To appear in JSPI (1999).

Orthogonal arrays of strength three from regular 3-wise balanced designs

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October 26, 1999

Abstract

The construction given in [4] is extended to obtain new infinite families of orthogonal arrays of strength 3. Regular 3-wise balanced designs play a central role in this construction.

1 Introduction

An *orthogonal array* of *size* N, with k *constraints* (or of *degree* k), s *levels* (or of *order* s), and *strength* t, denoted $\mathrm{OA}(N, k, s, t)$, is a $k \times N$ array with entries from a set of $s \geq 2$ symbols, having the property that in every $t \times N$

submatrix, every $t \times 1$ column vector appears the same number $\lambda = N/s^t$ times. The parameter λ is the *index* of the orthogonal array. An $\mathrm{OA}(N, k, s, t)$ is also denoted by $\mathrm{OA}_\lambda(t, k, s)$; in this notation, if t is omitted it is understood to be 2, and if λ is omitted it is understood to be 1. A *parallel class* in an $\mathrm{OA}_\lambda(t, k, s)$ is a set of s columns so that each row contains all s symbols within these s columns. A *resolution* of the orthogonal array is a partition of its columns into parallel classes, and an OA with such a resolution is termed *resolvable*. An $\mathrm{OA}_\lambda(t, k, n)$ is *class-regular* or *regular* if some group Γ of order n acts regularly on the symbols of the array. A class-regular $\mathrm{OA}_\lambda(t, k, n)$ is resolvable. See [1] for a brief survey on orthogonal arrays of strength at least 3.

An *ordered design* $\mathrm{OD}(N, k, s, t)$ is a $k \times N$ array with entries from a set of $s \geq 2$ symbols, having the property that in every $t \times N$ submatrix, every $t \times 1$ column vector of distinct symbols appears the same number λ of times, where $N = \lambda s(s-1)\cdots(s-t+1)$. We also use the notation $\mathrm{OD}_\lambda(t, k, s)$ to describe such an array.

In [4] a construction for orthogonal arrays of strength 3 is given that starts from resolvable 3-(v, k, λ) designs and uses 3-transitive groups. The conditions on the resolvable 3-design ingredient can be relaxed and a more general theorem can be stated using a resolvable set system $(X, \mathcal{B})$ such that:

1. the number of blocks containing three points $x, y, z \in X$, $x \neq y \neq z \neq x$, is a constant λ_3 that does not depend on the choice of x, y, z;

2. the number of blocks containing two points $x, y \in X$ but disjoint from a third point $z \in X$, $x \neq y \neq z \neq x$, is a constant b_2^1 that does not depend on the choice of x, y, z.

We allow $(X, \mathcal{B})$ to contain blocks of any size, including 1, 2, 3 and $|X|$.

If $x, y \in X$, $x \neq y$, then the number of blocks containing x and y is $\lambda_2 = b_2^1 + \lambda_3$ independent of the choice of x and y. These set systems need not be balanced for points. For example, the set system

$$\left\{\begin{matrix} \{1,2,3,4\}, \{1,2,\infty\}, \{1,3,\infty\}, \{1,4,\infty\}, \{2,3,\infty\}, \{2,4,\infty\}, \\ \{3,4,\infty\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\} \end{matrix}\right\}$$

has $\lambda_3 = 1$, $\lambda_2 = 3$, points $1,2,3,4$ each occur in 7 blocks, but ∞ in 6 blocks. If resolvability is required, then every point must occur in the same number $\lambda_1 = r$ of blocks. Kageyama [3] called a t-wise balanced design that is also i-balanced for each $i < t$ a *regular* t-wise balanced design.

If $\lambda_3 \neq 0$, and the block size is constant, then such a design is a 3-design. But these conditions are not necessary. For example, the edges of the complete graph K_v when v is even have $\lambda_3 = 0$, $\lambda_2 = 1$, and $\lambda_1 = v - 1$. Furthermore K_v has a 1-factorization and so this set system is resolvable.

A 3-$(v, \mathcal{K}, \Lambda)$ *design of width* w is a pair $(X, \mathcal{B})$ where X is a v-element set of *points* and $\mathcal{B}$ is a collection of subsets of X called *blocks* satisfying:

1. the size of every block is in $\mathcal{K}$;
2. $\Lambda = [\lambda_1, \lambda_2, \lambda_3]$ and every i-element subset is in λ_i blocks, $i = 1, 2, 3$ and
3. the blocks can be partitioned into λ_1 resolution classes using no more than w blocks in any one class.

We further generalize the theorem from [4] by replacing the 3-transitive group by a suitable ordered design. The revised theorem is as follows:

Theorem 1.1 *Suppose there exists an* $OD_m(3, w, n+1)$. *If a* 3-$(v, \mathcal{K}, \Lambda)$ *design of width* w *exists such that* $n = (\lambda_1 - \lambda_3)/(\lambda_2 - \lambda_3) - 2$ *and* $\lambda_3(n+1) \leq \lambda_2$, *then an* $OA_{m(n-1)(\lambda_2-\lambda_3)}(3, v, n+1)$ *also exists.*

Proof: The proof is similar to the proof of Theorem 2.1 in [4]. Let $(X, \mathcal{B})$ be a 3-$(v, \mathcal{K}, \Lambda)$ design of width w, where $X = \{x_i : 1 \leq i \leq v\}$. Denote the resolution classes of this design by $\mathcal{B}_j$, $1 \leq j \leq \lambda_1$, and let the blocks in $\mathcal{B}_j$ be denoted $B_{j,k}$, $1 \leq k \leq w_j$, where $w_j \leq w$. Construct the matrix v by λ_1 matrix M as in [4]:

$$M(i, j) = k \Leftrightarrow x_i \in B_{j,k},$$

$1 \leq i \leq v$, $1 \leq j \leq \lambda_1$.

Now let $r_1, \ldots, r_w$ be the rows of an $\mathrm{OD}_m(3, w, n+1)$ on symbols $1, \ldots, n+1$. Then replace every entry k of M by the row vector r_k. Call the resulting matrix M'.

Finally, let C be the matrix containing $m(n-1)(\lambda_2 - \lambda_3(n+1))$ copies of each "constant" column $x\,x\,\ldots\,x$, for $1 \leq x \leq n+1$. Then the matrix $[M', C]$ can be shown to be the desired $\mathrm{OA}_{m(n-1)(\lambda_2-\lambda_3)}(3, v, n+1)$, as in [4].

□

Results on ordered designs can be found in Teirlinck [5]. The only ordered designs we use in this paper are those that arise from 3-transitive permutation groups, as follows: Let G act 3-transitively on an $(n+1)$-element set Ω and let $m(n^3 - n)$ be the order of G. Then it is clear that there is an $\mathrm{OD}_m(3, n+1, n+1)$. In particular, if $n = q$ is a prime power, then the sharply 3-transitive group $PGL_2(q)$ yields an $\mathrm{OD}_1(3, q+1, q+1)$. Deleting any $q + 1 - w$ rows of this ordered design yields an $\mathrm{OD}_1(3, w, q+1)$. We therefore have the following corollary:

Corollary 1.2 *If a* 3-$(v, \mathcal{K}, \Lambda)$ *design of width* w *exists such that* $n = (\lambda_1 - \lambda_3)/(\lambda_2 - \lambda_3) - 2$ *is a prime power,* $\lambda_3(n+1) \leq \lambda_2$ *and* $w \leq n+1$, *then an* $OA_{(n-1)(\lambda_2-\lambda_3)}(3, v, n+1)$ *also exists.*

2 Applications of the Construction

Here is our first application of the construction.

Theorem 2.1 *Let q be an odd prime power. Then there exists an $OA_{q-1}(3, q+3, q+1)$.*

Proof: Set $v = q + 3$. Then a 1-factorization of K_v is a 3-$(v, \{2\}, [v-1, 1, 0])$ design of width $w = v/2 = (q+3)/2$. Then $w-1 \leq q$ and $(\lambda_1 - \lambda_3)/(\lambda_2 - \lambda_3) - 2 = (v-1-0)/(1-0) - 2 = v - 3 = q$, and the result follows from Corollary 1.2. □

The next lemma gives a construction for regular 3-designs having more than one block size.

Lemma 2.2 *For all $x \geq 2$ there exists a 3-$(4x, \{2, 4, 2x\}, [1 + 2(x-1)(2x-1), 2x-1, 1])$ design of width $2(x-1)$.*

Proof: The construction is essentially the doubling construction for Steiner quadruple systems (see [2], for example). Let A and B be two disjoint sets of size $2x$ and let $\{a_1, a_2, \ldots, a_{2x-1}\}$ and $\{b_1, b_2, \ldots, b_{2x-1}\}$ be 1-factorizations of A and B respectively. Take as blocks

1. the sets A and B each of size $2x$;
2. the x^2 4-element subsets in each of the $2x-1$ families: $\{\alpha \cup \beta : \alpha \in a_i, \beta \in b_i\}$, for all $i = 1, 2, \ldots, 2x-1$; and
3. all the $2\binom{2x}{2}$ pairs that are either in A or in B each repeated $x-2$ times.

It is not difficult to show that the blocks can be arrranged into resolution classes with at most $2(x-1)$ blocks each, to produce the required design. □

If we use the 3-$(v, \mathcal{K}, \Lambda)$ designs constructed in Theorem 2.2 as ingredients to Corollary 1.2 then

$$\begin{aligned} q &= \frac{(\lambda_1 - \lambda_3)}{(\lambda_2 - \lambda_3)} - 2 \\ &= \frac{(1 + 2(x-1)(2x-1) - 1)}{2(x-1)} - 2 \\ &= 2x - 3. \end{aligned}$$

Consequently, the following orthogonal arrays are obtained.

Theorem 2.3 *An $OA_{q^2-1}(3, 2(q+3), q+1)$ exists for every odd prime power q.*

If $\mathcal{D}_i$ is a 3-$(v, \mathcal{K}_i, \Lambda_i)$ design of width w_i, for $i = 1, 2, \ldots, n$ then for natural numbers α_i, the union with repeated blocks $\cup_{i=1}^{n} \alpha_i \mathcal{D}_i$ of α_i copies of $\mathcal{D}_i$, $1, 2, \ldots, n$ is a 3-$(v, \cup_i \mathcal{K}_i, \sum_i \alpha_i \Lambda_i)$ design of width $w = \max\{w_i\}$. We illustrate this idea next.

Theorem 2.4 *Let q be a prime power and choose integers $a, b, m \geq 1$ such that*

1. $q + 3 = m(a + b)$;
2. $ma \geq 4$;
3. $m(a + 2b) \equiv 0 \pmod 4$; *and*
4. $(m(a + 2b) - 4)/4 \equiv 0 \pmod b$.

Then an $OA_{\frac{(a+b)}{4b}(q-1+mb)(q-1)}\left(3, \left(\frac{a+2b}{a+b}\right)(q+3), q+1\right)$ *exists.*

Proof: Let $x = m(a + 2b)/4$. Then $x \geq 2$ is a positive integer and by Theorem 2.2 there is 3-$(4x, \{2, 4, 2x\}, [1 + 2(x - 1)(2x - 1), 2x - 1, 1])$ design $\mathcal{D}_1$ of width $2(x - 1)$. Also the edges of the complete graph K_{4x} (see the proof of Corollary 2.1) form a 3-$(4x, \{2\}, [4x - 1, 1, 0])$ design D_2 of width $w = 2x$. Take *one* copy of D_1 and $\frac{a}{b}(x - 1)$ copies of D_2 to form a

$$3\text{-}\left(4x, \{2, 4, 2x\}, \left[1 + \frac{(x-1)(4(a+b)x - (a+2b))}{b}, 1 + \frac{(x-1)(a+2b)}{b}, 1\right]\right)$$

design D of width $w = 2x$. The conditions of Theorem 1.1 are satisfied. □

The main applications of Theorem 1.1 rest on finding suitable regular 3-wise balanced designs. We have illustrated in this section the applications of some easily constructed designs of this type, but expect that further constructions can lead to more existence results for orthogonal arrays. We close by observing that, conversely, orthogonal arrays can be used to construct 3-$(v, \mathcal{K}, \Lambda)$ designs.

Theorem 2.5 *If there exists an $OA_\mu(3, n, yw)$, then there exists a 3-$(n, \mathcal{K}, \Lambda)$ design of width w with*

$$\Lambda = [\mu y^3 w^3, \mu y^3 w^2, \mu y^3 w].$$

Proof: Let A be an $\mathrm{OA}_\mu(3, n, v)$, where $v = yw$. We think of A as an $n \times \mu v^3$ array defined on symbol set X, $|X| = yw$. Partition X into subsets Y_i, $i = 1, 2, \ldots, w$ with each $|Y_i| = y$. We define a 3-$(n, \mathcal{K}, \Lambda)$ design of width w with $w\mu v^3$ blocks, as follows: for $i = 1, 2, \ldots, w$ and each column j of A, define a block $B_{i,j} = \{h : A[h, j] \in Y_i\}$. Then $(\{1, \ldots, n\}, \{B_{ij}\})$ is a 3-$(n, \mathcal{K}, \Lambda)$ design of width w with $\Lambda = [\mu y^3 w^3, \mu y^3 w^2, \mu y^3 w]$. □

Acknowledgments

The authors' research is supported as follows: ARO grant DAAG55-98-1-0272 (Colbourn), NSA grant MDA904-97-1-0072 (Kreher) and NSERC grant RGPIN 203114-98 (Stinson).

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