13.0 BICYCLE EXAMPLE

(Updated Spring 2001)

Interested in time it takes to pedal up a hill. 7 variables of interest:

<table>
<thead>
<tr>
<th>Variable</th>
<th>Low</th>
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<tr>
<td>7-Tires</td>
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<td>Soft</td>
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Introduce variables 4,5,6, & 7 as follows:
4=12  I=124
5=13  =>  I=135  Generators
6=23  I=236
7=123  I=1237

Selected generator member of family: I= ± 124 = ± 135 = ± 236 = ± 1237

Defining relationship:
I=124=135=236=1237
2 @ a time  = 2345=1346=347=1256=257=167
3 @ a time  = 456=1457=2467=3567
Run the experiment.

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Use Base design calc. matrix to calc l_i’s

\[ l_1 = \frac{(-69 + 52 - 60 + 83 - 71 + 50 - 59 + 88)}{4} = 3.5 \]
\[ l_{123} = \frac{(-69+52+60-83+71-50-59+88)}{4} = 2.5 \]
\[ l_1 = 66.5 \text{ est } 1 \]
\[ l_1 = 3.5 \text{ est } 1+24+35+67 \]
\[ l_2 = 12.0 \text{ est } 2+14+36+57 \]
\[ l_3 = 1.0 \text{ est } 3+15+26+47 \]
\[ l_{12} = 22.5 \text{ est } 12+4+37+56 \]
\[ l_{13} = 0.5 \text{ est } 13+5+27+46 \]
\[ l_{23} = 1.0 \text{ est } 23+6+17+45 \]
\[ l_{123} = 2.5 \text{ est } 34+25+16+7 \]

- Could plot l’s on NPP to find important ones
- In this case, we know from past experiments that \( \sigma_y = 3 \)

\[ \sigma_e^2 = \sigma_{\text{effect}}^2 = \frac{4\sigma_y^2}{N} \]

Therefore

\[ \sigma_e = \left[ \frac{4 \cdot 3^2}{8} \right]^{\frac{1}{2}} = \sqrt{4.5} \approx 2.1 \]

It can be easily found out that \( l_2 \) and \( l_{12} \) are statistically significant with 95% confidence.
Simple conclusion: $\mu_2$ & $\mu_4$ are important
Could also be: $\mu_2$ & $\mu_{12}$ important or $\mu_2$ & $\mu_{14}$ important or...

To unconfound, sort out what is/is not important, let’s run another 8 tests.Overlaying our knowledge on the experiment results suggests that probably 4 & 14 is important, 2nd experiment will unconfound 4 and its interactions.

<table>
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Signs of column 4 in this recipe matrix are opposite/flipped/ “folded” from those of 1st recipe matrix. All the other signs are the same.
Let’s examine the recipe matrix to see what the generators are for this 2nd $2^7$ design. $4= -12$, $5=13$, $6=23$, $7=123$. So the defining relation for the 2nd design is:

\[
I=-124=135=236=1237
=\pm 2345=\pm 1346=\pm 1256=\pm 257=\pm 167
=\pm 456=\pm 1457=\pm 2467=\pm 3567=\pm 1234567
\]

Therefore,

- $l_1' = 67.0$ is the estimate of the effect of $I$
- $l_1' = -2.0$ is the estimate of the effect of $1-24+35+67$
- $l_2' = -12.5$ is the estimate of the effect of $2-14+36+57$
- $l_3' = -1.5$ is the estimate of the effect of $3+15+26-47$
- $l_{12}' = -21.5$ is the estimate of the effect of $12-4+37+56$
- $l_{13}' = -1.5$ is the estimate of the effect of $13+5+27-46$
- $l_{23}' = -3.0$ is the estimate of the effect of $23+6+17-45$
\( l_{123}' = -2.0 \) is the estimate of the effect of -34+25+16+7

\((l_1 + l_1')/2 = 66.75 \text{ est I} \)

\((l_1 - l_1')/2 = -0.5 \text{ est Higher Order Terms} \)

\((l_2 + l_2')/2 = 0.75 \text{ est 1+35+67} \)

\((l_2 - l_2')/2 = 12.25 \text{ est 14} \)

\((l_3 + l_3')/2 = -0.25 \text{ est 3+15+26} \)

\((l_3 - l_3')/2 = 1.25 \text{ est 47} \)

\((l_{12} + l_{12}')/2 = 0.5 \text{ est 12+37+56} \)

\((l_{12} - l_{12}')/2 = 22.0 \text{ est 4} \)

\((l_{13} + l_{13}')/2 = -0.5 \text{ est 13+5+27} \)

\((l_{13} - l_{13}')/2 = 1.0 \text{ est 46} \)

\((l_{23} + l_{23}')/2 = -1.0 \text{ est 23+6+17} \)

\((l_{23} - l_{23}')/2 = 2.0 \text{ est 45} \)

\((l_{123} + l_{123}')/2 = 0.25 \text{ est 25+16+7} \)

\((l_{123} - l_{123}')/2 = 2.25 \text{ est 34} \)

It can be seen that all interactions and main effects of 4 are unconfounded.

If \( \sigma_y = 3, \sigma_{eff} = \left( \frac{4\sigma^2}{N} \right)^{\frac{1}{2}} = \left( \frac{4 \cdot 3^2}{16} \right)^{\frac{1}{2}} = 1.5 \). So, average (I), 14, and 4 are important and estimated to be 66.75, 12.25, and 22.0 respectively.

Model for response is:

\[ \hat{y} = 66.75 + \frac{22.0}{2} x_4 + \frac{12.25}{2} x_1 x_4 \]

**Summary:**

- First we conducted the “principal fraction” from the family of generators
  \[ I = \pm 124 = \pm 135 = \pm 236 = \pm 1237, \text{ i.e., we used the generators with all “+” signs:} \]
  \[ 4 = 12,5 = 13,6 = 23,7 = 123 \]

- Second we conducted another 2\(^7\) design with generators 4=\(-12,5=13,6=23,7=123\). We noted that the recipe matrix for this 2nd design was identical to that of our first design except that column 4 was folded. We saw that the results of this fold-over design when combined with those of the 1st design unconfounded the effects of 4 and all of its 2-factor interactions.

Another type of follow-up design - to clear up confounding left by a 1st design is the Mirror Image Design which flips all signs in the recipe matrix of the 1st design.
The mirror image design when combined with the 1st design will unconfound all the main effects. Let’s examine the recipe matrix to see what the generators happen to be:

\[ 4 = -12, 5 = 13, 6 = 23, 7 = 123. \]

So defining relation is:

\[ I = -124 = -135 = -236 = 1237 \]
\[ = 2345 = 1346 = 1256 = -257 = -167 \]
\[ = -456 = 1457 = 2467 = 356 \]
\[ = -1234567 \]

Therefore, assuming 3-factor interactions and higher are negligible,

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1st Design

- \( l_1 \) est I
- \( l_1 \) est 1+24+35+67
- \( l_2 \) est 2+14+36+57
- \( l_3 \) est 3+15+26+47
- \( l_{12} \) est 12+4+37+56
- \( l_{13} \) est 13+5+27+46
- \( l_{23} \) est 23+6+17+45
- \( l_{123} \) est 34+25+16+7

2nd Design: Mirror Image Design

- \( l_1' \) est I
- \( l_1' \) est 1-24-35-67
- \( l_2' \) est 2-14-36-57
- \( l_3' \) est 3-15-26-47
- \( l_{12}' \) est 12-4+37+56
- \( l_{13}' \) est 13-5+27+46
- \( l_{23}' \) est 23-6+17+45
- \( l_{123}' \) est -34-25-16+7
By combining the two designs, we can obtain

\[
\frac{1}{2} \left( l_I + l''_I \right) \text{ est I} \quad \frac{1}{2} \left( l_I - l''_I \right) \text{ est Higher Order Eff.}
\]

\[
\frac{1}{2} \left( l_1 + l''_1 \right) \text{ est 1} \quad \frac{1}{2} \left( l_1 - l''_1 \right) \text{ est 24+35+67}
\]

\[
\frac{1}{2} \left( l_2 + l''_2 \right) \text{ est 2} \quad \frac{1}{2} \left( l_2 - l''_2 \right) \text{ est +14+36+57}
\]

\[
\frac{1}{2} \left( l_3 + l''_3 \right) \text{ est 3} \quad \frac{1}{2} \left( l_3 - l''_3 \right) \text{ est 15+26+47}
\]

\[
\frac{1}{2} \left( l_{12} + l''_{12} \right) \text{ est 12+37+56} \quad \frac{1}{2} \left( l_{12} - l''_{12} \right) \text{ est 4}
\]

\[
\frac{1}{2} \left( l_{13} + l''_{13} \right) \text{ est 13+27+46} \quad \frac{1}{2} \left( l_{13} - l''_{13} \right) \text{ est 5}
\]

\[
\frac{1}{2} \left( l_{23} + l''_{23} \right) \text{ est 23+17+45} \quad \frac{1}{2} \left( l_{23} - l''_{23} \right) \text{ est 6}
\]

\[
\frac{1}{2} \left( l_{123} + l''_{123} \right) \text{ est 7} \quad \frac{1}{2} \left( l_{123} - l''_{123} \right) \text{ est 34+25+16}
\]