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A hole-size bound for incomplete t-wise balanced designs

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Abstract

An incomplete t-wise balanced design of index λ is a triple $(X, H, \mathcal{B})$ where X is a v–element set, H is a subset of X is called the hole, and $\mathcal{B}$ is a collection of subsets of X called blocks, such that every t-element subset of X is either in H or in exactly λ blocks, but not both. If H is a hole in an incomplete t-wise balanced design of order v and index λ, then $|H| \le v/2$ if t is odd and $(v-1)/2$ if t is even. In particular, this result establishes the validity of Kramer's conjecture that the maximal size of a block in a Steiner t-wise balanced design is at most $v/2$ if t is odd and at most $(v-1)/2$ when t is even.

Keywords: t-wise balanced designs, designs with holes.

1 Introduction

A t*–wise balanced design* (tBD) of type t–$(v, \mathcal{K}, \lambda)$ is a pair $(X, \mathcal{B})$ where X is a $v-$element set of points and $\mathcal{B}$ is a collection of subsets of X called blocks, with the property that the size of every block is in $\mathcal{K}$ and every t–element subset of X is contained in exactly λ blocks. If $\mathcal{K}$ is a set of positive integers strictly between t and v, then we say the tBD is *proper*. A t–$(v, \mathcal{K}, \lambda)$ design is also denoted by $S_\lambda(t, \mathcal{K}, v)$. If $|\mathcal{K}| = 1$, then the tBD is called a t–(v, k, λ) design, where $\mathcal{K} = \{k\}$. If $\lambda = 1$, then the notation $S(t, \mathcal{K}, v)$ is often used and the design is a Steiner system.

An *incomplete* t*–wise balanced design* (ItBD) of type t–$(v, h, \mathcal{K}, \lambda)$ is a triple $(X, H, \mathcal{B})$ where X is a v–element set of points, H is an h–element subset $H \subseteq X$ (called the *hole*), and $\mathcal{B}$ is a collection of subsets of X called *blocks*, such that every t–element subset of points is either contained in the hole or in exactly λ blocks, but not both. Thus, a ItBD of type t–$(v, h, \mathcal{K}, \lambda)$ is

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equivalent to a tBD of type t–$(v, \mathcal{K} \cup \{h\}, \lambda)$ having a block of size h which is repeated λ times. In particular, when $\lambda = 1$, a tBD of type t–(v, k, λ) is a ItBD of type t–$(v, h, \mathcal{K}, 1)$ for any $h \in \mathcal{K}$, provided of course that the tBD actually has a block of size h.

In 1983, Kramer [1] posed the following conjecture:

> *Let $(X, \mathcal{B})$ be a proper tBD with $t \geq 2$, and $\lambda = 1$. If k is the size of any block in $(X, \mathcal{B})$, then $k \leq (v-1)/2$ when t is even; while $k \leq v/2$ when t is odd.*

Kramer verified this conjecture for $t \leq 5$ and was able to show that in each case the bound was best possible by constructing an infinite family of tBDs meeting the bound. Recently, Ira and Kramer [3] considered this conjecture for $t = 6$, and were able to show the following.

Theorem 1.1 *If B is a block in a proper Steiner 6–wise balanced design, then $|B| \leq v/2$.*

This falls just short of Kramer's conjecture for $t = 6$. On the otherhand, as the authors in [3] point out, no proper Steiner tBD has been constructed when $t \geq 6$. In this paper we will prove the following result:

Theorem 1.2 *Let $(X, H, \mathcal{B})$ be a proper ItBD with $t \geq 2$. If $h = |H| \geq t$ is the size of the hole in $(X, H, \mathcal{B})$, then $h \leq (v-1)/2$ when t is even, while $h \leq v/2$ when t is odd.*

Setting $\lambda = 1$, we verify Kramers's conjecture for all $t \geq 2$:

Corollary 1.3 *In any proper tBD $(X, \mathcal{B})$ of type t–$(v, \mathcal{K}, 1)$, we have $k \leq (v-1)/2$ when t is even, while $k \leq v/2$ when t is odd, where k is the size of any block in $(X, \mathcal{B})$.*

To obtain Theorem 1.2, we use as an essential tool the following result, which allows us to consider only proper ItBDs of type t–$(v, h, \{t+1\}, \lambda)$:

Theorem 1.4 *Suppose there exists a proper ItBD of type t–$(v, h, \mathcal{K}, \lambda)$ with $2 \leq t \leq h < v$. Then there exists a proper ItBD of type t–$(v, h, \{t+1\}, \lambda')$ where*

$$\lambda' = \lambda \prod_{k \in \mathcal{K}} (k - t).$$

Proof: Let $(X, H, \mathcal{B})$ be a proper tBD of type t–$(v, h, \mathcal{K}, \lambda)$ with $2 \leq t \leq h < v$. Let $\mathcal{K} = \{k_1, k_2, \ldots, k_\ell\}$. For each $i = 1, 2, \ldots, \ell$ and each block $B \in \mathcal{B}$ of size k_i, take

$$c_i = \prod_{j=1, j \neq i}^{\ell} (k_j - t)$$

copies of B and then construct on each copy a t–$(k_i, t+1, k_i - t)$ design (obtained by simply taking all $(t+1)$–element subsets of B). We claim that the resulting design is a ItBD of type t–$(v, h, \{t+1\}, \lambda')$. Indeed, let T be any t–element subset of our point set X. If $T \subseteq H$, then T was not contained in any block in the original design and so is not contained in any block in the new design. Otherwise, suppose that T is contained in r_i blocks of size k_i in the original design, $i = 1, 2, \ldots, \ell$. Then

$$r_1 + r_2 + \cdots + r_\ell = \lambda.$$

In the new design, T is contained in

$$\begin{aligned} r_1c_1(k_1-t)+r_2c_2(k_2-t)+\cdots+r_\ell c_\ell(k_\ell-t) &= \sum_{i=1}^{\ell}\left\{r_i\prod_{j=1,j\neq i}^{\ell}(k_j-t)\right\}(k_i-t) \\ &= (r_1+r_2+\cdots+r_\ell)\prod_{j=1}^{\ell}(k_j-t) \\ &= \lambda\prod_{k\in\mathcal{K}}(k-t) \end{aligned}$$

blocks, as required. ▮

Thus, it suffices to prove Theorem 1.2 in the particular case $\mathcal{K}=\{t+1\}$. We can simplify the problem further by making the following observation, as noted in the case $\lambda=1$, [1]:

Lemma 1.5 *Suppose that Theorem 1.2 is true when t is even. Then it is also true when t is odd.*

Proof: Suppose to the contrary that we have a proper ItBD of type t–$(v,h,\mathcal{K},\lambda)$ with $h>v/2$, where t is odd, $t\geq 3$. Then $v\leq 2h-1$. Now take the derived design through a point in the hole to obtain a proper I$(t-1)$BD of type $(t-1)$–$(v-1,h-1,\mathcal{K}-1,\lambda)$ with $v-1\leq(2h-1)-1=2h-2=2(h-1)$, which is a contradiction to the hypothesis because $t-1$ is even. ▮

Thus, we need only consider the case t even; finally, we reduce the problem to that of proving the non-existence of proper ItBDs of types t–$(2h-1,h,\{t+1\},\lambda)$ and t–$(2h,h,\{t+1\},\lambda)$, as follows. We first prove Theorem 1.2 for $t=2$ and $\mathcal{K}=\{3\}$.

Lemma 1.6 *If there is an incomplete 2–$(v,h,\{3\},\lambda)$ design with $h<v$, then $v\geq 2h+1$.*

Proof: Let $(X,H,\mathcal{B})$ be the indicated incomplete 2–$(v,h,\{3\},\lambda)$ design, and let $x\in H$. Then the derived design with respect to X yields a λ–regular multi-graph on the vertex set $X\setminus H$; taking derived designs over all $x\in H$ yields a λh–regular multi-graph on $X\setminus H$. Now let $v\in X\setminus H$. Then for any $v'\in X\setminus H$ with $v'\neq v$, the number of times the pair $\{v,v'\}$ occurs in our multi-graph cannot exceed λ, for otherwise the pair $\{v,v'\}$ would have appeared in more than λ triples in $(X,H,\mathcal{B})$. Hence, $\lambda h\leq\lambda(v-h-1)$ and so $v\geq 2h+1$ as desired. ▮

Now suppose $t'\geq 4$ is even and that we have established the non-existence of proper It'BDs of types t'–$(2h-1,h,\{t'+1\},\lambda)$ and t'–$(2h,h,\{t'+1\},\lambda)$, and that we have established Theorem 1.2 for $t=t'-2$. Then there cannot exist a proper It'BD of type t'–$(v,h,\{t'+1\},\lambda)$ for any $v\leq 2h-2$, for if otherwise, then by deriving through two points in the hole we would obtain a proper ItBD of type t–$(v-2,h-2,\{t+1\},\lambda)$, where $v-2\leq 2h-4=2(h-2)$, contrary to our assumption. Hence Theorem 1.2 holds for t'. By inductive reasoning, starting with Lemma 1.6, we can summarize the above discussion as follows:

Lemma 1.7 *If there do not exist proper* ItBD*s of type t–$(2h-1,h,\{t+1\},\lambda)$ or t–$(2h,h,\{t+1\},\lambda)$ for any λ and any $2\leq t\leq h$, where t is even, then Theorem 1.2 holds (for all $2\leq t\leq h<v$, $\mathcal{K}$ and λ).*

In the next section, we will establish the non-existence of ItBDs of the type given in Lemma 1.7, and thereby establish Theorem 1.2. Then in section 3 we will construct some families of designs meeting the bounds of Theorem 1.2. In particular, for each odd $t \geq 3$ and each $h \geq t+1$ we will construct a ItBD of type t–$(2h, h, \{t+1\}, (2h-t)t!\binom{h-1}{t})$; by deriving through a point in the hole in these designs we will obtain for each even $t \geq 2$ and each $h \geq t+1$ a ItBD of type t–$(2h+1, h, \{t+1\}, (2h-t+1)(t+1)!\binom{h}{t+1})$. We will then show that there cannot exist any proper ItBD of type t–$(v, h, \mathcal{K}, \lambda)$ meeting the bounds of Theorem 1.2 when $\min\{k : k \in \mathcal{K}\} \geq t+2$.

We conclude this section by pointing out that if there exists a proper ItBD of type t–$(v, h, \mathcal{K}, \lambda)$ and a proper tBD type t–$(h, \mathcal{K}, \lambda)$, then we can construct the tBD on the points of the hole in the ItBD to obtain a proper tBD of type t–$(v, \mathcal{K}, \lambda)$ having a tBD of type t–$(h, \mathcal{K}, \lambda)$ as a (proper) sub-design. The reverse construction also holds: just remove the blocks (but not the points) of the sub-design to obtain an incomplete tBD. Thus Theorem 1.2 yields the following result concerning the maximal size of a sub-design in a tBD of type t–$(v, \mathcal{K}, \lambda)$:

Corollary 1.8 *Suppose that there is a proper tBD of type t–$(v, \mathcal{K}, \lambda)$ containing a tBD of type t–$(w, \mathcal{K}, \lambda)$ as a proper sub-design. Then $v \geq 2w$ when t is odd, while $v \geq 2w+1$ when t is even.*

2 The nonexistence of certain ItBDs.

In this section we show that proper ItBDs of types t–$(2h-1, h, \{t+1\}, \lambda)$ and t–$(2h, h, \{t+1\}, \lambda)$ cannot exist for any even t. We begin by proving two combinatorial identities. We use the notation

$$g_t(h) = h(h-1)(h-2)\cdots(h-t+1) = t!\binom{h}{t}.$$

(Note $g_0(h) = 1$, the empty product.)

Lemma 2.1 *For every even integer $t \geq 2$, and $h \geq t$,*

$$\frac{1}{t!}\sum_{j=1}^{t}(-1)^j g_{j-1}(h-1)g_{t-j+1}(h-1) = \binom{h-1}{t-1}.$$

Proof:

$$\begin{aligned} & \frac{1}{t!}\sum_{j=1}^{t}(-1)^j g_{j-1}(h)g_{t-j+1}(h-1) \\ = \quad & \frac{1}{t!}\left[-g_t(h-1) + g_1(h)g_{t-1}(h-1) + \sum_{j=3}^{t}(-1)^j g_{j-1}(h)g_{t-j+1}(h-1)\right] \\ = \quad & -\binom{h-1}{t} + \binom{h}{t} + \frac{1}{t!}\sum_{j=3}^{t}(-1)^j g_{j-1}(h)g_{t-j+1}(h-1) \end{aligned}$$

Now if $t = 2$, then the sum term is vacuous; otherwise by observing that

$$\begin{aligned} g_{j-1}(h)g_{t-j+1}(h-1) &= [h(h-1)\cdots(h-j+2)][(h-1)(h-2)\cdots(h-t+j-1)] \\ &= [h(h-1)(h-2)\cdots(h-t+j-1)][(h-1)\cdots(h-j+2)] \\ &= g_{t-j+2}(h)g_{j-2}(h-1), \end{aligned}$$

we see that for each $j = 3, \ldots, \frac{t}{2}+1$, the j and $t-j+3$ terms in the sum cancel. (They have opposite sign, because t is even). Thus in either case the sum term is equal to 0, and we get

$$\frac{1}{t!}\sum_{j=1}^{t}(-1)^j g_{j-1}(h)g_{t-j+1}(h-1) = -\binom{h-1}{t} + \binom{h}{t} = \binom{h-1}{t-1}$$

as desired. ∎

Lemma 2.2 *For every even integer $t \geq 2$, and $h \geq t$*

$$\sum_{j=1}^{t}(-1)^{j-1}g_{j-1}(h)g_{t-j+1}(h) = -\frac{tg_t(h)}{2h-t}.$$

Proof: We proceed by induction on t. When $t = 2$ we get

$$\sum_{j=1}^{2}(-1)^{j-1}g_{j-1}(h)g_{2-j+1}(h) = h(h-1) - h^2 = -\frac{2h(h-1)}{2h-2},$$

as required. Now define

$$f_t(h) = \sum_{j=1}^{t}(-1)^{j-1}g_{j-1}(h)g_{t-j+1}(h)$$

and let $t \geq 4$. Then

$$\begin{aligned} f_t(h) &= \sum_{j=1}^{t}(-1)^{j-1}g_{j-1}(h)g_{t-j+1}(h) \\ &= \sum_{j=0}^{t-1}(-1)^{j}g_{j}(h)g_{t-j}(h) \\ &= g_t(h) + \sum_{j=1}^{t-2}(-1)^{j}g_{j}(h)g_{t-j}(h) - g_{t-1}(h)h \\ &= g_t(h) - g_{t-1}(h)h + h^2\sum_{j=1}^{t-2}(-1)^{j}g_{j-1}(h-1)g_{t-j-1}(h-1) \\ &= g_t(h) - g_{t-1}(h)h - h^2\sum_{j=1}^{t-2}(-1)^{j-1}g_{j-1}(h-1)g_{t-j-1}(h-1) \end{aligned}$$

$$\begin{aligned}
&= g_t(h) - g_{t-1}(h)h - h^2 f_{t-2}(h-1) \\
&= g_t(h) - g_{t-1}(h)h + h^2 \frac{(t-2)g_{t-2}(h-1)}{2(h-1)-(t-2)} \qquad (1) \\
&= g_{t-1}(h)(h-t+1) - g_{t-1}(h)h + h\frac{(t-2)g_{t-1}(h)}{2h-t} \\
&= g_{t-1}(h)\left(\frac{(-t+1)(2h-t)+h(t-2)}{2h-t}\right) \\
&= g_{t-1}(h)(h-t+1)\left(\frac{-t}{2h-t}\right) \\
&= g_t(h)\left(\frac{-t}{2h-t}\right),
\end{aligned}$$

as desired. (The induction assumption was used in step (1).) ▮

Now suppose that we have a proper ItBD $(X, H, \mathcal{B})$ of type t–$(2h-1, h, \{t+1\}, \lambda)$ with $h \geq t$. For each $j = 0, 1, \cdots, t-1$, let

$$\mathcal{B}_j = \{B \in \mathcal{B} : |B \cap H| = j\}$$

and

$$\mathcal{T}_j = \{T \subseteq X : |T| = t \text{ and } |T \cap H| = j\}.$$

Let $t_i = |\mathcal{T}_i|$. Then $t_i = \binom{h}{i}\binom{h-1}{t-i}$. If we set $x_j = |\mathcal{B}_j|$, then counting pairs (T, B), such that $T \subset X$, $|T| = t$, and $T \subset B \in \mathcal{B}$ in two ways we see that the following equations hold.

$$(t+1-i)x_i + (i+1)x_{i+1} = \lambda t_i$$

for $i = 0, 1, \cdots, t-2$, and

$$2x_{t-1} = \lambda t_{t-1}.$$

The coefficient matrix of this system of equations is

$$A = \begin{bmatrix} t+1 & 1 & & & & \\ & t & 2 & & & \\ & & t-1 & 3 & & \\ & & & \ddots & \ddots & \\ & & & & 3 & t-1 \\ & & & & & 2 \end{bmatrix}.$$

If we set

$$\vec{x} = [x_0, x_1, \ldots, x_{t-1}]$$

and

$$\vec{t} = [t_0, t_1, \ldots, t_{t-1}],$$

then because each t–element subset of X not contained in H must occur in λ blocks, the matrix equation

$$A\vec{x} = \lambda\vec{t}$$

must hold. The matrix A is upper triangular with non-zero main diagonal. Thus the matrix A is invertible, whence

$$\vec{x} = A^{-1}(\lambda\vec{t}) = \lambda A^{-1}\vec{t}.$$

Indexing the rows and columns of A^{-1} by $1, 2, \ldots, t$ it can be readily verified that

$$A^{-1}[1,j] = (-1)^{j-1}\frac{1}{j\binom{t+1}{j}}$$

for $j = 1, 2, \ldots, t$. Therefore

$$\begin{aligned} x_0 &= \lambda\sum_{j=1}^{t}(-1)^{j-1}\frac{1}{j\binom{t+1}{j}}\binom{h}{j-1}\binom{h-1}{t-j+1} \\ &= -\frac{\lambda}{t+1}\sum_{j=1}^{t}(-1)^{j}\frac{1}{\binom{t}{j-1}}\binom{h}{j-1}\binom{h-1}{t-j+1} \\ &= -\frac{\lambda}{t+1}\left\{\frac{1}{t!}\sum_{j=1}^{t}(-1)^{j}g_{j-1}(h)g_{t-j+1}(h-1)\right\}. \end{aligned}$$

Thus, applying Lemma 2.1 we see that

$$x_0 = \frac{-\lambda}{t+1}\binom{h-1}{t-1} < 0.$$

This is a contradiction, because x_0 counts the number of blocks disjoint from the hole H and consequently cannot be negative. Therefore, no proper ItBD of type t–$(2h-1, h, \{t+1\}, \lambda)$ with $h \geq t$ can exist.

Now consider a proper ItBD of type t–$(2h, h, \{t+1\}, \lambda)$ with $h \geq t$. Using the same analysis as before, we arrive at the matrix equation

$$A\vec{x} = \lambda\vec{t},$$

except this time

$$t_i = \binom{h}{i}\binom{h}{t-i}$$

for $i = 0, 1, \ldots, t-1$. So in this case we get

$$\begin{aligned} x_0 &= \lambda\sum_{j=1}^{t}(-1)^{j-1}\frac{1}{j\binom{t+1}{j}}\binom{h}{j-1}\binom{h}{t-j+1} \\ &= \frac{\lambda}{t+1}\sum_{j=1}^{t}(-1)^{j-1}\frac{1}{\binom{t}{j-1}}\binom{h}{j-1}\binom{h}{t-j+1} \\ &= \frac{\lambda}{(t+1)!}\left\{\sum_{j=1}^{t}(-1)^{j-1}g_{j-1}(h)g_{t-j+1}(h)\right\} \end{aligned}$$

Thus, applying Lemma 2.2 we see that

$$x_0 = \frac{\lambda}{(t+1)!}\left(-\frac{tg_t(h)}{2h-t}\right) = \frac{-\lambda t}{(t+1)(2h-t)}\binom{h}{t} < 0,$$

again a contradiction. Thus, no proper ItBD of type t–$(2h, h, \{t+1\}, \lambda)$ with $h \geq t$ can exist. We have therefore established the following:

Theorem 2.3 *There do not exist proper* ItBDs *of types* t–$(2h-1, h, \{t+1\}, \lambda)$ *or* t–$(2h, h, \{t+1\}, \lambda)$ *for any* λ *and any* $2 \leq t \leq h$*, when* t *is even.*

Hence, by Lemma 1.7 we have established Theorem 1.2 and so the validity of Kramer's conjecture (Corollary 1.3). In particular, setting $\mathcal{K} = \{k\}$ in Corollary 1.3 we obtain the following:

Theorem 2.4 *A Steiner system* $S(t, k, v)$ *has* $k \leq v/2$ *when* t *is odd and* $k \leq (v-1)/2$ *when* t *is even.*

3 Meeting the bounds of Theorem 1.2

In this section we will show that for every t and every $h \geq t+1$ there exists a ItBD that meets the bound given by Theorem 1.2. Hence, these bounds are sharp! We require the following results.

Lemma 3.1 *Let* M *be the* t *by* t *matrix whose* $[i, j]$*–entry is given by*

$$M[i,j] = \begin{cases} h-t+i & \text{if } j = i, \\ h-i & \text{if } j = i+1, \\ 0 & \text{if } j \neq i \text{ or } i+1, \end{cases}$$

for $0 \leq i, j \leq t-1$*, and where* $h \geq t+1$*. Then the (unique) solution* $\vec{v}$ *to the matrix equation*

$$M\vec{v} = \vec{w},$$

where $\vec{w} = [0, 0, \ldots, 0, h-t+1]^T$ *has*

$$v_{t-i} = (-1)^{i-1}\frac{g_i(h-t+i)}{g_i(h-1)}$$

for $i = 1, 2, \ldots, t$*, and where* g_i *are the functions defined in Section 2.*

Proof: We proceed by induction on i. For clarity, we first explicitly write out the matrix equation $M\vec{v} = \vec{w}$:

$$\begin{bmatrix} h-t & h & & & & \\ & h-t+1 & h-1 & & & \\ & & h-t+2 & h-2 & & \\ & & & \ddots & \ddots & \\ & & & & h-2 & h-t+2 \\ & & & & & h-1 \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ \vdots \\ v_{t-2} \\ v_{t-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ h-t+1 \end{bmatrix}.$$

Clearly we have $(h-1)v_{t-1} = h-t+1$, and so

$$v_{t-1} = \frac{h-t+1}{h-1}.$$

Consequently, the result is established for $i = 1$. Suppose $2 \leq i \leq t$. Then from the matrix equation we see that

$$(h-i)v_{t-i} + (h-t+i)v_{t-(i-1)} = 0,$$

and so by induction we have

$$v_{t-i} = \frac{-(h-t+i)}{h-i} v_{t-(i-1)} = \frac{-(h-t+i)}{h-i} \left((-1)^{i-2} \frac{g_{i-1}(h-t+i-1)}{g_{i-1}(h-1)} \right).$$

Hence,

$$v_{t-i} = (-1)^{i-1} \frac{g_i(h-t+i)}{g_i(h-1)},$$

as desired. ∎

Lemma 3.2 *Let $J = [1,1,\ldots,1]^T$ and let $\vec{v}$ be the vector of Lemma 3.1. If t is odd, then $g_t(h-1)(J+\vec{v})$ is a non-negative integer column vector $\vec{u}$ (where $u_{t-i} = g_t(h-1)(1+v_{t-i})$, $i = 1,2,\ldots,t$) in which $u_{t-i} = 0$ if and only if $i = t-1$.*

Proof: For each $i = 1,2,\ldots t$,

$$\begin{aligned} u_{t-i} &= g_t(h-1)(1+v_{t-i}) \\ &= g_t(h-1)\left(1 + (-1)^{i-1}\frac{g_i(h-t+i)}{g_i(h-1)}\right). \end{aligned}$$

Thus the vector $\vec{u}$ is integral, because $g_i(h-1)$ divides $g_t(h-1)$. Now we must show that

$$1 + (-1)^{i-1}\frac{g_i(h-t+i)}{g_i(h-1)} \quad \geq \quad 0. \tag{2}$$

It suffices to consider only even i , $2 \leq i \leq t-1$. When i is even inequality (2), is equivalent to

$$g_i(h-1) \geq g_i(h-t+i),$$

which follows easily because $i \leq t-1$. Note in particular that there is equality in (2) if and only if $i = t-1$. The result follows. ∎

We can now present our constructions. For any finite set Y let $Sym(Y)$, denote the symmetric group on Y.

Theorem 3.3 *For each odd $t \geq 3$ and each $h \geq t+1$ there exists a* ItBD $(H\dot{\cup}Y, H, \mathcal{B})$ *of type t–$(2h, h, t+1, (2h-t)t!\binom{h-1}{t})$, with $Sym(H) \times Sym(Y)$ as an automorphism group.*

Proof: The point set of the design is $X = H\dot{\cup}Y$, where H and Y are disjoint sets of cardinality h. The hole is H. There are precisely t orbits $\Delta_0, \Delta_1, \ldots, \Delta_{t-1}$ of t-element subsets that need to be covered. The orbit Δ_i is the set of t-element subsets that intersect the hole H in exactly i points, $i = 0, 1, \ldots, t-1$. Similarly, there are precisely t orbits $\Gamma_0, \Gamma_1, \ldots, \Gamma_{t-1}$ of possible blocks ($(t+1)$–element subsets) that are available. The orbit Γ_j is the set of all $(t+1)$–element subsets that intersect the hole in exactly j points. Thus $|\Gamma_j| = \binom{h}{j}\binom{h}{t+1-j}$. Now consider the matrix M whose $[i, j]$–entry is

$$|\{B \in \Gamma_j : T \subseteq B\}|,$$

where T is any fixed representative of Δ_i. Then there is a ItBD $(H\dot{\cup}Y, H, \mathcal{B})$ of type t–$(2h, h, t+1, (2h-t)t!\binom{h-1}{t})$ with $Sym(H) \times Sym(Y)$ as an automorphism group if and only if there is a non-negative integer vector $\vec{u}$ such that

$$M\vec{u} = (2h-t)t!\binom{h-1}{t}J.$$

Furthermore, M is the matrix appearing in Lemma 3.1. Let $\vec{u}$ be the vector given in Lemma 3.2. Applying Lemmas 3.1 and 3.2 we see that

$$\begin{aligned} M\vec{u} &= M[g_t(h-1)(J+\vec{v})] \\ &= g_t(h-1)M(J+\vec{v}) \\ &= g_t(h-1)(MJ+M\vec{v}) \\ &= g_t(h-1)\left(\begin{bmatrix} 2h-t \\ 2h-t \\ \vdots \\ 2h-t \\ h-1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ h-t+1 \end{bmatrix}\right) \\ &= g_t(h-1)(2h-t)J \\ &= (2h-t)t!\binom{h-1}{t}J. \end{aligned}$$

The result follows. ∎

Remark 3.4 From Lemma 3.2, the vector $\vec{u}$ has $u_1 = 0$. This means that in the design $(H\dot{\cup}Y, H, \mathcal{B})$ constructed in Theorem 3.3, there are *no* blocks in the design that intersect H in exactly one point! That is, the block orbit Γ_0 is not used. Hence, the t–element subsets in Δ_0 are covered $\lambda = (2h-t)t!\binom{h-1}{t}$ times by the u_0 copies of the orbit Γ_0. (This is a tBD $(Y, \mathcal{B}‘)$ of type t–$(h, \{t+1\}, \lambda)$. Note that $\mathcal{B}‘$ is just the set of all $t+1$-element subsets of Y each repeated $u_0 = \lambda/(h-t)$ times.) We can remove this sub-design and replace it with lambda copies of the block Y to obtain a ItBD of type t–$(2h, h, \{t+1, h^\star\}, (2h-t)t!\binom{h-1}{t})$. This gives the surprising result that the bound of Theorem 1.2 can be achieved without all blocks having size $t+1$.

Theorem 3.5 *For each even $t \geq 2$ and each $h \geq t+1$ there exists a* ItBD *of type t-$(2h+1, h, t+1, (2h-t+1)(t+1)!\binom{h}{t+1})$.*

Proof: From Theorem 3.3, there is a I$(t+1)$BD $(H\dot{\cup}Y, H, \mathcal{B})$ of type $(t+1)$–$(2(h+1), h+1, t+2, (2(h+1)-(t+1))(t+1)!\binom{(h+1)-1}{t+1})$. Take the derived design through a point $x \in H$ to get the desired ItBD. ∎

Remark 3.6 Recall that $Sym(H) \times Sym(Y)$ was a group of automorphisms of the design constructed in Theorem 3.3. If x is any point in the design, then its stabilizer in $Sym(H) \times Sym(Y)$ is an automorphism group of the derived design through x. Hence the designs constructed in Theorem 3.5 have $Sym(H \setminus \{x\}) \times Sym(Y)$ as a group of automorphisms.

In Remark 3.4 we noted that the bound of Theorem 1.2 can be achieved without all blocks having $t+1$ points (at least when t is odd). We conclude this section by showing that whether t is even or odd, the bound cannot be achieved if the smallest block has $t+2$ points (that is, there *must* be blocks of size $t+1$ present in order to meet the bound). Note that it suffices to prove this theorem for even t, for if t is odd and there exists a proper ItBD of type t–$(2h, h, \mathcal{K}, \lambda)$ with $\min\{k : k \in \mathcal{K}\} \geq t+2$, then the derived design through a point in the hole would be a I$(t-1)$BD of type $t-1$–$(2h-1, h-1, \mathcal{K}-1, \lambda)$ (meeting the bound of Theorem 1.2) with minimum block size at least $t+1 = (t-1)+2$. We use the same strategy here as in Section 2. First of all, if there exists a proper ItBD of type t–$(2h+1, h, \mathcal{K}, \lambda)$ with $\min\{k : k \in \mathcal{K}\} \geq t+2$, then by the obvious generalization of Theorem 1.4, there exists a proper ItBD of type t–$(2h+1, h, \{t+2\}, \lambda')$ for some λ' (in fact $\lambda' = \lambda \prod_{k \in \mathcal{K}} \binom{k-t}{2}$); thus, we can restrict ourselves to the case $\mathcal{K} = \{t+2\}$. That is, it suffices to prove the non-existence of a ItBD of type t–$(2h+1, h, \{t+2\}, \lambda)$ for each even t. An induction proof similar to Lemma 2.2 will establish the following combinatorial identity.

Lemma 3.7 *For every even integer $t \geq 2$, and $h \geq t$*

$$\sum_{j=1}^{t} (-1)^{j-1} j g_{j-1}(h) g_{t-j+1}(h+1) = -\frac{t g_{t+1}(h+1)}{2h-t}.$$

Proof: We proceed by induction on t. When $t = 2$, we get

$$\sum_{j=1}^{2} (-1)^{j-1} j g_{j-1}(h) g_{2-j+1}(h+1) = (h+1)h - 2h(h+1) = -(h+1)h = \frac{-2(h+1)h(h-1)}{2h-2}$$

as required. Now define

$$f_t(h) = \sum_{j=1}^{t} (-1)^{j-1} j g_{j-1}(h) g_{t-j+1}(h+1)$$

and let $t \geq 4$. Then

$$\begin{aligned} f_t(h) &= \sum_{j=1}^{t} (-1)^{j-1} j g_{j-1}(h) g_{t-j+1}(h+1) \\ &= \sum_{j=0}^{t-1} (-1)^{j} (j+1) g_{j}(h) g_{t-j}(h+1) \end{aligned}$$

$$\begin{aligned}
&= g_t(h+1) + \sum_{j=1}^{t-2}(-1)^j(j+1)g_j(h)g_{t-j}(h+1) - tg_{t-1}(h)(h+1) \\
&= g_t(h+1) + h(h+1)\sum_{j=1}^{t-2}(-1)^j(j+1)g_{j-1}(h-1)g_{t-j-1}(h) - tg_{t-1}(h)(h+1) \\
&= g_t(h+1) + h(h+1)\sum_{j=1}^{t-2}(-1)^j g_{j-1}(h-1)g_{t-j-1}(h) \\
&\quad -h(h+1)\sum_{j=1}^{t-2}(-1)^{j-1} j g_{j-1}(h-1)g_{t-j-1}(h) - tg_{t-1}(h)(h+1). \qquad (3)
\end{aligned}$$

Now applying Lemma 2.1 we see that

$$\begin{aligned}
\sum_{j=1}^{t-2}(-1)^j g_{j-1}(h-1)g_{t-j-1}(h) &= (t-2)!\binom{h-1}{(t-2)-1} - g_{t-2}(h) + g_{t-2}(h-1) \\
&= (t-2)g_{t-3}(h-1) - g_{t-2}(h) + g_{t-2}(h-1) \\
&= (t-2-h)g_{t-3}(h-1) + g_{t-2}(h-1) \\
&= -g_{t-2}(h-1) + g_{t-2}(h-1) \\
&= 0.
\end{aligned}$$

Hence, by induction Equation (3) becomes

$$\begin{aligned}
f_t(h) &= g_t(h+1) - h(h+1)f_{t-2}(h-1) - tg_{t-1}(h)(h+1) \\
&= g_t(h+1) - h(h+1)\left\{\frac{-(t-2)g_{t-1}(h)}{2(h-1)-(t-2)}\right\} - tg_{t-1}(h)(h+1) \\
&= g_t(h+1)\left(\frac{2h-t+(t-2)h-(2h-t)t}{2h-t}\right) \\
&= g_t(h+1)\left(\frac{-t(h-t+1)}{2h-t}\right) \\
&= -\frac{tg_{t+1}(h+1)}{2h-t},
\end{aligned}$$

as desired. ∎

Now suppose that we have a proper ItBD of type t–$(2h+1, h, \{t+2\}, \lambda)$ for t even and $h \geq t$. Using the analysis of Section 2 we conclude that there must be a non-negative integer vector $\vec{x} = [x_0, x_1, \ldots, x_{t-1}]$ satisfying the matrix equation

$$A\vec{x} = \lambda\vec{t}$$

where $\vec{t} = [t_0, t_1, \ldots, t_{t-1}]$, $t_i = \binom{h}{i}\binom{h+1}{t-i}$, and

$$A = \begin{bmatrix} \binom{t+2}{2} & 1\cdot(t+1) & \binom{2}{2} & & & & \\ & \binom{t+1}{2} & 2\cdot t & \binom{3}{2} & & & \\ & & \binom{t}{2} & 3\cdot(t-1) & \binom{4}{2} & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & \binom{5}{2} & (t-2)\cdot 4 & \binom{t-1}{2} \\ & & & & & \binom{4}{2} & (t-1)\cdot 3 \\ & & & & & & \binom{3}{2} \end{bmatrix}.$$

Hence, $\vec{x} = \lambda A^{-1}\vec{t}$. Indexing the rows and columns of A^{-1} by $1, 2, 3, \ldots, t$, the first row of A^{-1} can easily be shown to be

$$A^{-1}[1, j] = \frac{(-1)^{j-1} j}{\binom{t+2}{2}\binom{t}{j-1}}$$

for $j = 1, 2, \ldots, t$, and so applying Lemma 3.7 (in the anti-penultimate step)

$$\begin{aligned} x_0 &= \frac{\lambda}{\binom{t+2}{2}} \sum_{j=1}^{t} (-1)^{j-1} j \frac{\binom{h}{j-1}\binom{h+1}{t-j+1}}{\binom{t}{j-1}} \\ &= \frac{\lambda}{t!\binom{t+2}{2}} \sum_{j=1}^{t} (-1)^{j-1} j g_{j-1}(h) g_{t-j+1}(h+1) \\ &= \frac{\lambda}{t!\binom{t+2}{2}} \left(\frac{-t g_{t+1}(h+1)}{2h-t} \right) \\ &= \frac{-2\lambda t g_{t+1}(h+1)}{(t+2)!(2h-t)} \\ &< 0, \end{aligned}$$

and we have a contradiction to the requirement $x_0 \geq 0$. Hence, no ItBD of the given type can exist. To summarize,1 we have shown the following:

Theorem 3.8 *There do not exist proper* ItBD*s of types*

$$t\text{–}(2h+1, h, \mathcal{K}, \lambda) \qquad (t \textit{ even})$$

or

$$t\text{–}(2h, h, \mathcal{K}, \lambda) \qquad (t \textit{ odd})$$

for any λ *and any* $2 \leq t \leq h$*, with* $\min\{k : k \in \mathcal{K}\} \geq t + 2$.

Acknowledgments

References

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