k_1\right) = \Delta\left(\frac{u}{n-1}\right) + \frac{n}{n-1}\Delta u + \Delta w := F$$

If n is smooth, the right hand side is in $H^{-1}(\Omega)$, and

$$\|u\|_{H^{2+2s}(\Omega)} \leq C\|F\|_{H^{-2+2s}(\Omega)} \leq C\|F\|_{H^{-1}(\Omega)}$$

for $0 < s < \min(1/2, s_0/2)$ and where $s_0 > 0$ is the regularity limit given by [2, Section 4]. If Ω is convex, $s = 1/2$. So $k_1 \in H^{2+2s}(\Omega)$ where s depends on the interior angles of the domain.

Choosing $v_h = P_{2,h}k_1$ where $P_{2,h}$ is the $H^2(\Omega)$ projection into X_h we have

$$\begin{aligned}\|k_1 - P_{2,h}k_1\|_{H^2(\Omega)} &\leq Ch^{2s}\|t_1\|_{H^{2+2s}(\Omega)} \leq Ch^{2s}\|F\|_{H^{-1}(\Omega)} \\ &\leq Ch^{2s}\left(\|u\|_{H^2(\Omega)} + \|w\|_{H^{1+\alpha}(\Omega)}\right).\end{aligned} \tag{9}$$

Putting together the estimates from (8) and (9) we have proved that

$$\inf_{v_h, z_h \in X_h \times Y_h} \|T(u,w) - (v_h, z_h))\|_{H^2(\Omega)\times H^1(\Omega)} \leq Ch^{\min(\alpha, 2s)}\left((\|u\|_{H^2(\Omega)} + \|w\|_{H^{1+\alpha}(\Omega)}\right).$$

Using this in (7) proves the first estimate of the lemma.

Now consider the adjoint $T^* : H_0^2(\Omega) \times H_0^1(\Omega) \to H_0^2(\Omega) \times H_0^1(\Omega)$. For $(v,z) \in H_0^2(\Omega) \times H_0^1(\Omega)$, This is defined by

$$A((u,w), T^*(v,z)) = b_1(u,v) + b_2(w,v) + c(w,z) \text{ for all } (u,w)) \in H_0^2(\Omega) \times H_0^1(\Omega)$$

Letting $T^*(v,z) = (t_1^*, t_2^*)$, the strong form of the this equation is

$$\Delta\left(\frac{1}{n-1}\Delta t_1^*\right) \;=\; \frac{1}{n-1}\Delta v + \Delta\frac{n}{n-1}v + \frac{n}{n-1}z := G \tag{10}$$

$$\Delta t_2^* \;=\; \Delta v \tag{11}$$

In the same way as before, since $v \in H^2(\Omega)$, we have that $t_2^* \in H^{1+\alpha}(\Omega)$ and so choosing $z_h = P_{1,h}t_2^*$ gives

$$\|t_2^* - z_h\|_{H^1(\Omega)} \leq Ch^\alpha\|t_2^*\|_{H^{1+\alpha}(\Omega)} \leq Ch^\alpha\|v\|_{H^2(\Omega)}.$$

In addition since $n/(n-1)$ is smooth, the right hand side of (10) has the regularity $G \in L^2(\Omega)$, and again

$$\|t_1^* - P_{2,h}t_1^*\|_{H^2(\Omega)} \leq Ch^{2s}\|t_1^*\|_{H^{2+2s}(\Omega)} \leq Ch^{2s}\|G\|_{L^2(\Omega)} \leq Ch^{2s}\left(\|v\|_{H^2(\Omega)} + \|w\|_{L^2(\Omega)}\right)$$

□

Theorem 2. *Under the assumptions of Lemma 1, there is a constant C_λ such that*

$$|\lambda - \hat{\lambda}_h| = O(h^{2\min(\alpha,2s)})$$

where α and s are the exponents in Lemma 1.

Remark 2. *From [2, Figure 1] we expect that s can be chosen so that $s > 1/2$ so the theorem predicts at least $O(h)$ convergence for the eigenvalues. If the domain is convex we predict quadratic convergence.*

Proof. Suppose we have m eigenfunctions

$$T(u_j, v_j) = \lambda(u_j, v_j)$$

together with a dual basis for $R(E)$ denoted $(u_j^*, v_j^*) \in H_0^2(\Omega) \times H_0^1(\Omega)$

$$A((u_j, v_j), (u_\ell^*, v_\ell^*)) = \delta_{j,\ell}.$$

We apply Theorem 1 using $\phi = (u, v) \in H_0^2(\Omega) \times H_0^1(\Omega)$ and $S\phi = T(u, v)$ (similarly for T^*). By Lemma 1 we have the norm convergence of the operators. It remains to estimate the term $[(S - S_h)\phi_j, \phi_j^*]$. In our case

$$[(S - S_h)\phi_j, \phi_j^*] = A((T - T_h)(u_j, v_j), (u_j^*, v_j^*)) = A((T - T_h)(u_j, v_j), T^*(u_j^*, v_j^*)).$$

By Galerkin orthogonality this implies that

$$[(S - S_h)\phi_j, \phi_j^*] = A((T - T_h)(u_j, v_j), (T^* - T_h^*)(u_j^*, v_j^*)).$$

Using the error estimate from Lemma 1 completes the proof. $\square$

3 Numerical Examples

Now we show some simple examples. Let V_h be the finite element space generated by using Argyris elements on a regular triangular mesh of Ω. Let $X_h \subset V_h \cap H_0^2(\Omega)$. We choose Y_h to be the standard continuous piecewise linear Lagrange element such that $Y_h \subset H_0^1(\Omega)$. To describe the method in more detail, let $\{\phi_i\}_{i=1}^{N_h}$ be the basis for X_h and $\{\psi_i\}_{i=1}^{M_h}$ be the basis for Y_h. We define the following matrices

$$A_{ij} = (\triangle\phi_j, \triangle\phi_i)_{n-1},\ S_{ij}^1 = (\triangle\phi_j, \phi_i)_{n-1},\ S_{ij}^2 = (n\phi_j, \triangle\phi_i),$$

$$S_{ij} = (\nabla\psi_j, \nabla\phi_i),\ S'_{ij} = (\nabla\psi_j, \nabla\psi_i),\ M_{ij} = (\psi_j, n\phi_i)_{n-1},$$

where $\mathbf{u} = (u_1, \ldots, u_{N_h})^T$ such that $u_h = \sum_{i=1}^{N_h} u_i\phi_i$ and $\mathbf{w} = (w_1, \ldots, w_{M_h})^T$ such that $w_h = \sum_{i=1}^{M_h} w_i\psi_i$. The matrix eigenvalue problem is given by $\mathcal{A}\mathbf{x} = \tau\mathcal{B}\mathbf{x}$ where

$$\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & S' \end{pmatrix}, \quad \mathcal{B} = -\begin{pmatrix} S^1 + S^2 & -S \\ M & 0 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} \mathbf{u} \\ \mathbf{w} \end{pmatrix}.$$

To compute the generalized eigenvaues of this system, we use the Matlab *eigs* command.

Shape	Base mesh	1 refinement	2 refinements	3 refinements
unit square	1.877313	1.879039	1.879455	1.879557
Number of DoFs	1587	6407	25767	103367
L-shaped	2.971278	2.964095	2.958426	2.955279
Number of DoFs	1187	4807	19367	77767
circle	1.989962	1.988407	1.988088	1.988017
Number of DoFs	1245	5023	20199	81031

Table 1: The first (real) transmission eigenvalues for the test domains on a series of uniformly refined meshes. The index of refraction is $n = 16$. DoFs refers to the total number of degree of freedoms $(M_h + N_h)$.

We choose three test domains: the unit square, an L-shaped domain, and the disk with radius 1/2 centered at the origin. The unit square and the L-shaped domain are given by

$$(-1/2, 1/2) \times (-1/2, 1/2)$$

and

$$(-1/2, 1/2) \times (-1/2, 1/2) \setminus ([0, 1/2] \times [-1/2, 0])$$

respectively.

For simplicity, we choose the function $n(x) = 16$ since we can then compare the results computed here to those in the literature [11, 22]. For each domain we generate a coarse triangular mesh and then uniformly refine the mesh to perform a convergence study. In the case of the circle each refinement gives a better and better polygonal approximation of the curved boundary. So we do not use curved elements for the circular domain, and this may have a major effect on the convergence rates in that case.

The computed transmission eigenvalues are shown in Table 1. They are consistent with the values in [11, 22].

In Fig. 1, we plot the relative error in the first real transmission eigenvalue against the mesh size h when linear elements are used to discretize $H_0^1(\Omega)$. For the circle we can compute the true relative error using precise estimates of the transmission eigenvalue computed via special functions. For the other domains we compare the difference between the eigenvalues on successive meshes. The results then indicate convergence rates for each domain. The convergence orders for the unit square and the circle are 2 and for the L-shaped domain are less than 1/2. This is to be expected since even for smooth eigenfuctions the order of convergence is limited by the piecewise linear space. An interesting observation is that the eigenvalues converges from below for the unit square while from above for the L-shaped domain and the circle.

Using linear elements to discretize $H_0^1(\Omega)$ and Argyris elements for the biharmonic terms limits the maximum possible convergence rate to that of the lower order space. In Fig. 2 we show results using piecewise quadratic elements to discretize $H_0^1(\Omega)$. As expected, the convergence rate for the first eigenvalue on the L-shaped domain does not change compared to that in Fig. 1 because we expect this eigenfunction to be singular near the reentrant

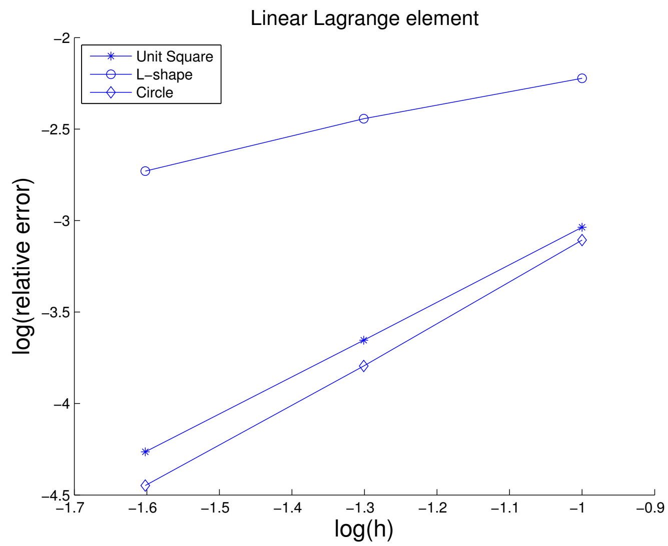

Figure 1: Relative errors in the first real transmission eigenvalue as a function of the mesh size h in the discrete problem using piecewise linear elements to discretize $H_0^1(\Omega)$. As expected the convergence rate for the circle and square is second order, while for the L-shaped domain it is slower.

corner. For the square the convergence rate is now fourth order, again as might be expected if the first eigenfunction is smooth. The convergence rate for eigenfunctions on the circular domain does not increase to fourth order despite the fact that the eigenfunctions are smooth in this case. This is likely because we approximate the circular domain with a mesh of triangles so there is a geometric error that pollutes the eigenvalue calculation. This example suggests that using a higher order space to discretize $H_0^1(\Omega)$ improves the convergence rate for smooth eigenfunctions.

4 Conclusion

We have proved convergence of a new conforming finite element method for approximating transmission eigenvalues. This was obtained by modifying an existing scheme to obtain a computationally tractable problem. Numerical results suggest that our theory gives the correct convergence rates at least for convex domains.

Obviously the use of conforming finite elements in H^2 is quite complicated and would be difficult to implement in $\mathbb{R}^3$ or for Maxwell's equations. Instead we might prefer other methods, for example discontinuous Galerkin schemes, that avoid using smoother elements. Efforts to expand the theory in this paper to that case are a next step in providing a reliable and convenient method for computing transmission eigenvalues.

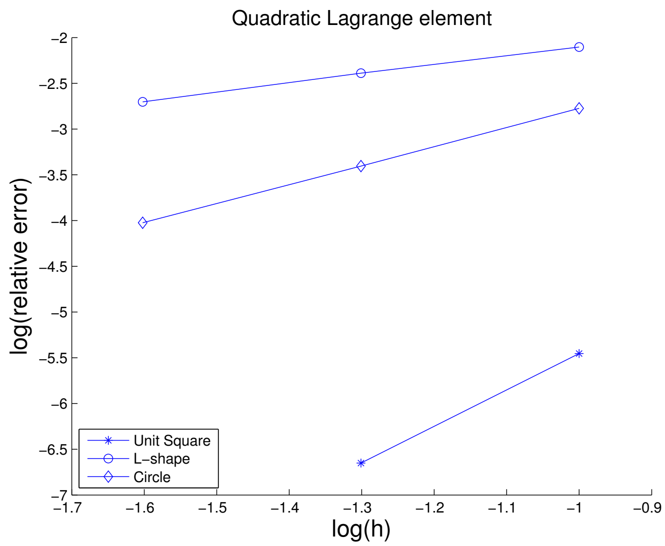

Figure 2: Relative errors in the first real transmission eigenvalue as a function of the total number of degrees of freedom in the discrete problem using piecewise quadratic elements to discretize $H_0^1(\Omega)$. Compared to Fig. 1 the convergence rate for the L-shaped domain is unchanged reflecting the low regularity of the the eigenfunction in that case. For the square domain the convergence rate increases to $O(h^4)$. For the circle a corresponding increase in the convergence rate is not seen (see the text for more discussion).

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