Class room note: Drawing the pentagram

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Abstract

Using a strait-edge, string and chalk we show how to draw the pentagram

1 History and motivation

To be written.

2 The algebra

Consider five equally spaced line segments meeting at a point B. The angels formed will be $360/5 = 72^\circ$ angles. Thus we will need to construct $\cos(72^\circ)$ using our straight-edge, sring and chalk. Recall that in radians $72^\circ = \frac{2\pi}{5}$. Let

$$\alpha = e^{\frac{2\pi}{5}i} = \cos(\frac{2\pi}{5}) + i\sin(\frac{2\pi}{5}),$$

then $\alpha^5 = (e^{\frac{2\pi}{5}i})^5 = e^{2\pi i} = 1$. Hence α is a root of the polynomial $X^5-1 = (X-1)(1+X+X^2+X^3+X^4)$. Thus, because $\alpha \neq 1$, we have

$$1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 = 0$$

Also $\alpha^4\alpha = \alpha^5 = 1$, So

$$\alpha^4 = \alpha^{-1} = \cos(-\frac{2\pi}{5}) + i\sin(-\frac{2\pi}{5}) = \cos(\frac{2\pi}{5}) - i\sin(\frac{2\pi}{5}),$$

because $\cos(-x) = \cos(x)$ and $\sin(-x) = -\sin(x)$. Let $\beta = \alpha + \alpha^4$. Then

$$\beta = 2\cos(\frac{2\pi}{5}) = 2\cos(72^\circ).$$

We need to construct $\beta/2$. Now

$$\beta^2+\beta-1 = (\alpha+\alpha^4)^2+(\alpha+\alpha^4)-1 = \alpha^2+2\alpha^5+\alpha^8+\alpha+\alpha^4-1 = \alpha^2+2+\alpha^3+\alpha+\alpha^4-1 = \alpha^2+1+\alpha^3+\alpha+\alpha^4 = 0$$

Therefor $\beta/2$ and hence $\cos(72^\circ)$ is a root of

$$4X^2 + 2X - 1$$

a quadratic. Thus $\cos(72^\circ)$ can be drawn bye inscribing straight lines and circles. Note in particular applying the quadratic formula we have

$$\cos(72^\circ) = \frac{\sqrt{5}-1}{4}.$$

Thus we need to construct this length.

3 The geometry

In Section 2 we showed that we need to construct

$$\cos(72^\circ) = \frac{\sqrt{5}-1}{4}.$$

The only tools we have available are a straight edge with no marks some string and a piece of chalk. Thus given two points we can draw a straight line through them and we can draw a circle centered at one of the points that passes through the other. Furthermore using the compass (string and chalk) we can

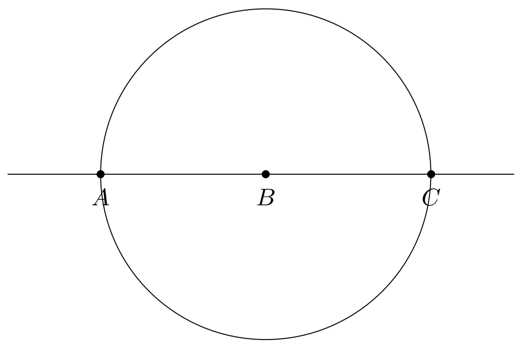
i erect a perpendicular to line at a point on the line;

ii bisect a line segment; and

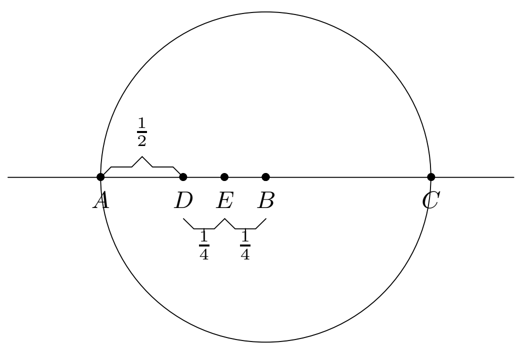
iii copy the distance between two marked points on one line to another line.

The 7 steps to construct the pentagram are:

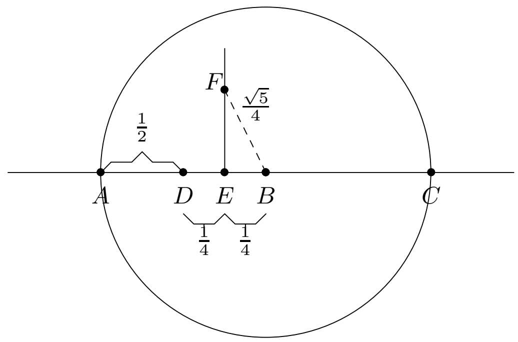
1. Mark any two points A and B and draw a line ℓ through them. Draw a circle of radius $|AB|$ centered at B. Let C be the the point other than A where the circle intersects ℓ. Take AB to be our unit length, i.e. the length $|AB| = 1$.



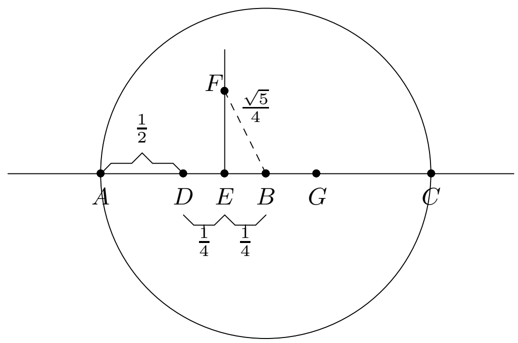
2. Bisect AB at D and DB at E.



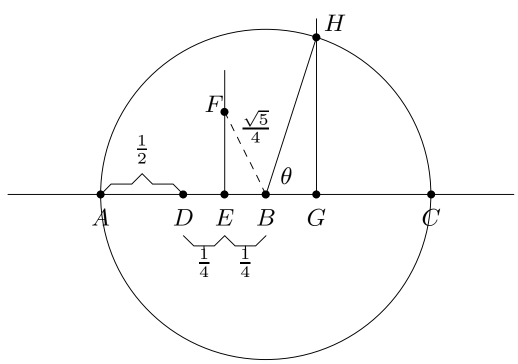
3. Erect a line perpendicular to ℓ at E and mark off F on it such that $|EF| = |AD|$. Note that the Pythagorian formula shows that $|BF| = \sqrt{5}/4$.



4. Mark G on ℓ such that $|EG| = |BF|$



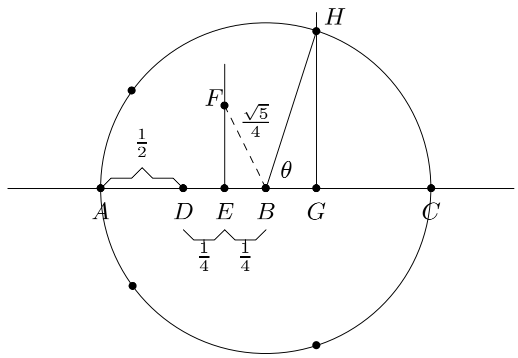
5. Erect a line perpendicular to ℓ at G and Let H be the point where it intersects the circle.



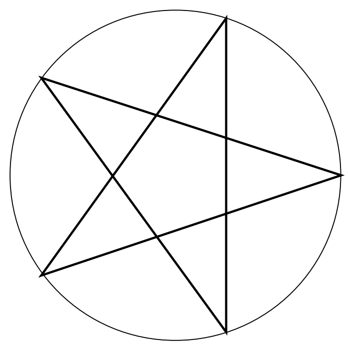
The angle $\theta = HBG$ is 72°s. To see this observe that

$$\cos(\theta) = \frac{|BG|}{|BH|} = \frac{|BG|}{1} = |BG| = |EG| - \frac{1}{4} = |BF| - \frac{1}{4} = \frac{\sqrt{5}}{4} - \frac{1}{4} = \frac{\sqrt{5}-1}{4}.$$

6. Using arc CH mark off the vertices of the pentagram.



7. Draw the pentagram.



Acknowledgements

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