A note on $\{4\}$-GDDs of type 2^{10}

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Abstract

Motivated by a problem on resolvable designs we compute all non-isomorphic group divisible designs of type 2^{10} with block-size 4 and show that none has a parallel class.

A *group-divisible design* (or GDD) is a triple $(X, \mathcal{G}, \mathcal{A})$, which satisfies the following properties:

1. X is a set of elements called *points*,
2. $\mathcal{G}$ is a partition of X into subsets called *groups*,
3. $\mathcal{A}$ is a set of subsets of X (called *blocks*) such that a group and a block contain at most one common point,
4. every pair of points from distinct groups occurs in a unique block.

To avoid trivial cases, we will require that a GDD have more than one group.

The *group-type* or *type* of the GDD $(X, \mathcal{G}, \mathcal{A})$ is defined to be the multi-set $\{|G| : G \in \mathcal{G}\}$. We sometimes use an exponential notation to denote types: the type ${t_1}^{u_1} \dots {t_k}^{u_k}$ denotes u_i occurrences of t_i, $1 \leq i \leq k$. A GDD $(X, \mathcal{G}, \mathcal{A})$ is said to be a K-GDD if $|A| \in K$ for every $A \in \mathcal{A}$. A GDD of type t^u is said to be *uniform.* A subset $\mathcal{P} \subseteq \mathcal{A}$ of the blocks is said to be an *α-parallel class* if every point appears in exactly α blocks of $\mathcal{P}$. A 1-parallel classes is simply called a parallel class. In this paper we report that there are exactly 5 non-isomorphic $\{4\}$-GDDs of type 2^{10} and that none of them have a parallel class of blocks.

In Table 1 we present the 5 non-isomorphic $\{4\}$-GDDs of type 2^{10} as a list of base blocks which when developed with the automorphisms in the group $\langle \alpha, \beta \rangle$ will construct the designs. In each case the full automorphism group is $\langle \alpha, \beta \rangle$ and the groups of the GDD are :

$$\mathcal{G} = \{\{0,10\},\{1,11\},\{2,12\},\{3,13\},\{4,14\},\{5,15\},\{6,16\},\{7,17\},\{8,18\},\{9,19\}\}\,.$$

Table 1: The 5 non-isomorphic $\{4\}$-GDDs of type 2^{10}

Design 1:

Group order: 8

Generators: $\alpha = (0,3,1,2)(4,17,5,16)(6,14,7,15)(8,18)(9)(10,13,11,12)(19)$

$\beta = (0,16,1,17)(2,4,3,5)(6,11,7,10)(8,18)(9,19)(12,14,13,15)$

Base blocks: $\{6,7,8,19\}$, $\{0,8,15,17\}$, $\{0,6,11,14\}$, $\{0,4,13,19\}$, $\{0,1,2,3\}$

Design 2:

Group order: 12

Generators: $\alpha = (0,3)(1,2)(4)(5)(6,7)(8,9)(10,13)(11,12)(14)(15)(16,17)(18,19)$

$\beta = (0,1,2)(3)(4,7,8)(5,6,9)(10,11,12)(13)(14,17,18)(15,16,19)$

Base blocks: $\{4,11,12,15\}$, $\{4,5,16,17\}$, $\{0,15,17,18\}$, $\{0,5,7,8\}$, $\{0,4,13,19\}$, $\{0,1,2,3\}$

Design 3:

Group order: 16

Generators: $\alpha = (0,2)(1,3)(4,5)(6)(7)(8,19)(9,18)(10,12)(11,13)(14,15)(16)(17)$

$\beta = (0,4,2,16,1,5,3,17)(6,11,15,13,7,10,14,12)(8,19,18,9)$

Base blocks: $\{6,7,8,19\}$, $\{0,6,11,14\}$, $\{0,4,13,19\}$, $\{0,1,2,3\}$

Design 4:

Group order: 72

Generators: $\alpha = (0,9,17,13)(1,16,14,15)(2)(3,10,19,7)(4,5,11,6)(8,18)(12)$

$\beta = (0,3,1,2)(4,17,5,16)(6,14,7,15)(8)(9,19)(10,13,11,12)(18)$

Base blocks: $\{0,8,12,17\}$, $\{0,1,2,3\}$

Design 5:

Group order: 720

Generators: $\alpha = (0,6,17,4,8,9,12,1)(2,11,10,16,7,14,18,19)(3,15,13,5)$

$\beta = (0,5,8,4,1)(2,7,6,3,9)(10,15,18,14,11)(12,17,16,13,19)$

Base block: $\{0,1,2,3\}$

None of the designs in Table 1 has a parallel class.

Three different backtracking approaches were independently written to enumerate the $\{4\}$-GDDs of type 2^{10}. A discussion of the backtracking techniques used can be found in [5] and in [6]. All programs had the same conclusion.

> *The non-isomorphic $\{4\}$-GDDs of type 2^{10} are exactly those in Table 1 and none of these designs has a parallel class.*

Non-isomorphism was determined with the the graph-isomorphism program described in [5] and independently with the graph-isomorphism program NAUTY written by McKay [7].

The motivation for our study comes from a problem on resolvable designs. It has been known for some time that there is no resolvable $\{3\}$-GDD of type 2^6 (i.e. no Nearly Kirkman Triple system on 12 points, see [4]). As a consequence the "six-groups" case in the study of resolvable $\{3\}$-GDDs, of type g^u proved to be intractable when studied in [1, 9]; indeed in these papers it was proved that the necessary conditions for the existence of resolvable $\{3\}$-GDDs of type g^u were sufficient, except for $(g, u) = (2, 3), (6, 3)$ or $(2, 6)$, and except possibly for $u = 6$. In [8], the author developed a new construction to circumvented the non-existence of a Nearly Kirkman Triple system on 12 points by showing that it would suffice to partition the blocks of some $\{3\}$-GDD of type 2^6 into an α-parallel class and a β-parallel class where $\gcd(\alpha, \beta) = 1$. Now deleting a point from the cyclic Steiner triple system of order 13 yields a $\{3\}$-GDD of type 2^6 with a 2-parallel class, whence one can set $\alpha = 2$ and $\beta = 3$. With this, the "six-groups" case, and hence the spectrum for resolvable $\{3\}$-GDDs of type g^u, was settled in [8].

It has been speculated for some time that no resolvable $\{4\}$-GDD of type 2^{10} exists. This paper confirms this. Work has already been done on determining the spectrum for resolvable $\{4\}$-GDDs of type g^u (see e.g. [3] and [10]) and the non-existence of the case $(g, u) = (2, 10)$ raises concerns for the "ten-groups" case analogous to the "six-groups"' case described above. The construction in [8] would go a long way in settling this case if we could partition the blocks of some $\{4\}$-GDD of type 2^{10} into an α-parallel and a β-parallel class where $\gcd(\alpha, \beta) = 1$. As the replication number of this *GDD* is 6, this would require $\alpha = 1$ and $\beta = 5$, i.e. we require a $\{4\}$-GDD of type 2^{10} having a parallel class of blocks. Unfortunately, this paper also confirms that no such design exists. We anticipate then that the "ten-groups" case for resolvable $\{4\}$-GDDs of type g^u will prove to be very challenging to complete.

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