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On the maximum size of a hole in an incomplete t-wise balanced design with specified minimum block size

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Abstract. We derive a general upper bound on the size of a hole in an incomplete t-wise balanced design of order v and index λ, given that its minimum block size is $k \geq t+1$: if h is the size of the hole, then $h \leq (v+(k-t)(t-2)-1)/(k-t+1)$. We then show that this bound is sharp infinitely often when $t=2$ or 3, in that for each $h \geq t$ and each $k \geq t+1$, $(t,h,k) \neq (3,3,4)$, there exists an ItBD meeting the bound.

Keywords: t-wise balanced designs, designs with holes.

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1. Introduction

A *t–wise balanced design* (tBD) of type t–$(v, \mathcal{K}, \lambda)$ is a pair $(X, \mathcal{B})$ where X is a v–element set of points and $\mathcal{B}$ is a collection of subsets of X called blocks, with the property that the size of every block is in $\mathcal{K}$ and every t–element subset of X is contained in exactly λ blocks. If $\mathcal{K}$ is a set of positive integers strictly between t and v, then we say the tBD is *proper*.

An *incomplete t–wise balanced design* (ItBD) of type t–$(v, h, \mathcal{K}, \lambda)$ is a triple $(X, H, \mathcal{B})$ where X is a v–element set of points, H is an h–element subset $H \subseteq X$ (called the *hole*), and $\mathcal{B}$ is a collection of subsets of X called *blocks*, such that every t–element subset of points is either contained in the hole or in

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exactly λ blocks, but not both. Thus, a ItBD of type t–$(v,h,\mathcal{K},\lambda)$ is equivalent to a tBD of type t–$(v,\mathcal{K}\cup\{h\},\lambda)$ having a block of size h which is repeated λ times. In particular, when $\lambda=1$, a tBD of type t–$(v,\mathcal{K},\lambda)$ is a ItBD of type t–$(v,h,\mathcal{K},1)$ for any $h\in\mathcal{K}$, provided of course that the tBD actually has a block of size h.

In a recent article **[KR]** , the authors prove the following result:

Theorem 1.1. **[KR, Theorem 1.2]** *Let $(X,H,\mathcal{B})$ be a proper* ItBD *with $t\geq 2$. If $h=|H|\geq t$ is the size of the hole in $(X,H,\mathcal{B})$, then $h\leq (v-1)/2$ when t is even, while $h\leq v/2$ when t is odd.*

Setting $\lambda=1$, we verify Kramer's conjecture for all $t\geq 2$:

Corollary 1.2. **[KR, Corollary 1.3]** *In any proper tBD $(X,\mathcal{B})$ of type t–$(v,\mathcal{K},1)$, we have $k\leq (v-1)/2$ when t is even, while $k\leq v/2$ when t is odd, where k is the size of any block in $(X,\mathcal{B})$.*

Moreover, in **[KR, Theorem 3.3,3.5,3.8]** it was shown that for every $t\geq 2$ and every $h\geq t+1$, there exists an ItBD (with λ as a function of h and t) meeting the bounds of Theorem 1.1, and that *any* ItBD meeting this bound must have $k=t+1$ as its minimum block size. This of course raises the question regarding what happens if we prescribe the minimum block size in the ItBD to be something larger than $t+1$. In this article we derive the following upper bound:

If $(X,H,\mathcal{B})$ is a proper ItBD *of type t–$(v,h,\mathcal{K},\lambda)$ with $h\geq t\geq 2$ and $\min\mathcal{K}=k\geq t+1$, then*

$$h\leq\frac{v+(k-t)(t-2)-1}{k-t+1}.$$

We will show that this bound is sharp when $t=2$ or 3 in that for each $h\geq t$ and each $k\geq t+1$, $(t,h,k)\neq(3,3,4)$, there exists an ItBD (with λ a function of h and k) meeting this bound.

2. The upper bound

In this section we prove (Theorem 2.1) the bound mentioned in Section 1. This is in fact a generalization of Lemma 1.6 in **[KR]** and we generalize the technique used therein to prove our result here. We will require the following terminology: if $(X,\mathcal{B})$ is a tBD, then an *α-parallel class of blocks* in $(X,\mathcal{B})$ is a subset $\mathcal{B}'\subseteq\mathcal{B}$ with the property that each point $x\in X$ is contained in exactly α of the blocks in $\mathcal{B}'$.

Theorem 2.1. *If* $(X, H, \mathcal{B})$ *is a proper* ItBD *of type* t–$(v, h, \mathcal{K}, \lambda)$ *with* $h \geq t \geq 2$ *and* $\min \mathcal{K} = k \geq t+1$, *then*

$$h \leq \frac{v + (k-t)(t-2) - 1}{k-t+1}.$$

Proof. Let S be a fixed $(t-2)$–element subset of H and let

$$H = S \cup \{x_1, x_2, x_3, \dots, x_{h-t+2}\}.$$

Consider the derived design with respect to $S \cup \{x_i\}$, where i is a fixed element of $\{1, 2, \dots, h-t+2\}$. Now because H is a hole in the original ItBD, the blocks in the derived design form a λ–parallel class of blocks, each of size at least $k-t+1$ on the $v-h$ points of $X \setminus H$; call this set of blocks $\mathcal{B}_i$. Then again, because H is a hole in the original ItBD; we have $\mathcal{B}_i \cap \mathcal{B}_j = \emptyset$ for all $1 \leq i < j \leq h-t+2$. (It may well be that as *sets* there is a block $B_i \in \mathcal{B}_i$ and a block $B_j \in \mathcal{B}_j$ that are equal; however, they will have arisen from *distinct* blocks in $(X, H, \mathcal{B})$ and so as blocks are distinct.) Thus, as i ranges over $\{1, 2, \dots, h-t+2\}$, we obtain $h-t+2$ λ-parallel classes of blocks, each of size at least $k-t+1$, on the $v-h$ points of $X \setminus H$. Now because $k \geq t+1$, we have $k-t+1 \geq 2$, so let $v, v' \in X \setminus H$, $v \neq v'$. The pair v, v' cannot occur together in more than λ blocks in $\mathcal{B}_S = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_{h-t+2}$, for otherwise the t–element set $\{v, v'\} \cup S$ would have occurred in more than λ blocks in $(X, H, \mathcal{B})$. Thus, by considering the blocks in $\mathcal{B}_S$ which contain the fixed point $v \in X \setminus H$, we have

$$\lambda(k-t)(h-t+2) \leq \lambda(v-h-1)$$

from which we have

$$h \leq \frac{v + (k-t)(t-2) - 1}{k-t+1}.$$

□

Remark 2.2. Note that since S was a fixed but arbitrary $(t-2)$–element subset of H, we have equality in Theorem 2.1 if and only if every block that intersects the hole in exactly $t-1$ points has size k and that among these blocks each *pair* of elements from $X \setminus H$ is covered exactly $\lambda\binom{h}{t-2}/(t-1)$ times. This completely characterizes the case for $t = 2$:

Corollary 2.3. *In any incomplete* 2–$(v, h, \mathcal{K}, \lambda)$ *design with* $2 \leq h < v$ *and* $\min \mathcal{K} = k \geq 3$, *we have*

$$h \leq \frac{v-1}{k-1},$$

with equality occurring if and only if every block has size k *and intersects the hole (in exactly one point).*

Corollary 2.4. *In any incomplete* 3–$(v,h,\mathcal{K},\lambda)$ *design with* $3 \le h < v$ *and* $\min \mathcal{K} = k \ge 4$*, we have*

$$h \le \frac{v+k-4}{k-2},$$

with equality occurring only if every block that intersects the hole does so in exactly two points and has size k.

Proof. From Theorem 2.1 and Remark 2.2 we need only show that when $h = (v+k-4)/(k-2)$ no block intersects the hole in exactly one point. Suppose, to the contrary, that such a block B exists, and let x be the unique point in the intersection of B with the hole. Then taking the derived design through x yields an incomplete 2–$(v-1,h-1,\mathcal{K}-1,\lambda)$ design with $2 \le h-1 < v-1$ and $\min(\mathcal{K}-1) = k-1 \ge 2$, with

$$\begin{aligned} h-1 &= \frac{v+k-4}{k-2} - 1 \\ &= \frac{v-2}{k-2} \\ \text{i.e. } h-1 &= \frac{(v-1)-1}{(k-1)-1}. \end{aligned}$$

But in this derived design there is a block $B \setminus \{x\}$ that does not intersect the hole, contradicting Corollary 2.3. □

Remark 2.5. With regards to Corollary 2.4, it is *not* necessary that every block intersects the hole in order for equality to occur. For example, let $X = \{a,b,c\} \cup \{1,2,3,4,5\}$, $H = \{a,b,c\}$, and

$$\begin{aligned} \mathcal{B} = &\{\{1,2,3,4,5\},\{1,2,3,4,5\},\{1,2,3,4,5\}\} \\ &\cup \{\{x,y,i,j,k\} : x,y \in H \text{ and } i,j,k \in X \setminus H\}. \end{aligned}$$

This is an incomplete 3–$(8,3,\{5\},6)$ design (meeting the bound of Corollary 2.4) with the three copies of $\{1,2,3,4,5\}$ disjoint from the hole $H = \{a,b,c\}$.

We conclude this section by observing that the argument used in the proof of Corollary 2.4 can be easily generalized to show that in any incomplete t–$(v,h,\mathcal{K},\lambda)$ design meeting the bound of Theorem 2.1 the number of blocks which intersect the hole in exactly $t-2$ points is zero.

3. Meeting the bounds for $t = 2$ and $t = 3$

In this section, we will show that the bound of Theorem 2.1 is sharp infinitely often when $t = 2$ or 3. We begin with $t = 3$, using the technique in [**KR, Section 3**] to construct our designs. If $Y \subseteq X$, let $Sym(Y)$ denote the symmetric group on Y.

Theorem 3.1. *For each $h \geq 3$ and each $k \geq 4$, $(h,k) \neq (3,4)$, there exists a* I3BD $(X, H, \mathcal{B})$ *of type* 3–$(v, h, \{k\}, \lambda)$ *where* $v = (k-2)h - (k-4)$ *and*

$$\lambda = \binom{v-3}{k-3}(k-1)!\binom{v-h-1}{k-1},$$

having $Sym(H) \times Sym(X \setminus H)$ as an automorphism group.

Proof. There are three orbits $\Delta_0, \Delta_1, \Delta_2$ of 3–element subsets that need to be covered, where Δ_i is the set of 3–element subsets that intersect the hole in exactly i points. Similarly, there are three orbits $\Gamma_0, \Gamma_1, \Gamma_2$ of blocks (k–element subsets) that are available, where Γ_j is the set of all k–element subsets that intersect the hole in exactly j points. Thus,

$$|\Gamma_j| = \binom{h}{j}\binom{v-h}{k-j}.$$

Now, consider the 3 by 3 matrix M whose $[i,j]$-entry is

$$M[i,j] = |\{B \in \Gamma_j : T \subseteq B\}|,$$

where T is any fixed representative of Δ_i. Then the design whose existence is asserted by the statement of the theorem exists if and only if there is a non–negative integer vector $\vec{u}$ such that

$$M\vec{u} = \lambda J,$$

$J = [1,1,1]^{\mathsf{T}}$. We now proceed to show that such a vector $\vec{u}$ exists. The case $k = 4$ is handled in [**KR, Theorem 3.3**] , so we can henceforth assume that $k \geq 5$. The matrix M is given explicitly by

$$M = \begin{bmatrix} \binom{v-h-3}{k-3} & h\binom{v-h-3}{k-4} & \binom{h}{2}\binom{v-h-3}{k-5} \\ 0 & \binom{v-h-2}{k-3} & (h-1)\binom{v-h-2}{k-4} \\ 0 & 0 & \binom{v-h-1}{k-3} \end{bmatrix}.$$

Now observe that for $i = 0, 1, 2$, the sum along row i of M is

$$\binom{v-3}{k-3} - \sum_{\alpha=3-i}^{k-3}\binom{h-i}{\alpha}\binom{v-h-(3-i)}{k-3-\alpha}.$$

Hence, we first solve

$$M\vec{v} = \vec{w},$$

where

$$\vec{w} = \left[\sum_{\alpha=3}^{k-3}\binom{h}{\alpha}\binom{v-h-3}{k-3-\alpha}, \sum_{\alpha=2}^{k-3}\binom{h-1}{\alpha}\binom{v-h-2}{k-3-\alpha}, \sum_{\alpha=1}^{k-3}\binom{h-2}{\alpha}\binom{v-h-1}{k-3-\alpha}\right]^{\mathsf{T}}.$$

We see that

$$\begin{aligned}
v_2 &= \frac{1}{\binom{v-h-1}{k-3}}\sum_{\alpha=1}^{k-3}\binom{h-2}{\alpha}\binom{v-h-1}{k-3-\alpha}\\
&= \frac{1}{\binom{v-h-1}{k-3}}\left\{\binom{v-3}{k-3}-\binom{v-h-1}{k-3}\right\}\\
&= \frac{\binom{v-3}{k-3}}{\binom{v-h-1}{k-3}} - 1,\\
v_1 &= \frac{1}{\binom{v-h-2}{k-3}}\left\{\sum_{\alpha=2}^{k-3}\binom{h-1}{\alpha}\binom{v-h-2}{k-3-\alpha} - (h-1)\binom{v-h-2}{k-4}v_2\right\}\\
&= \frac{1}{\binom{v-h-2}{k-3}}\left\{\sum_{\alpha=1}^{k-3}\binom{h-1}{\alpha}\binom{v-h-2}{k-3-\alpha} - (h-1)\binom{v-h-2}{k-4}\frac{\binom{v-3}{k-3}}{\binom{v-h-1}{k-3}}\right\}\\
&= \frac{1}{\binom{v-h-2}{k-3}}\left\{\binom{v-3}{k-3}-\binom{v-h-2}{k-3}-\frac{(h-1)(k-3)}{v-h-1}\binom{v-3}{k-3}\right\}\\
&= -1, \text{ because } v-h-1=(h-1)(k-3),
\end{aligned}$$

and

$$\begin{aligned}
v_0 &= \frac{1}{\binom{v-h-3}{k-3}}\left\{\sum_{\alpha=3}^{k-3}\binom{h}{\alpha}\binom{v-h-3}{k-3-\alpha} - h\binom{v-h-3}{k-4}v_1 - \binom{h}{2}\binom{v-h-3}{k-5}v_2\right\}\\
&= \frac{1}{\binom{v-h-3}{k-3}}\left\{\sum_{\alpha=1}^{k-3}\binom{h}{\alpha}\binom{v-h-3}{k-3-\alpha} - \binom{h}{2}\binom{v-h-3}{k-5}\frac{\binom{v-3}{k-3}}{\binom{v-h-1}{k-3}}\right\}\\
&= \frac{1}{\binom{v-h-3}{k-3}}\left\{\binom{v-3}{k-3}-\binom{v-h-3}{k-3}-\frac{h(h-1)(k-3)(k-4)}{2(v-h-1)(v-h-2)}\binom{v-3}{k-3}\right\}\\
&= \frac{\binom{v-3}{k-3}}{\binom{v-h-3}{k-3}}\left\{1-\frac{h(k-4)}{2(v-h-2)}\right\} - 1\\
&= \frac{\binom{v-3}{k-3}}{\binom{v-h-3}{k-3}}\left\{\frac{(h-2)(k-2)}{2(v-h-2)}\right\} - 1.
\end{aligned}$$

Thus, we take $\vec{v} = [v_0, v_1, v_2]^{\mathsf{T}}$. Now because $h \geq 3$, it is easy to see that $v_0 > -1$, $v_1 = -1$, and $v_2 > -1$. Hence $\vec{v} + J$ is a non-negative rational vector which, by our choice of $\vec{w}$, is the unique solution to

$$M(\vec{v}+J) = \binom{v-3}{k-3} J.$$

Then setting

$$\vec{u} = (k-1)!\binom{v-h-1}{k-1}(\vec{v}+J),$$

we see that $\vec{u}$ is a non-negative *integer* vector for which

$$M\vec{u} = \binom{v-3}{k-3}(k-1)!\binom{v-h-1}{k-1} J = \lambda J$$

as desired. The result follows. $\square$

Remark 3.2. Note that in the solution vector $\vec{u}$ in the proof of Theorem 3.1, we have $u_1 = 0$. This means that in each design constructed by this result the orbit Γ_1 is never used. That is, there are no blocks which intersect the hole in exactly one point, as must be the case by Corollary 2.4. With regards to the parameters $(h,k) = (3,4)$, it is easy to show that no I3BD of type 3–$(6,3,\{4\},\lambda)$ exists, for any $\lambda > 0$.

Designs meeting the bound of Theorem 2.1, for $t = 2$, are now easily obtained.

Theorem 3.3. *For each $h \geq 2$ and each $k \geq 3$, $(h,k) \neq (2,3)$, there exists an* I2BD $(X,H,\mathcal{B})$ *of type* 2–$(v,h,\{k\},\lambda)$ *where* $v = (k-1)h+1$ *and*

$$\lambda = \binom{v-2}{k-2} k! \binom{v-h-1}{k}.$$

Proof. From Theorem 3.1, there is a I3BD of type 3–$(v+1,h+1,\{k+1\},\lambda)$, where

$$v+1 = ((k+1)-2)(h+1) - ((k+1)-4) = (k-1)h+2$$

and

$$\lambda = \binom{v+1-3}{k+1-3}(k+1-1)!\binom{v+1-(h+1)-1}{k+1-1}.$$

Take the derived design through a point in the hole to get the desired I2BD. $\square$

Remark 3.4. One can of course obtain infinite classes of 2–designs with $\lambda = 1$ meeting the bound of Theorem 2.1 by starting with resolvable BIBDs with $\lambda = 1$ (e.g. one–factorizations, Kirkman Triple Systems, etc.). With regard to the parameters $(h,k) = (2,3)$ in Theorem 3.3, it is a simple matter to construct I2BDs of type 2–$(5,2,\{3\},\lambda)$ for any even $\lambda > 0$.

4. Conclusion

It would be of great interest to determine the effectiveness of the bound of Theorem 2.1 for $t \geq 4$. Note that when $k = t + 1$, this bound reduces to $h \leq (v + t - 3)/2$ (which, incidentally, is equivalent to Theorem 2.1 in [**K**]) and so *cannot* be sharp in the case $t \geq 4$ (see Theorem 1.1). That is, one must restrict oneself to ItBDs with $\min \mathcal{K} = k \geq t + 2$.

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