A CONTINUUM MECHANICS MODEL TO PREDICT SHEAR ANGLE AND CUTTING FORCES IN ORTHOGONAL CUTTING

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ABSTRACT

The mechanics of the orthogonal cutting process are examined. Under an assumed streamline and shear plane curvilinear system, the Eulerian strain and strain rate tensor in the primary deformation zone are obtained analytically. When the material behavior is dependent on strain, strain rate and temperature, the temperature distribution in machining is numerically estimated using an iterative incremental method. The results obtained are compared with those from the classical shear plane theory. The shear angle as a function of undeformed chip thickness and cutting velocity, and cutting forces in orthogonal cutting are predicted by the total work minimization principle and are seen to agree with the experimental results from the literature.

INTRODUCTION

Machining, or material removal, operations are widely employed in industry for the production of a variety of engineered products. The performance of these cutting operations is often characterized by such measures as machined surface finish, cutting force, tool life, and the part dimensions produced by the process. There has been considerable interest for many years in the development of predictive models for the mechanics of cutting processes, but to date no reliable and generally accepted model seems to have been developed.

The orthogonal cutting model proposed by Merchant [1945] has often been used as a basis for analyzing various machining processes. In this model, based on bulk equilibrium analyses, a set of relationships among cutting forces in orthogonal cutting has been established. However, the shear angle determined in Merchant’s orthogonal cutting theory relies on several assumptions that may not be satisfied in practice. Due to a vital role of the shear angle in the classical orthogonal cutting theory, much attention has focused on the predicting it as a function of the tool geometry (e.g., rake angle), material type, friction conditions at the chip/tool interface [Merchant, 1945; Lee and Shaffer, 1951; Oxley and Hastings, 1977, etc.], little success has been made in predicting the relationships among shear angle, undeformed chip thickness and cutting velocity. Although the total shear strain can be calculated from the tool geometry, the shear strain rate becomes infinite and the strain distribution in the shear zone is not clear.

Many efforts have been conducted to predict the strain and strain rate distributions as well as temperature distribution in the finite shear zone. As early as 1950-60s, the shear strain rate was experimentally estimated [Shaw, 1950; Choo and Bisacca, 1951; Gorani and Kobayashi, 1966; Keccecioglu, 1968; Stevenson and Oxley, 1969-70]. In their analyses, strains and strains rates were numerically calculated from experimentally determined streamlines. Later, a model of the machining process was developed by Tay, et al. [1976] to describe the velocity, strain and strain rate distributions within the primary and secondary zones as a function of cutting conditions, in which a family of hyperbolic streamlines was proposed to describe the flow of chip material. The heat source strength was assumed to be proportional to the strain rate in the deformed zone. The temperature distribution was then calculated by the finite element method. A limitation of the previous work, such as that as Tay, et al. [1976], is that the strain and strain rate are defined based on small deformation gradient theory. In fact, the strains in cutting are actually very large. Another limitation of the previous work is that the velocity distribution obtained does not satisfy the basic continuity equation at any point in the deformation zone.

Another category of attempts to analyze cutting processes is to simulate cutting processes by the finite element method [Iwata, et al., 1974; Strenkowski, et al., 1985; Lin and Pan, 1993; Marusich and Ortiz, 1995]. Theoretically the finite element method can take many factors in cutting processes into account, but it does not take any advantage of the fact that for a typical cutting process that the deformation is dominated by shearing.

The third widely used approach to predict the shear angle and cutting forces in cutting processes is to employ the minimum work principle. In the work of Stephenson and Wu [1988], a computer model
for three dimensional cutting processes was developed. Strain hardening and temperature effects were not considered in their model. Cordebois and Constantin [1994] developed a model for an orthogonal process in which the fracture energy was included and shown to be insignificant. In this model the shear angle was found by minimizing the intrinsic dissipation work in the shear zone so that the shear angle obtained was much higher than that in reality. In the above two models the shear zone was assumed to be bounded by two concentric circular arcs. Zheng, et al. [1996], under the streamline curvilinear coordinate system, analytically obtained the velocity distribution and the deformation rate tensors in the primary shear zone by using Mathematica [Wolfram, 1993]. The secondary deformation zone was treated as a boundary layer. A viscoplastic constitutive equation with linear viscous term was used, so the strain hardening effect was not considered, and the strain rate effect of the material was not well described.

In this work, the finite Eulerian strain and the Eulerian strain rate are obtained analytically based on the finite deformation theory of continuum mechanics. A general form for the constitutive equation of work materials may be employed to estimate the shear angle, cutting forces and temperature distribution in the shear zone. The temperature distribution in machining is numerically estimated using an iterative incremental method. The results obtained are compared with those from classical shear plane theory. The total work minimization principle is used to estimate unknown streamline parameters, leading to predictions of the shear angle and cutting forces in orthogonal cutting. Model predictions are compared with results from the literature.

THEORETICAL APPROACH

Geometry and Coordinate Systems

Consider the orthogonal cutting and plane strain problem situated as shown in the Cartesian coordinate systems $(x^1, x^2)$ in Figure 1. The curvilinear coordinate system $(q^1, q^2)$ is set on the assumed streamline and in the direction parallel to the so-called shear plane. The relationship between the coordinates $(q^1, q^2)$ and $(x^1, x^2)$ satisfy the following equations:

$$(x^2 - q^2 t_o)^2 \tan \alpha - (x^2 - q^2 t_o)(x^1 + q^2 t_o \cot \phi) = a t_o^2 (1 + \eta q^2 \csc \phi),$$

$$t_o q^1 = x^2 + \tan \phi x^1,$$  

where $t_o$ is the uncut chip thickness, $a$ and $\eta$ are constants related to the cut material and cutting conditions, $\alpha$ is the rake angle, and $\phi$ is the so-called shear plane angle. The coordinates $(x^1, x^2)$ can be expressed in terms of the dimensionless curvilinear coordinates $(q^1, q^2)$ as follows,

$$x^1 = t_o \left[ \frac{(\Omega - 1)q^1 - \sqrt{\Delta}}{\beta \tan \phi} - q^2 \cot \phi \right],$$

$$x^2 = t_o \left[ \frac{q^1 + \sqrt{\Delta} + q^2}{\beta} \right],$$  

where $\beta = 2(\tan \alpha \tan \phi + 1), \Delta = (q^1)^2 + 2\alpha \beta \tan \phi(1 + \eta q^2 \csc \phi)$.  

![Figure 1. Orthogonal cutting geometry and coordinate system](image)

Velocity

For steady state deformation conditions, the continuity equation is,

$$\nabla \cdot (\rho \mathbf{v}) = 0,$$  

where $\rho$ is the density of work material, and $\mathbf{v}$ is the velocity vector [Malvern, 1969]. In the curvilinear coordinate system, due to the assumption that $q^2 = \text{constant}$ is a family of streamlines, the contravariant component of the velocity vector in $q^2$ direction will vanish. From Eq.(3), we can have,

$$v^{(1)} = \frac{\rho_0 V_o}{\rho} \frac{\partial \xi}{\partial q^1} \frac{\partial}{\partial q^1} + \frac{1}{\rho} \frac{1}{\cos \phi} \sqrt{\beta - 1 - q^1 \sqrt{\Delta}}^2 + \tan \phi(1 + q^1 \sqrt{\Delta})^2,$$  

where $v^{(1)}$ is the physical component of velocity vector in $q^1$ direction, $\phi$ and $V_o$ are the shear angle and cutting speed. $\rho_0$ and $\rho$ are the initial and instantaneous densities of work material respectively, $g_{ij}$ is one of the covariant components of metric tensor $\tilde{g}$, and $g$ the determinant of tensor $\tilde{g}$. The covariant components of the metric tensor, denoted $g_{ij}$, are defined as follows:

$$g_{ij} = \tilde{g}_{i \cdot j} \cdot \text{and} \quad g_{z} = \frac{\partial x}{\partial q^1} t_o.$$  

The limits of the velocity, when $q^1$ goes to $\pm \infty$, are given by,

$$\lim_{q^1 \to \infty} V^{(1)} = V_o, \quad \lim_{q^1 \to -\infty} V^{(1)} = \frac{\rho_0 V_o \sin \phi}{\rho \cos (\alpha - \phi)} = V_c,$$  

where $V_c$ and $\rho_c$ are the velocity and final density of the chip. It may be noted that if $\rho_c = \rho_0$, Eq.(6) is consistent with the velocity in the classical shear plane model.

Eulerian Strain

Consider $s$ to be the length of a streamline from an initial reference point $(Q^1, Q^2)$ to point $(q^1, q^2)$. The derivative of $s$ with
respect to time $t$ is the physical component of velocity vector in the $q^1$ direction. That is,
\[
\frac{dq^i}{dt} = v^{(1)} \text{ or } \sqrt{g} dq^i = i_q V_q dt.
\]  
(7)

The relationship between the initial reference point ($Q^1, Q^2$) in the material coordinates and the current one ($q^1, q^2$), which is needed to calculate the deformation gradient tensor, is then,
\[
f(q^1, q^2) - f(Q^1, Q^2) = V_q f,
\]  
(8)

where the function $f$ is given by,
\[
f(q^1, q^2) = q^1 i_q \cot \phi + a \mu \ln \csc \phi \log(q^1 + \sqrt{\Delta}).
\]  
(9)

In examining Eq. (8), it is clear that the position of a point on a streamline, $(q^1, q^2)$, is dependent on the starting position $(Q^1, Q^2)$. This dependence may be expressed by two implicit functions, $Q^1 = Q^1(q^1, q^2)$ and $Q^2 = Q^2(q^1, q^2)$. From the derivative chain rule, the inverse of the two-point deformation gradient tensor, $F^{-1}$, is obtained as,
\[
F^{-1} = \frac{\partial Q^1}{\partial q^1}.
\]  
(10)

Therefore, the Eulerian strain tensor $\varepsilon$, expressed in covariant components, can be calculated using the following formula,
\[
e_{ij} = g_{ij} - (F^{-1}_{ij}) G_{ij} (F^{-1}_{ij}),
\]  
(11)

where $G_{ij}$ is the covariant components of the metric tensor at the reference point in the material coordinates.

The Eulerian strain tensor is dependent on both the current configuration point $(q^1, q^2)$ and initial point $(Q^1, Q^2)$. Considering Eq. (8), $Q^2$ can be replaced by $q^2$ whenever it appears. In addition, $Q^1$ can be chosen to be negative infinity. In this case, Eq. (11) will be used to be dependent only on the current coordinates $(q^1, q^2)$.

The effective strain, by its definition, is
\[
\varepsilon = \frac{1}{2} (\varepsilon^T e_{ij} - tr^2(\varepsilon)), \quad \text{where} \quad tr(\varepsilon) = g^{ij} e_{ji}.
\]  
(12)

The analytical expression of effective strain is both lengthy and complicated and will not be listed here. Using Mathematica, the effective strain when $q^1$ goes to infinity, can be shown as:
\[
\varepsilon = \frac{\Delta q^1}{\eta} (2\beta^2 - 2\beta^2 \cos 4\Phi) \csc^4 \Phi \sec^4 \Phi.
\]  
(13)

Replacing $\beta$ by $2(\tan \Phi \tan \alpha - 1)$, results in,
\[
\varepsilon = \frac{\cos \alpha}{2 \cos (\Phi - \alpha) \sin \Phi}.
\]  
(14)

Comparing with the value from the classical shear plane theory, we can conclude the final effective strain obtained here is same as the total shear strain defined in the classical shear plane model.

**Eulerian Strain Rate**

The material behavior is dependent on the strain rate or deformation rate as well as the strain. The deformation rate tensor, $\dot{D}$, is defined by,
\[
D_{ij} = \frac{1}{2} (v_{i,j} + v_{j,i}),
\]  
(15)

where $D_{ij}$ is the covariant component of the tensor $\dot{D}$, and $v_{i,j}$ is the covariant derivative of covariant component $v_i$ of velocity with respect to coordinate $q_j$.

If $\rho$ is constant, then the continuity equation becomes,
\[
\nabla \cdot \mathbf{v} = \rho \frac{d}{dt} (\rho) = 0,
\]  
(16)

therefore, the second invariant of tensor $\dot{D}$ is
\[
II_D = \frac{1}{2} (\dot{D} : \dot{D} - tr^2 \dot{D}) = \frac{1}{2} \eta_D \eta_D,
\]  
(17)

where $\eta_D$ is the contravariant component of tensor $\dot{D}$.

In general, the deformation rate tensor can be regarded as the finite Eulerian strain rate. Strictly speaking, the finite Eulerian strain rate tensor is given by,
\[
\dot{\varepsilon} = \dot{D} - \dot{L} - \dot{L} \cdot \varepsilon - \varepsilon \cdot \dot{L},
\]  
(18)

where the tensor $\dot{L}$ is the gradient of the velocity vector. The difference between the deformation rate and the Eulerian strain rate defined above is negligible.

**Temperature Estimation**

The energy equation for a steady state process is,
\[
\rho C_v \nabla \cdot \nabla \theta = \nabla \cdot (k \nabla \theta) + \Gamma \psi_p,
\]  
(19)

where $C_v$ and $k$ are the specific heat capacity and thermal conductivity, $\Gamma$ is the friction of plastic work converted into heat. $\psi_p$ here is designated as the rate of plastic work per unit volume. Because, in a typical machining process, the conductive term in Eq. (19) is much smaller than the other two terms, it can be safely ignored when we estimate the temperature distribution in the primary cutting zone. After that, along a streamline, the temperature distribution can be numerically obtained by using an incremental method. At each step, since the dissipated work is dependent upon temperature and the temperature distribution is dependent upon the dissipated work, an iterative method has to be applied. In detail, a starting point is chosen at which the temperature is equal to the ambient temperature and an end point is chosen near the end of the primary zone. Between the two chosen points, a streamline is divided equally into an even number of sections. A central difference scheme may be used to approximate Eq. (19) without the conduction term:
\[
- \theta_i + \theta_{i+1} + \frac{\Gamma \Delta q^1}{\rho C_v \psi_p} i \left( \dot{\varepsilon}, \frac{1}{2} (\theta_{i-1} + \theta_{i+1}) \right) = 0,
\]  
(20)

where $i$ indicates an odd-number element, $\psi_i$ is the covariant component of velocity at the $i$-th node on a streamline, and $\Delta q^1 = q^i_{i+1} - q^i_{i-1}$. Equation (20) may be solved using an iterative method.

The heat source term in Eq. (20) is an unknown. In general, the
heat source is assumed to be the dissipated power per unit volume. The dissipated work in the primary zone or the source term of the energy equation, which depends on both the deformation rate and temperature, may be calculated using a general constitutive equation,

\[ \bar{\sigma} = G_0 \bar{\varepsilon}^n. \]  

(21)

In this paper, a constitutive equation of the following form is applied:

\[ \bar{\sigma} = \sigma_1 \bar{\varepsilon}^n, \]  

(22)

where \( \sigma_1 \) and \( n \) are functions of both temperature and strain rate, which were determined for a low carbon steel by Tay, et al. [1974], based on the machining data provided by Stevenson and Oxley [1973]. That is,

\[ \sigma_1 = 986.6 - 3.6130 \times 10^{-2} + 100.705 \log \dot{\varepsilon} \quad (\text{MPa}), \]

\[ n = 0.469 + 0.000126 \times 0.096 \log \dot{\varepsilon}. \]

The dissipated rate of work, which finally converts into heat, is then given by:

\[ \Psi_p = \frac{\bar{\sigma} \dot{\varepsilon}}{1 + \frac{C}{C_p \dot{\varepsilon}}}. \]  

(23)

In addition, there is a temperature rise, \( \Delta \theta \), in the formed chip because the friction power, \( P_f \), generated at the tool/chip interface is converted into heat. Therefore

\[ \Delta \theta = \frac{P_f}{P_{ci} \dot{\varepsilon} V_o}, \quad \text{and} \quad P_f = w f \int_0^{\pi} \tau f V_c dx, \]  

(24)

where \( w \) is the chip width, \( h \) is the tool/chip contact length, and \( m \) is the friction factor.

**Total Power and Its Minimization**

The overall power per unit width of cut, \( P \), done in machining may be characterized as the sum of the power consumed in the shear zone as well as at the chip/tool interface, that is,

\[ P = \int \Psi_p J d \psi d q^2 + \int \Psi_f dx, \]  

(25)

where \( J \) is the Jacobian determinant and given by

\[ J = \frac{d(\dot{\psi}, \dot{q})}{d(\psi, \dot{q})}. \]

The total power, \( P \), is a function of four variables: \( \psi, a, \eta, \) and \( h \), but these variables are not independent. It is possible to determine the values of shear angle \( \psi \) and the parameter \( \eta \) if some constraints are imposed on both \( a \) and \( h \). Although, strictly speaking, the upper bound method is only applied to problems with prescribed surface configurations, this restriction was ignored by many researchers who used the upper bound method to find the steady-state shape of the deformed surface [Stephenson and Wu, 1988; Azarkin and Richmond, 1991; Cordebois and Constantin, 1994].

The total power, \( P \), is calculated numerically using Gaussian quadrature integration. As the shearing power is largely consumed near the shear plane, care must be exercised in the numerical integration of Eq. (25). One might consider using additional points in the Gaussian quadrature integration to improve accuracy. However, these points would tend to be focused on improving accuracy near the limits of the integral rather than on the region near the shear plane. In this work the shear zone is divided into 4 elements with two of them having a common edge on the shear plane. Sample calculations show that sufficient accuracy of the numerical integral can be obtained when 9 Gaussian points are used in each element.

The optimization method used to minimize the total power is Davidon-Fletcher-Powell (DFP) method. "This method is the best general purpose unconstrained optimization method making use of the derivatives of an objective function." [Rao, 1995] It can be thought of as a conjugate gradient method and converges quadratically when the objective function is of a quadratic form. The gradients of the objective function are evaluated numerically. The unidirectional minima are obtained by the quadratic interpolation method. Practical calculations show that the minimum can be reached within several number of iterations in most cases.

**Determination of The Parameter "a" for a Streamline**

Based on the experimental work of Stevenson and Oxley [1970], Tay et al. related \( a \) in Eq. (1) to \( \psi \) in the following way,

\[ a = \frac{1}{16 C^2 \sin^4 \phi (\tan \alpha + \cot \phi)}, \]  

(26)

where \( C \) is a so-called strain rate constant determined by experiment. Following the work of Stevenson and Oxley [1970], \( C \) was taken as equal to 5.9 for all the steels considered. The strain rate constant is believed to be a material constant, that has to be experimentally determined. In this paper, rather than to determine the parameter \( a \) by using Eq. (26), a geometric method to estimate the parameter \( a \) will be proposed.

![Figure 2. The sketch of a chip](image)

Considering the sketch of the chip in Figure 2, CBD is the streamline of \( q^2 = 1.0 \), which corresponds to the top of the chip. The lines EA and AF are two asymptotes of the streamline CBD. The line OA divides evenly the angle of \( LEAF \). The line EB is tangent to the streamline CBD at the point B. Consequently the streamline is oriented at an angle of \( \delta = \pi/4 - \alpha/2 \) at the point B relative to the initial horizontal direction. The radius of curvature of the streamline at the point B is designated as \( R \).

From Eq. (1), \( R \) is shown to be,
\[
\frac{1}{R} = \frac{\gamma^*}{[1 + (\gamma^*)^2]^{3/2}} \quad (27)
\]
\[
= \frac{2a_\gamma^*(1 + \eta)}{[4a_\gamma^2(1 + \eta)\tan\alpha + (\gamma_2 - \gamma_1)^2 + (\gamma_3 - \gamma_1)^2]^{3/2}}
\]
Considering the geometrical relationship given in Figure 2, Eq. (27) can be expressed in the form,
\[
\frac{1}{R} = \frac{2(\tan\alpha + \tan\delta)^{3/2}}{t_\alpha/\sqrt{[(2\tan\alpha + \tan\delta)^2 + 1]}^{3/2}}.
\]
(28)
Cordebois and Constantin [1994] suggested that when the streamline was oriented counterclockwise at an angle of \(\delta = \pi/4 - \alpha/2\) relative to the initial horizontal direction, the radius of curvature near the chip root is given by
\[
R = \frac{t_\alpha \cos(\phi - \alpha)}{\cos\alpha \cdot \sin^3(\delta + \phi)}.
\]
(29)
If the expressions for the radius of curvature given by Eqs. (28) and (29) are equated, an equation for the parameter \(a\) is obtained:
\[
a = \frac{4(\tan\alpha + \tan\delta)^3 \cos^2(\phi - \alpha)}{(1 + \eta)[(2\tan\alpha + \tan\delta)^2 + 1]} \cos^2\alpha \cdot \sin^6(\delta + \phi)
\]
(30)
Equation (30) expresses the parameter \(a\) as a function of both the rake angle and the shear angle.

**Calculation of Chip/Tool Contact Length**

In previous work [Hastings, et al., 1974; Oxley and Hastings, 1977], the chip/tool contact length was expressed as,
\[
h = C_\eta \cdot \frac{t_\alpha \sin(\phi + \lambda - \alpha)}{\cos\lambda \cdot \sin\alpha},
\]
(31)
where \(\lambda\) is the mean friction angle over the chip/tool interface and \(C_\eta\) is a correction coefficient. \(C_\eta\) was found to be 1.0 by drawing a line through the stress free surface on the shear plane parallel to the resultant cutting force. When \(C_\eta = 1.0\), Eq. (31) tends to underestimate the value of contact length. Oxley and Hastings, by taking moments of the normal stresses on the chip/tool interface about the tool tip to find the position of the resultant force and making an assumption that the resultant force will intercept the chip/tool interface at a distance of half the contact length, \(h\), from the tool tip, shows that \(C_\eta\) is given by,
\[
C_\eta = 1 + \frac{C_\eta}{3(1 + \pi/2 - 2\phi - C_\eta)},
\]
(32)
where \(C\) is the strain rate constant mentioned above, and \(n\) is the strain hardening exponent. \(C_\eta\) reduces to 1.0 when \(n\) equals zero.

**Based on the machining theory developed by Oxley and Hastings [1977],** the correction coefficient of the contact length may be expressed in terms of the cutting conditions to be described shortly. The normal force at the chip/tool interface, \(N\), is shown to be,
\[
N = \frac{k_{AB} \cdot \dot{\gamma} \cos^2\lambda}{\sin\phi \cdot \cos(\phi + \lambda - \alpha)}.
\]
(33)
where \(k_{AB}\) is the shear flow stress along shear plane. It is noted that \(k_{AB}\) is assumed to be constant along the shear plane. From the stress free boundary condition on the chip surface at the shear plane and the geometric relationships of orthogonal cutting, the shear angle is shown to satisfy the following equation,
\[
\tan(\phi + \lambda - \alpha) = 1 + \pi/2 - 2\phi - C_\eta.
\]
(34)
For a uniform normal stress distribution, the normal stress on the chip/tool interface, \(\sigma_N\), is given by,
\[
\sigma_N = N/(hw),
\]
(35)
where \(w\) is the width of chip. In addition, from the stress boundary condition at the tool tip, the normal stress on the tool/chip interface, \(\sigma'_{N}\) can be found as,
\[
\sigma'_{N} = k_{AB}(1 + \pi/2 - 2\alpha - 2C_\eta).
\]
(36)
By imposing the condition \(\sigma_N = \sigma'_{N}\) and combining Eqs. (31) to (36), the following result is obtained,
\[
C_\eta = \frac{1}{3}(2\alpha_\epsilon + 6\phi_\epsilon - 3\xi + (2\alpha_\epsilon + 6\phi_\epsilon - 3\xi)^2 - 48\phi_\epsilon(\alpha_\epsilon - \xi)),
\]
(37)
where \(\alpha_\epsilon = 1 + \pi/2 - 2\alpha\), \(\phi_\epsilon = 1 + \pi/2 - 2\phi\), and \(\xi = \cos^2\lambda/\sin(\phi + \lambda - \alpha)\cos(\phi + \lambda - \alpha)\). For 4 different values of the average friction angle, \(\lambda\), Eqs. (34) and (37) with \(\alpha = 20^\circ\) are plotted in Figure 3.

![Figure 3](image)

**Figure 3. Variation of \(C_\eta\) with both \(\phi\) and \(\lambda\) when \(\alpha = 20^\circ\)**

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**Eq. (34); - - Eq. (37-); - - Eq. (37+)**

Obviously Eq. (37) with a negative sign is a physically feasible solution. Therefore, once \(\lambda\) and \(\alpha\) are known, both the shear angle \(\phi\) and the product of \(C\) and \(n\) can be obtained by solving Eq. (34) and Eq. (37) with a negative sign. It seems both \(\phi\) and \(Cn\) will decrease when the friction coefficient is reduced. After substituting the value of \(Cn\) obtained into Eq. (32), \(C_\eta\), hence \(h\) can be calculated. It should be noted that the shear angle predicted in this way is still overestimated as compared with the experimental data from Tay, et al. [1974, 1976].

**NUMERICAL SIMULATION AND DISCUSSION**

A series of numerical simulations were performed to predict the cutting force \(F_c\), the thrust force \(F_t\), and the friction force \(F_f\) between the chip and the tool rake face. The cutting pressures \(K_c\), \(K_t\), and \(K_f\) were also calculated (forces \(F_c\), \(F_t\) and \(F_f\) divided by the chip load). For
the numerical simulations both the friction coefficient and the friction factor were 0.60 in this numerical simulation and the cutting velocity was $U_0 = 78.9$ m/min. The cutting pressures $K_c$, $K_t$, $K_f$ and shear angle $\phi$ under different cutting conditions are shown in Figures 4. In general the cutting pressures will decrease when either the undeformed chip thickness or the rake angle increases. These trends are consistent with experimental observations widely reported in the literature.

![Cutting Pressures and Shear Angle Variation](image)

Figure 4. Cutting Pressures and Shear Angle Vary with Undeformed Chip Thickness

It is noted that the friction factor $m$ will directly influence the friction power generated on the chip/tool interface. Unfortunately the friction factor $m$ under various cutting conditions as well as for various tool and workpiece materials are both theoretically and experimentally difficult to determine.

![Shear Angle as a Function of Rake Angle](image)

Figure 5 Shear Angle as a Function of rake Angle

Figure 4d shows that the shear angle, $\phi$, increases slightly as the undeformed chip thickness increases. Form Figure 5, the variation of $\phi$ with rake angle $\alpha$ is seen to be nonlinear as opposed to the linear behavior of classical theories. The linear relationship between $\phi$ and $\alpha$ in Oxley and Hastings’ theory is not clear intuitively. In their method the shear angle is obtained by solving Eqs. (34) and (37) simultaneously. At a first look, it seems that $\phi$ predicted by this way is dependent upon the behavior of chip material. This dependence of $\phi$ on the behavior of material is very weak in Oxley and Hastings’ theory. The contour of friction angle as a function of $\alpha$ and $\phi$ is shown in Figure 6.

The following general expression for the shear angle may be used to characterize the classical theories mentioned previously:

$$\phi = \pi/4 + \alpha \Phi_\alpha - \lambda \Phi_\lambda,$$

where both $\Phi_\alpha$ and $\Phi_\lambda$ are constants ($\Phi_\alpha = 0.5$ for the Merchant theory, and $\Phi_\alpha = 0.5$ and $\Phi_\lambda = 1.0$ in the theory of Lee and Shaffer).

From Figure 6, it is evident that $\Phi_\alpha$ and $\Phi_\lambda$ may be estimated as 0.81 and 0.79 for the Oxley and Hastings’ theory. In addition, there is an instability included in this theory, where small changes in the shear angle for a given rake angle seem to produce significantly different angles. Cordebois and Constantin [1994] also predicted the shear angle by minimization of the rate of dissipation work. Because the friction at chip/tool interface was considered zero, the shear angle predicted was much higher than that observed from experiment.

![Relationship among $\alpha$, $\phi$ and $\lambda$ from Oxley and Hastings’ theory](image)

Figure 6 Relationship among $\alpha$, $\phi$ and $\lambda$ from Oxley and Hastings’ theory

The verification of the continuum mechanics approach described herein can be made based on the experimental data from Tay, et al. [1974, 1976]. The cutting conditions, which were constant for all three cases considered, were as follows: the rake angle, width of cut and undeformed chip thickness are 20°, 9.50 mm and 0.274 mm respectively. The experimental data are listed in the upper part of Table 1. The results and comparisons between them are summarized in lower part of Table 1. The parameters chosen or determined in this model are listed in the slightly shaded bottom area in Table 1.

The shear angles under different cutting velocities predicted based on both Merchant’s and Oxley’s theory are much higher than the experimental results. Lee and Shaffer’s theory tends to underestimate the shear angles. Among them, Oxley and Hastings’ method yields better results. The shear angles predicted in this work are slightly higher than the measured shear angles. Based on Cordebois and
Constantin’s idea, the shear angle $\phi$ predicted will be much higher than that from Merchant’s theory when the friction over the chip/tool interface is ignored.

### Table 1: Verification of the model under different cutting velocities

<table>
<thead>
<tr>
<th>Cutting Velocity (m/min)</th>
<th>29.6</th>
<th>78.0</th>
<th>155.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Contact Length (mm)</td>
<td>0.68</td>
<td>0.82</td>
<td>0.91</td>
</tr>
<tr>
<td>Cutting Force, $F_c$ (N)</td>
<td>3240</td>
<td>3360</td>
<td>3750</td>
</tr>
<tr>
<td>Thrust Force, $F_t$ (N)</td>
<td>550</td>
<td>666</td>
<td>1190</td>
</tr>
<tr>
<td>Shear Angle, $\phi$ (degree)</td>
<td>30.0</td>
<td>28.5</td>
<td>27.8</td>
</tr>
</tbody>
</table>

- $\phi$ : Merchant’s Theory
- $\phi$ : Lee and Shaffer’s Theory
- $\phi$ : Oxley and Hastings Theory
- $\phi$ : Predicted in This Work

<table>
<thead>
<tr>
<th>$F_c$ Predicted (N)</th>
<th>3011</th>
<th>3396</th>
<th>3978</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_t$ Predicted (N)</td>
<td>569.1</td>
<td>618.6</td>
<td>1102</td>
</tr>
</tbody>
</table>

- $T_c$ : Tay et al’s Method (°C)
- $T_c$ : Boothroyd’s Procedure (°C)
- $T_c$ : Calculated in This Work (°C)

<table>
<thead>
<tr>
<th>Temperature Rise due to Friction (°C)</th>
<th>34.6</th>
<th>118.0</th>
<th>314.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average Chip Temperature (°C)</td>
<td>302.0</td>
<td>412.9</td>
<td>637.4</td>
</tr>
<tr>
<td>Contact Length Predicted (10^-4 m)</td>
<td>5.885</td>
<td>6.395</td>
<td>8.897</td>
</tr>
<tr>
<td>Parameter $a$ in Eq. (1)</td>
<td>0.1630</td>
<td>0.1767</td>
<td>0.2114</td>
</tr>
<tr>
<td>Parameter $n$ in Eq. (1)</td>
<td>0.7504</td>
<td>0.7504</td>
<td>0.7506</td>
</tr>
<tr>
<td>Friction Factor $m$ in this Calculation</td>
<td>0.61</td>
<td>0.60</td>
<td>0.60</td>
</tr>
</tbody>
</table>

The temperatures on the shear plane, designated as $T_c$, at different cutting velocities, are 175.2, 196.8, and 199.4°C respectively. This result is compared with those from Tay et al’s finite element method, and from Boothroyd’s calculation procedure. In case that $V_o = 29.6$ m/min, the temperature on the shear plane is less than that from both Tay et al’s method and that from Boothroyd’s calculation procedure. In the other two cases, the temperatures on the shear plane obtained here are very close to the values from Tay et al’s method and much lower than the ones from Boothroyd’s procedure. The temperature rises due to friction work and the average chip temperatures are also calculated. The average chip temperature is the sum of the final chip temperature along a streamline and the temperature rise due to the friction work done over the chip/tool interface, which also is an underestimate of the average temperature at the chip/tool interface. The approach to estimate the temperature herein is not able to calculate the temperature distribution over the chip/tool interface.

In addition, both the cutting force $F_c$ and the thrust force $F_t$ predicted are very close to the experimental results. However the contact lengths in these cases are generally underestimated.

### CONCLUSION

The velocity, Eulerian strain, Eulerian strain rate, and deformation rate distributions in the deformation zone, based on the finite deformation theory of continuum mechanics, are analytically obtained and consistent with some results from the shear plane theory of orthogonal cutting in certain limit states. A simple iterative incremental method is able to predict the temperature on the shear plane as well as the average chip temperature when the behavior of cut material is dependent upon strain, strain rate and temperature. The shear angle is predicted by minimization of the total rate of work using DFP method. In a wide range of cutting conditions the cutting pressures and the shear angle as a function of undeformed chip thickness and cutting velocity can be predicted reasonably. A future work is to develop a theoretical or empirical model of friction at the chip/tool interface, which can predict the friction characteristics under different cutting conditions. The model to predict the contact length also needs to be refined if possible. Additionally the approach developed in this paper may be applied to determine the constitutive equation of materials by cutting tests, which is dependent upon strain, strain rate and temperature.

### ACKNOWLEDGMENTS

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### REFERENCES


