Further results on the maximum size of a hole in an incomplete t-wise balanced design with specified minimum block size

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ABSTRACT

Kreher and Rees [3] proved that if h is the size of a hole in an incomplete t-wise balanced design of order v and index λ having minimum block size $k \geq t+1$, then

$$h \leq \frac{v+(k-t)(t-2)-1}{k-t+1}.$$

They showed that when $t = 2$ or 3, this bound is sharp infinitely often in that for each $h \geq t$ and each $k \geq t+1$, $(t,h,k) \neq (3,3,4)$, there exists an ItBD meeting the bound. In this paper, we show that this bound is sharp infinitely often for every t, viz: for each $t \geq 4$ there exists a constant $C_t > 0$ such that whenever $(h-t)(k-t-1) \geq C_t$ there exists an ItBD meeting the bound for some $\lambda = \lambda(t,h,k)$. We then describe an algorithm by which it appears that one can obtain a reasonable upper bound on C_t for any given value of t. © 2001 John Wiley & Sons, Inc.

1. INTRODUCTION

A *t–wise balanced design* (tBD) of type t–$(v, \mathcal{K}, \lambda)$ is a pair $(X, \mathcal{B})$ where X is a v–element set of points and $\mathcal{B}$ is a collection of subsets of X called blocks, with the property that the size of every block is in $\mathcal{K}$ and every t–element subset of X

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is contained in exactly λ blocks. If $\mathcal{K}$ is a set of positive integers strictly between t and v, then we say the tBD is *proper*.

An *incomplete t–wise balanced design* (ItBD) of type t–$(v, h, \mathcal{K}, \lambda)$ is a triple $(X, H, \mathcal{B})$ where X is a v–element set of points, H is an h–element subset $H \subseteq X$ (called the *hole*), and $\mathcal{B}$ is a collection of subsets of X called *blocks*, such that every t–element subset of points is either contained in the hole or in exactly λ blocks, but not both. Thus, an ItBD of type t–$(v, h, \mathcal{K}, \lambda)$ is equivalent to a tBD of type t–$(v, \mathcal{K} \cup \{h\}, \lambda)$ having a block of size h which is repeated λ times. In particular, when $\lambda = 1$, a tBD of type t–$(v, \mathcal{K}, 1)$ is a ItBD of type t–$(v, h, \mathcal{K}, 1)$ for any $h \in \mathcal{K}$, provided of course that the tBD actually has a block of size h.

In a recent article [3], the authors gave an upper bound on the size of a hole in an ItBD with specified minimum block size $k \geq t+1$. In what follows, an α-parallel class of blocks in a tBD $(X, \mathcal{B})$ is a subset $\mathcal{B}' \subset \mathcal{B}$ with the property that each point $x \in X$ is contained in exactly α of the blocks in B'.

Theorem 1.1. [3, Theorem 2.1] *If* $(X, H, \mathcal{B})$ *is a proper* ItBD *of type* t–$(v, h, \mathcal{K}, \lambda)$ *with* $h \geq t \geq 2$ *and* $\min \mathcal{K} = k \geq t+1$, *then*

$$h \leq \frac{v + (k-t)(t-2) - 1}{k - t + 1}.$$

Proof. Let S be a fixed $(t-2)$–element subset of H and let

$$H = S \cup \{x_1, x_2, x_3, \ldots, x_{h-t+2}\}.$$

Consider the derived design with respect to $S \cup \{x_i\}$, where i is a fixed element of $\{1, 2, \ldots, h-t+2\}$. Now because H is a hole in the original ItBD, the blocks in the derived design form a λ–parallel class of blocks, each of size at least $k-t+1$ on the $v-h$ points of $X \setminus H$; call this set of blocks $\mathcal{B}_i$. Then again, because H is a hole in the original ItBD; we have $\mathcal{B}_i \cap \mathcal{B}_j = \emptyset$ for all $1 \leq i < j \leq h-t+2$. (It may well be that as *sets* there is a block $B_i \in \mathcal{B}_i$ and a block $B_j \in \mathcal{B}_j$ that are equal; however, they will have arisen from *distinct* blocks in $(X, H, \mathcal{B})$ and so as blocks are distinct.) Thus, as i ranges over $\{1, 2, \ldots, h-t+2\}$, we obtain $h-t+2$ λ-parallel classes of blocks, each of size at least $k-t+1$, on the $v-h$ points of $X \setminus H$. Now because $k \geq t+1$, we have $k-t+1 \geq 2$, so let $v, v' \in X \setminus H$, $v \neq v'$. The pair v, v' cannot occur together in more than λ blocks in $\mathcal{B}_S = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_{h-t+2}$, for otherwise the t–element set $\{v, v'\} \cup S$ would have occurred in more than λ blocks in $(X, H, \mathcal{B})$. Thus, by considering the blocks in $\mathcal{B}_S$ which contain the fixed point $v \in X \setminus H$, we have

$$\lambda(k-t)(h-t+2) \leq \lambda(v-h-1)$$

from which we have

$$h \leq \frac{v + (k-t)(t-2) - 1}{k - t + 1}.$$

□

Moreover, it was shown in [3] that this bound is sharp when $t = 2$ or 3 in that for each $h \geq t$ and each $k \geq t+1$, $(t, h, k) \neq (3, 3, 4)$, there exists an ItBD meeting the bound for some $\lambda = \lambda(t, h, k)$.

In this article, we examine the effectiveness of the above bound for $t \geq 4$. An ItBD meeting this bound will be called *tight*. We first make the following observation regarding tight designs.

Lemma 1.2. *If $(X, H, \mathcal{B})$ is a tight* ItBD *of type t-$(v, h, \mathcal{K}, \lambda)$ and $S \subseteq H$ with $|S| = s$ where $1 \leq s \leq t-2$, then the derived design with respect to S is a tight* I$(t-s)$BD *of type $(t-s)$-$(v-s, h-s, \mathcal{K}-s, \lambda)$.*

Proof. We have
$$h = \frac{v + (k-t)(t-2) - 1}{k-t+1},$$
which we rewrite as
$$v - h - 1 = (h - t + 2)(k - t). \tag{1}$$
Then
$$\begin{aligned}(v-s) - (h-s) - 1 &= v - h - 1\\ &= (h - t + 2)(k - t)\\ &= ((h-s) - (t-s) + 2)((k-s) - (t-s)),\end{aligned}$$
i.e.
$$h - s = \frac{(v-s) + ((k-s) - (t-s))((t-s) - 2) - 1}{(k-s) - (t-s) + 1},$$
thus the derived design is also tight. □

In Section 2., we will examine the cases $t = 4$, 5 and 6 in some detail and use the analysis to describe a general algorithm by which it appears that one can derive sufficient conditions for the existence of tight t-designs for any value of t (the analysis also tells us how to construct them). We obtain the following results.

Theorem 1.3. *(Theorem 2.2) Let $t = 4$ or 5. Then, for every h, k with $(h-t)(k-t-1) \geq 6$, there exists a tight* ItBD *of type t-(v, h, k, λ) for some $\lambda = \lambda(t, h, k)$.*

Theorem 1.4. *(Lemma 2.3) Let $t = 6$. Then, for every h, k with $(h-t)(k-t-1) \geq 9$, there exists a tight* ItBD *of type t-(v, h, k, λ) for some $\lambda = \lambda(t, h, k)$.*

We obtain analogous results for $7 \leq t \leq 9$. (See Theorem 2.5.)

In Section 3., we prove the following general asymptotic result that confirms that the upper bound of Theorem 1.1 is eventually sharp for every value of t:

Theorem 1.5. *For each $t \geq 4$ there exists a constant $C_t > 0$ such that whenever $(h-t)(k-t-1) \geq C_t$ there exists a tight* ItBD *of type t-(v, h, k, λ) for some $\lambda = \lambda(t, h, k)$.*

Note that by Theorems 1.3 and 1.4 we have $C_4 \leq 6$, $C_5 \leq 6$ and $C_6 \leq 9$. The subsequent results in Section 2. give us analogous upper bounds for C_t where $7 \leq t \leq 9$.

Incomplete block designs have been extensively studied for $t = 2$ and have a wide variety of applications to, e.g. the construction of pairwise balanced designs with subdesigns, optimum pair-packings and coverings and the construction of a wide variety of other combinatorial objects using Wilson-type constructions, to name only a few. In [2] Kreher and Rees gave the first general upper bound on the size of a hole in an incomplete t-wise balanced design, viz:

Theorem 1.6. [2, Theorem 1.2] *In any proper* ItBD *of type* t-$(v, h, \mathcal{K}, \lambda)$ *with* $h \geq t \geq 2$, *we have* $h \leq (v-1)/2$ *when* t *is even, while* $h \leq v/2$ *when* t *is odd.*

Setting $\lambda = 1$ in Theorem 1.6 the authors in [2] settled a long-standing conjecture posed by E. Kramer in [1]:

Theorem 1.7. [2, Corollary 1.3] *In any proper* tBD *of type* t-$(v, \mathcal{K}, 1)$, *we have* $k \leq (v-1)/2$ *when* t *is even, while* $k \leq v/2$ *when* t *is odd, where* k *is the size of any block in the* tBD.

Moreover it was shown in [2] that for every $t \geq 2$ and every $h \geq t+1$ there exists an ItBD meeting the bound of Theorem 1.6 for some $\lambda = \lambda(h, t)$ and that *any* ItBD meeting this bound must have $k = t + 1$ as its minimum block size. This latter condition naturally raises the problem of determining an effective upper bound on h in the case where we prescribe the minimum block size to be something larger than $t + 1$. It is this that motivated the development of such a bound in [3] and the current investigation of the general effectiveness of this bound.

We will use the notation $\mathrm{Sym}(Y)$ to denote the symmetric group on the set Y, the notation

$$g_n(z) = z(z-1)(z-2)\cdots(z-n+1) = n!\binom{z}{n},$$

where $g_0(z) = 1$, to denote the empty product, and the notation J_t to denote the t by 1 vector of all 1s.

2. CONSTRUCTING TIGHT DESIGNS FOR SOME SMALL VALUES OF t

We will follow the technique used in [3, Section 3] to construct our designs. We begin by constructing a tight I5BD $(X, H, \mathcal{B})$ of type 5-(v, h, k, λ) having $\mathrm{Sym}(H) \times \mathrm{Sym}(X \setminus H)$ as an automorphism group. There are five orbits $\Delta_0, \Delta_1, \ldots, \Delta_4$ of 5-element subsets that need to be covered, where Δ_i is the set of all 5–element subsets that intersect the hole in exactly i points. Similarly, there are five orbits $\Gamma_0, \Gamma_1, \ldots, \Gamma_4$ of blocks (k-element subsets) that are available, where Γ_j is the set of all k–element subsets that intersect the hole in exactly j points. Thus,

$$|\Gamma_j| = \binom{h}{j}\binom{v-h}{k-j}.$$

Now consider the 5 by 5 matrix M_5 whose $[i,j]$-entry is

$$M_5[i,j] \;=\; |\{B \in \Gamma_j : T \subset B\}| = \binom{h-i}{j-i}\binom{v-h-5+i}{k-5-j+i},$$

where T is any fixed representative of Δ_i. Then, the design under consideration exists if and only if there is a non-negative rational vector $\vec{u}$ such that

$$M_5\vec{u} \;=\; J_5,$$

where $J_5 = [1,1,1,1,1]^T$. The matrix M_5 is given by

$$M_5 = \begin{bmatrix} \binom{v-h-5}{k-5} & \binom{h}{1}\binom{v-h-5}{k-6} & \binom{h}{2}\binom{v-h-5}{k-7} & \binom{h}{3}\binom{v-h-5}{k-8} & \binom{h}{4}\binom{v-h-5}{k-9} \\ 0 & \binom{v-h-4}{k-5} & \binom{h-1}{1}\binom{v-h-4}{k-6} & \binom{h-1}{2}\binom{v-h-4}{k-7} & \binom{h-1}{3}\binom{v-h-4}{k-8} \\ 0 & 0 & \binom{v-h-3}{k-5} & \binom{h-2}{1}\binom{v-h-3}{k-6} & \binom{h-2}{2}\binom{v-h-3}{k-7} \\ 0 & 0 & 0 & \binom{v-h-2}{k-5} & \binom{h-3}{1}\binom{v-h-2}{k-6} \\ 0 & 0 & 0 & 0 & \binom{v-h-1}{k-5} \end{bmatrix}$$

where $\binom{z}{n} = 0$ whenever $n < 0$. Taking $\vec{u} = [u_0, u_1, u_2, u_3, u_4]^T$, we get the following solution:

$$u_4 \;=\; \binom{v-h-1}{k-5}^{-1};$$

$$u_3\binom{v-h-2}{k-5} + u_4\binom{h-3}{1}\binom{v-h-2}{k-6} = 1.$$

Therefore,

$$u_3 \;=\; \binom{v-h-2}{k-5}^{-1}\left\{1 - \frac{(h-3)(k-5)}{v-h-1}\right\} = 0$$

by Equation (1);

$$u_2\binom{v-h-3}{k-5} + u_3\binom{h-2}{1}\binom{v-h-3}{k-6} + u_4\binom{h-2}{2}\binom{v-h-3}{k-7} = 1.$$

Therefore,

$$\begin{aligned} u_2 \;&=\; \binom{v-h-3}{k-5}^{-1}\left\{1 - \binom{h-2}{2}\binom{v-h-3}{k-7}\binom{v-h-1}{k-5}^{-1}\right\} \\ &=\; \binom{v-h-3}{k-5}^{-1}\left\{1 - \frac{(h-2)(h-3)(k-5)(k-6)}{2(v-h-1)(v-h-2)}\right\} \\ &=\; \binom{v-h-3}{k-5}^{-1}\left\{\frac{(h-4)(k-4)}{2(v-h-2)}\right\} \end{aligned}$$

by Equation (1);

$$u_1\binom{v-h-4}{k-5}+u_2\binom{h-1}{1}\binom{v-h-4}{k-6}$$
$$+u_3\binom{h-1}{2}\binom{v-h-4}{k-7}+u_4\binom{h-1}{3}\binom{v-h-4}{k-8}=1.$$

Therefore,

$$\begin{aligned}
u_1 &= \tbinom{v-h-4}{k-5}^{-1}\Big\{\, 1-\tbinom{h-1}{3}\tbinom{v-h-4}{k-8}\tbinom{v-h-1}{k-5}^{-1} \\
&\qquad -(h-1)\tbinom{v-h-4}{k-6}\tbinom{v-h-3}{k-5}^{-1}\tfrac{(h-4)(k-4)}{2(v-h-2)}\Big\} \\
&= \tbinom{v-h-4}{k-5}^{-1}\Big\{\, 1-\tfrac{(h-1)(h-2)(h-3)(k-5)(k-6)(k-7)}{6(v-h-1)(v-h-2)(v-h-3)} \\
&\qquad -\tfrac{(h-1)(k-5)(h-4)(k-4)}{2(v-h-3)(v-h-2)}\Big\} \\
&= \tbinom{v-h-4}{k-5}^{-1}\Big\{\tfrac{(h-4)(k-4)(hk-5k-6h+24)}{3(v-h-2)(v-h-3)}\Big\}
\end{aligned}$$

by Equation (1);

$$u_0\binom{v-h-5}{k-5}+u_1\binom{h}{1}\binom{v-h-5}{k-6}+u_2\binom{h}{2}\binom{v-h-5}{k-7}$$
$$+u_3\binom{h}{3}\binom{v-h-5}{k-8}+u_4\binom{h}{4}\binom{v-h-5}{k-9}=1.$$

Therefore,

$$\begin{aligned}
u_0 &= \binom{v-h-5}{k-5}^{-1}\Bigg\{\, 1-\binom{h}{4}\binom{v-h-5}{k-9}\binom{v-h-1}{k-5}^{-1} \\
&\qquad -\tbinom{h}{2}\tbinom{v-h-5}{k-7}\tbinom{v-h-3}{k-5}^{-1}\tfrac{(h-4)(k-4)}{2(v-h-2)} \\
&\qquad -h\tbinom{v-h-5}{k-6}\tbinom{v-h-4}{k-5}^{-1}\tfrac{(h-4)(k-4)(hk-5k-6h+24)}{3(v-h-2)(v-h-3)}\Bigg\} \\
&= \tbinom{v-h-5}{k-5}^{-1}\Big\{\, 1-\tfrac{h(h-1)(h-2)(h-3)(k-5)(k-6)(k-7)(k-8)}{24(v-h-1)(v-h-2)(v-h-3)(v-h-4)} \\
&\qquad -\tfrac{h(h-1)(k-5)(k-6)(h-4)(k-4)}{4(v-h-2)(v-h-3)(v-h-4)} \\
&\qquad -\tfrac{h(k-5)(h-4)(k-4)(hk-5k-6h+24)}{3(v-h-2)(v-h-3)(v-h-4)}\Big\} \\
&= \tbinom{v-h-5}{k-5}^{-1} \\
&\quad \cdot\Big\{\tfrac{(h-4)(k-4)(3h^2k^2-25hk^2+54k^2-31h^2k+245hk-486k+82h^2-598h+1092)}{8(v-h-2)(v-h-3)(v-h-4)}\Big\}
\end{aligned}$$

by Equation (1).

Now recall that we want conditions under which all $u_i \geq 0$. Let us make the following substitutions:

$$x = h - t \quad (= h - 5) \quad \text{and} \quad y = k - t - 1 \quad (= k - 6), \tag{2}$$

whereupon by Equation (1) we have

$$v - h - 1 \ = \ (x + 2)(y + 1). \tag{3}$$

Then we can see that

$$\begin{aligned} u_4 &= \binom{(x+2)(y+1)}{y+1}^{-1}, \\ u_3 &= 0 \end{aligned}$$

and for each $m = 3, 4, 5$,

$$u_{5-m} = \binom{(x+2)(y+1)-(m-1)}{y+1}^{-1} \frac{(m-2)(x+1)(y+2)S_m(x,y)}{(m-1)!g_{m-2}((x+2)(y+1)-1)}, \tag{4}$$

where $S_3(x, y) = 1$, $S_4(x, y) = xy - 6$, and $S_5(x, y) = 3x^2y^2 + 5xy^2 + 4y^2 + 5x^2y - 5xy + 12y + 4x^2 + 12x + 80$. Note that $S_m(x, y)$ is a symmetric polynomial in x and y of degree $m - 3$. Now, clearly

$$\begin{aligned} S_3(x,y) &\geq 0 \text{ for all } x \geq 0, y \geq 0; \\ S_4(x,y) &\geq 0 \text{ for all } x \geq 0, y \geq 0 \text{ with } xy \geq 6, \end{aligned}$$

while

$$\begin{aligned} S_5(x,y) = (3(xy)^2 - 5(xy) + 80) + (x+y)(5(xy) + 12) \\ +(x^2+y^2)(4) \geq 0 \text{ for all } x \geq 0, y \geq 0. \end{aligned}$$

Hence, all $S_m(x, y) \geq 0$ when $xy \geq 6$. Furthermore, the condition $xy \geq 6$ guarantees that

$$\binom{(x+2)(y+1) - (m-1)}{y+1} \geq \binom{(x+2)(y+1) - 4}{y+1} > 0,$$

because $(x + 1)(y + 1) - 4 > 0$. We have therefore established the following.

Lemma 2.1. *Let M_5 be the matrix presented earlier in this section. Then for every h, k with $(h - t)(k - t - 1) = (h - 5)(k - 6) \geq 6$, there exists a non-negative rational vector $\vec{u} = \vec{u}(h, k)$ such that*

$$M_5\vec{u} = J_5.$$

As a corollary to Lemma 2.1 we obtain Theorem 2.2:

Theorem 2.2. *(Theorem 1.3) Let $t = 4$ or 5. Then for every h, k with $(h-t)(k-t-1) \geq 6$ there exists a tight* ItBD *of type t-(v, h, k, λ) for some $\lambda = \lambda(t, h, k)$.*

Proof. For $t = 5$, we apply Lemma 2.1, where λ is any positive integer making $\lambda\vec{u}$ integral. Now suppose $t = 4$, and let h, k be given with $(h - 4)(k - 5) \geq 6$. Then

$((h+1)-5)((k+1)-6) \geq 6$, so that from the foregoing and Equation (1) there exists a tight I5BD of type 5-$(v+1, h+1, k+1, \lambda)$, i.e. Equation (1) implies that the number of points in this (tight) I5BD is $v+1$. Now take the derived design through any point in the hole and apply Lemma 1.2 to obtain a tight I4BD of type 4-(v, h, k, λ). □

Now suppose that we had started this section by constructing a tight I4BD $(X', H', \mathcal{B}')$ of type 4-(v', h', k', λ) having $\mathrm{Sym}(H') \times \mathrm{Sym}(X' \setminus H')$ as an automorphism group. Let $v = v'+1$, $h = h'+1$ and $k = k'+1$. Then it is easy to see that the matrix M_4 that we would have constructed (analogously to M_5) is precisely the matrix M_5 with the first row and column removed:

$$M_4 = \begin{bmatrix} \binom{v'-h'-4}{k'-4} & \binom{h'}{1}\binom{v'-h'-4}{k'-5} & \binom{h'}{2}\binom{v'-h'-4}{k'-6} & \binom{h'}{3}\binom{v'-h'-4}{k'-7} \\ 0 & \binom{v'-h'-3}{k'-4} & \binom{h'-1}{1}\binom{v'-h'-3}{k'-5} & \binom{h'-1}{2}\binom{v'-h'-3}{k'-6} \\ 0 & 0 & \binom{v'-h'-2}{k'-4} & \binom{h'-2}{1}\binom{v'-h'-2}{k'-5} \\ 0 & 0 & 0 & \binom{v'-h'-1}{k'-4} \end{bmatrix}.$$

Hence, the solution to

$$M_4\vec{w} = J_4$$

is $\vec{w} = [w_0, w_1, w_2, w_3]^T$, where $w_i = u_{i+1}$, $i = 0, 1, 2, 3$. Thus, $(X', H', \mathcal{B}')$ is the derived design through a point in the hole in a tight I5BD $(X, H, \mathcal{B})$ of type 5-(v, h, k, λ) having $\mathrm{Sym}(H) \times \mathrm{Sym}(X \setminus H)$ as an automorphism group whenever this latter design exists; that is, when $u_0 \geq 0$ and λu_0 is integral.

Example: We construct a (tight) I4BD of type 4-$(27, 10, 6, 1560)$. Here $x = h - t = 6$ and $y = k - t - 1 = 1$; note that $v - h - 1 = 16 = (x+2)(y+1)$. By Equation (4), the solution to $M_4\vec{w} = J_4$ is $\vec{w} = [w_0, w_1, w_2, w_3]^T$, where

$$\begin{aligned} w_3 &= u_4 = \binom{(x+2)(y+1)}{y+1} = \frac{1}{120}, \\ w_2 &= u_3 = 0, \\ w_1 &= u_2 = \binom{(x+2)(y+1)-2}{y+1}^{-1} \frac{(x+1)(y+2)S_3(x,y)}{2g_1((x+2)(y+1)-1)} = \frac{1}{130}, \text{ and} \\ w_0 &= u_1 = \binom{(x+2)(y+1)-3}{y+1}^{-1} \frac{2(x+1)(y+2)S_4(x,y)}{6g_2((x+2)(y+1)-1)} = 0, \end{aligned}$$

as $S_3(x,y) = 1$ and $S_4(x,y) = xy - 6 = 0$. Thus,

$$\vec{w} = [0, \frac{1}{130}, 0, \frac{1}{120}]^T.$$

Now $\mathrm{Lcm}(130, 120) = 1560$ and

$$1560\vec{w} = [0, 12, 0, 13]^T.$$

So our I4BD $(X', H', \mathcal{B}')$ is constructed as follows. Let

$$X' = \{x_1, x_2, \ldots, x_{10}\} \cup \{1, 2, \ldots, 17\},$$

$$H' = \{x_1, x_2, \ldots, x_{10}\}$$

and take $\mathcal{B}'$ to be the

$$b = 12\binom{10}{1}\binom{17}{5} + 13\binom{10}{3}\binom{17}{3}$$

blocks that are generated by the action of $\mathrm{Sym}(H') \times \mathrm{Sym}(X' \setminus H')$ on the multiset that consists of 12 copies of $\{x_1, 1, 2, 3, 4, 5\}$ and 13 copies of $\{x_1, x_2, x_3, 1, 2, 3\}$. Thus, the blocks are the 6-element subsets that intersect the hole in 1 point each repeated 12 times, together with the 6-element subsets that intersect the hole in 3 points each repeated 13 times .

Now we construct a (tight) I5BD $(X, H, \mathcal{B})$ of type 5-$(28, 11, 7, \lambda)$. Again $x = h - t = 6$, $y = k - t - 1 = 1$ and $v - h - 1 = 16 = (x+2)(y+1)$. Then the solution to $M_5\vec{u} = J_5$ is $\vec{u} = [u_0, u_1, u_2, u_3, u_4]^T$, where u_1, u_2, u_3 and u_4 are as given above, and where by Equation (4) u_0 is given by

$$u_0 = \binom{(x+2)(y+1)-4}{y+1}^{-1} \frac{3(x+1)(y+2)S_5(x,y)}{24g_3((x+2)((y+1)-1)} = \frac{5}{572}.$$

Hence,

$$\vec{u} = \left[\frac{5}{572}, 0, \frac{1}{130}, 0, \frac{1}{120}\right]^T.$$

Now $1560\vec{u}$ is not integral. Instead $\mathrm{Lcm}(572, 120, 130) = 17160$ and

$$17160\vec{u} = [150, 0, 132, 0, 143]^T.$$

Thus, taking $\lambda = 17160$ yields an I5BD of type 5-$(28, 11, 7, 17160)$, constructed in analogous fashion to the foregoing I4BD. We take

$$\begin{aligned} X &= \{x_1, x_2, \ldots, x_{10}, x_{11}\} \cup \{1, 2, \ldots, 17\}, \\ H &= \{x_1, x_2, \ldots, x_{10}, x_{11}\} \end{aligned}$$

and take $\mathcal{B}$ to be the

$$b = 150\binom{11}{0}\binom{17}{7} + 132\binom{11}{2}\binom{17}{5} + 143\binom{11}{4}\binom{17}{3}$$

blocks that are generated by the action of $\mathrm{Sym}(H) \times \mathrm{Sym}(X \setminus H)$ on the multiset consisting of 150 copies of $\{1, 2, 3, 4, 5, 6, 7\}$, 132 copies of $\{x_1, x_{11}, 1, 2, 3, 4, 5\}$, and 143 copies of $\{x_1, x_2, x_3, x_{11}, 1, 2, 3\}$. Note that the derived design of $(X, H, \mathcal{B})$ with respect to the point x_{11} is an I4BD of type 4-$(27, 10, 6, 17160)$, which is in fact composed of 11 disjoint copies of $(X', H', \mathcal{B}')$.

More generally, suppose that M_t is the t by t analogue of M_5, i.e.

$$M_t[i, j] = |\{B \in \Gamma_j : T \subseteq B\}|,$$

where T is any fixed representative of Δ_i, $i = 0, 1, \ldots, t-1$ and $j = 0, 1, \ldots, t-1$. Then

$$M_t[i, j](v, h, k) = \binom{h-i}{j-i}\binom{v-h-(t-i)}{k-t-(j-i)}.$$

Let $1 \leq s \leq t-2$ and $i,j \geq s$. Then

$$\begin{aligned} & M_t[i,j](v,h,k) \\ & \quad = \binom{(h-s)-(i-s)}{(j-s)-(i-s)} \binom{(v-s)-(h-s)-((t-s)-(i-s))}{(k-s)-(t-s)-((j-s)-(i-s))} \\ & \quad = M_{t-s}[i-s,j-s](v-s,h-s,k-s). \end{aligned}$$

In particular, for $s=1$ we see that M_t is of the form

$$M_t(v,h,k) \;=\; \begin{bmatrix} a_0 & a_1 & \cdots & a_{t-1} \\ 0 & & & \\ \vdots & & \begin{bmatrix} M_{t-1}(v-1,h-1,k-1) \end{bmatrix} & \\ 0 & & & \end{bmatrix}, \tag{5}$$

where

$$a_j = \binom{h}{j}\binom{v-h-t}{k-t-j}, j = 0,1,\ldots,t-1.$$

We will henceforth employ the substitutions $x = h-t$, $y = k-t-1$ and $v-h-1 = (x+2)(y+1)$ introduced in Equations (2) and (3). We then obtain

$$M_t[i,j](x,y) = \binom{x+t-i}{j-i}\binom{(x+2)(y+1)-(t-1)+i}{y+1-(j-i)}.$$

Then M_t is defined recursively as in Equation (5), but now in the simpler form

$$M_t(x,y) \;=\; \begin{bmatrix} a_0 & a_1 & \cdots & a_{t-1} \\ 0 & & & \\ \vdots & & \begin{bmatrix} M_{t-1}(x,y) \end{bmatrix} & \\ 0 & & & \end{bmatrix}, \tag{6}$$

where

$$a_j = \binom{x+t}{j}\binom{(x+2)(y+1)-(t-1)}{y+1-j}, j = 0,1,\ldots,t-1.$$

Consider the following example.

Lemma 2.3. *(Theorem 1.4) Let $t=6$. Then for every h,k with*

$$(h-t)(k-t-1) \geq 9$$

there exists a tight ItBD *of type t-(v,h,k,λ) for some $\lambda = \lambda(t,h,k)$.*

Proof. From the foregoing discussion we have

$$M_6 \;=\; \begin{bmatrix} a_0 & a_1 & \cdots & a_5 \\ 0 & & & \\ \vdots & & \begin{bmatrix} M_5 \end{bmatrix} & \\ 0 & & & \end{bmatrix},$$

where M_5 is as given at the beginning of this section, i.e.

$$M_5[i,j] \;=\; \binom{x+5-i}{j-i}\binom{(x+2)(y+1)-4+i}{y+1-(j-i)}, 0 \leq i,j \leq 4$$

and

$$a_j = \binom{x+6}{j}\binom{(x+2)(y+1)-5}{y+1-j}, 0 \le j \le 5.$$

Now, we have already established that the unique solution

$$\vec{u} = [u_0, u_1, u_2, u_3, u_4]^T$$

to the matrix equation $M_5\vec{u} = J_5$ has all $u_i \ge 0$ whenever

$$((h-1)-(t-1))((k-1)-(t-1)-1) = (h-t)(k-t-1) = xy \ge 6.$$

(See the discussion preceding Lemma 2.1.) Hence, the unique solution to $M_6\vec{w} = J_6$ is given by the vector $\vec{w} = [w_0, w_1, w_2, w_3, w_4, w_5]^T$ where $w_j = u_{j-1}$ for $j = 1, 2, 3, 4, 5$ and where w_0 is the solution to

$$\sum_{j=0}^{5} a_j w_j = a_0 w_0 + \sum_{j=1}^{5} a_j u_{j-1} = 1.$$

This yields (from Equation (4))

$$\begin{aligned}
&\binom{(x+2)(y+1)-5}{y+1} w_0 + (x+6)\binom{(x+2)(y+1)-5}{y} \\
&\quad \cdot \binom{(x+2)(y+1)-4}{y+1}^{-1} \frac{(x+1)(y+2)S_5(x,y)}{8g_3((x+2)(y+1)-1)} \\
&+ \binom{x+6}{2}\binom{(x+2)(y+1)-5}{y-1} \\
&\quad \cdot \binom{(x+2)(y+1)-3}{y+1}^{-1} \frac{(x+1)(y+2)S_4(x,y)}{3g_2((x+2)(y+1)-1)} \\
&+ \binom{x+6}{3}\binom{(x+2)(y+1)-5}{y-2} \\
&\quad \cdot \binom{(x+2)(y+1)-2}{y+1}^{-1} \frac{(x+1)(y+2)S_3(x,y)}{2g_1((x+2)(y+1)-1)} \\
&+ \binom{x+6}{4}\binom{(x+2)(y+1)-5}{y-3} \cdot 0 \\
&+ \binom{x+6}{5}\binom{(x+2)(y+1)-5}{y-4}\binom{(x+2)(y+1)}{y+1}^{-1} \\
&= 1.
\end{aligned}$$

We get

$$w_0 = \binom{(x+2)(y+1)-5}{y+1}^{-1} \left\{ 1 - \frac{(x+6)(y+1)(x+1)(y+2)S_5(x,y)}{8g_4((x+2)(y+1)-1)} \right.$$
$$- \frac{(x+6)(x+5)(y+1)y(x+1)(y+2)S_4(x,y)}{6g_4((x+2)(y+1)-1)}$$
$$- \frac{(x+6)(x+5)(x+4)(y+1)y(y-1)(x+1)(y+2)S_3(x,y)}{12g_4((x+2)(y+1)-1)}$$
$$\left. - \frac{(x+6)(x+5)(x+4)(x+3)(x+2)(y+1)y(y-1)(y-2)(y-3)}{120(x+2)(y+1)g_4((x+2)(y+1)-1)} \right\},$$

where $S_3(x,y) = 1$, $S_4(x,y) = xy - 6$, and $S_5(x,y) = 3x^2y^2 + 5xy^2 + 4y^2 + 5x^2y - 5xy + 12y + 4x^2 + 12x + 80$. This yields

$$w_0 = \binom{(x+2)(y+1)-5}{y+1}^{-1} \frac{(x+1)(y+2)S_6(x,y)}{30g_4((x+2)(y+1)-1)}, \tag{7}$$

where

$$S_6(x,y) = 11x^3y^3 + 27x^2y^3 + 22xy^3 + 27x^3y^2 - 51x^2y^2 - 156xy^2$$
$$-180y^2 + 22x^3y - 156x^2y - 106xy - 540y - 180x^2 - 540x - 1800.$$

Here, $S_6(x,y)$ is a symmetric polynomial in x and y of degree 3. So we write

$$\begin{aligned} S_6(x,y) &= (11(xy)^3 - 51(xy)^2 - 106(xy) - 1800) \\ &\quad + (x+y)(27(xy)^2 - 156(xy) - 540) \\ &\quad + (x^2+y^2)(22(xy) - 180) + (x^3+y^3)\cdot 0, \end{aligned}$$

from which it is easily deduced that $S_6(x,y) \geq 0$ for all $x, y \geq 0$ with $xy \geq 9$. Furthermore, the condition $xy \geq 9$ guarantees that

$$\binom{(x+2)(y+1)-5}{y+1} > 0$$

because $(x+1)(y+1) > 5$. Hence, $\vec{w}$ is a non-negative rational vector when $xy = (h-t)(k-t-1) \geq 9$. Taking λ to be any positive integer making $\lambda\vec{w}$ integral establishes the result. □

Remark 1. No multiple of the tight I4BD of type 4-$(27, 10, 6, 1560)$ that was constructed following Theorem 2.2 can be extended to a tight I6BD $(X, H, \mathcal{B})$ of type 6-$(29, 12, 8, \lambda)$ for any $\lambda > 0$ having $\mathrm{Sym}(H) \times \mathrm{Sym}(X \setminus H)$ as an automorphism group. This is because here $x = h - t = 6$ and $y = k - t - 1 = 1$, whereupon $S_6(x,y) = -7200 < 0$. Thus from Equation (7) $w_0 < 0$, i.e. no such I6BD exists.

Note that w_0 (in Equation (7)) has the same form as u_0, u_1 and u_2 (from Equation (4)) where here $m = 6$. We make the following conjecture.

Conjecture 1. *Let $t \geq 4$, and let $\vec{u} = [u_0, u_1, \ldots, u_{t-1}]^T$ be the unique solution to the matrix equation*

$$M_t(x,y)\vec{u} = J_t,$$

where $M_t(x,y)$ is as defined in Equation (6), and where we assume

$$a_0 = \binom{(x+2)(y+1)-(t-1)}{y+1} > 0$$

(i.e. $(x+1)(y+1) \geq t-1$). Then $\vec{u}$ is given by

$$\begin{aligned} u_{t-1} &= \binom{(x+2)(y+1)}{y+1}^{-1}, \\ u_{t-2} &= 0, \end{aligned}$$

and for each each $m = 3, 4, \ldots, t$,

$$u_{t-m} = \binom{(x+2)(y+1)-(m-1)}{y+1}^{-1} \frac{(m-2)(x+1)(y+2)S_m(x,y)}{(m-1)!g_{m-2}((x+2)(y+1)-1)},$$

where $S_m(x,y)$ is a symmetric polynomial in x and y of degree $m-3$ with integer coefficients. □

Remark 2. Proving Conjecture 2. would be of considerable value, particularly if we could at the same time determine the coefficients of $S_m(x,y)$. This is because $u_{t-m} \geq 0$ if and only if $S_m(x,y) \geq 0$; now if $S_m(x,y)$ is symmetric there exists a sequence of polynomials

$$p_0, p_1, \ldots, p_{m-3},$$

where p_i is a polynomial of degree i in the single variable (xy), viz:

$$\begin{aligned} S_m(x,y) = p_{m-3}(xy) + (x+y)p_{m-4}(xy) + (x^2+y^2)p_{m-5}(xy) \\ + \cdots + (x^{m-4}+y^{m-4})p_1(xy) + (x^{m-3}+y^{m-3})p_0(xy). \end{aligned}$$

Moreover, the p_i are uniquely determined. Now for our purposes $x, y \geq 0$ and so determining conditions under which $S_m(x,y) \geq 0$ can be simplified to determining conditions under which each $p_i(xy) \geq 0$. In particular, what we might seek is the smallest non-negative integer P_m such that whenever $xy \geq P_m$ we have $p_i(xy) \geq 0$ for all $i = 0, 1, \ldots, m-3$, assuming of course that such an integer exists. (Thus for example we have shown that $P_3 = 0$, $P_4 = 6$, $P_5 = 0$ and $P_6 = 9$.) In this event we have $S_m(x,y) \geq 0$ whenever $xy \geq P_m$. Then, taking

$$\mathcal{P}_t = \max\{P_m : 3 \leq m \leq t\}, \tag{8}$$

we see that the solution vector $\vec{u}$ to $M_t\vec{u} = J_t$ would be non-negative whenever $xy = (h-t)(k-t-1) \geq \mathcal{P}_t$. This yields a tight I$t$BD of type t-(v,h,k,λ) where $v-h-1 = (x+2)(y+1)$ and λ is any positive integer making $\lambda\vec{u}$ integral. (Thus, for example, $\mathcal{P}_4 = \mathcal{P}_5 = 6$ and $\mathcal{P}_6 = 9$.)

We can go part of the way towards proving Conjecture 2., as follows.

Theorem 2.4. *Let $t \geq 4$. Then, under the hypothesis of Conjecture 2. we have $\vec{u} = [u_0, u_1, \ldots, u_{t-1}]^T$, where $u_{t-1} = \binom{(x+2)(y+1)}{y+1}^{-1}$, $u_{t-2} = 0$, and for each $m = 3, 4, \ldots, t$,*

$$u_{t-m} = \binom{(x+2)(y+1)-(m-1)}{y+1}^{-1} \frac{(x+1)(y+2)R_m(x,y)}{(m-1)!g_{m-2}((x+2)(y+1)-1)},$$

where $R_m(x,y)$ is a polynomial in x and y of degree $m-3$ with integer coefficients.

Proof. We use induction on t, the case $t=4$ already having been settled, where $R_3(x,y)=1$ and $R_4(x,y)=2(xy-6)$. Now we follow the method of Lemma 2.3. Let $t \geq 5$; from Equation (6) we write

$$M_t = \begin{bmatrix} a_0 & a_1 & \cdots & a_{t-1} \\ 0 & & & \\ \vdots & & \begin{bmatrix} M_{t-1} \end{bmatrix} & \\ 0 & & & \end{bmatrix},$$

where

$$a_j = \binom{x+t}{j}\binom{(x+2)(y+1)-(t-1)}{y+1-j}, j=0,1,\dots,t-1.$$

Let $\vec{w}=[w_0,w_1,\dots,w_{t-2}]^T$ be the solution to $M_{t-1}\vec{w}=J_{t-1}$. Then the solution to $M_t\vec{u}=J_t$ is given by the vector $\vec{u}=[u_0,u_1,\dots,u_{t-1}]^T$, where $u_j=w_{j-1}$ for $j=1,2,\dots,t-1$, and where u_0 is the solution to

$$\sum_{j=0}^{t-1} a_j u_j = a_0 u_0 + \sum_{j=1}^{t-1} a_j w_{j-1} = 1.$$

By induction we have:

$$\begin{aligned}
&\binom{(x+2)(y+1)-(t-1)}{y+1}u_0 + \sum_{j=1}^{t-3}\binom{x+t}{j}\binom{(x+2)(y+1)-(t-1)}{y+1-j} \\
&\cdot \binom{(x+2)(y+1)-(t-j-1)}{y+1}^{-1}\frac{(x+1)(y+2)R_{t-j}(x,y)}{(t-j-1)!g_{t-j-2}((x+2)(y+1)-1)} \\
&+\binom{x+t}{t-1}\binom{(x+2)(y+1)-(t-1)}{y+1-(t-1)}\binom{(x+2)(y+1)}{y+1}^{-1} = 1.
\end{aligned}$$

This yields

$$\begin{aligned}
&\binom{(x+2)(y+1)-(t-1)}{y+1}u_0 \\
&\quad + \frac{(x+1)(y+2)}{g_{t-2}((x+2)(y+1)-1)}\sum_{j=1}^{t-3}\frac{g_j(x+t)g_j(y+1)R_{t-j}(x,y)}{j!(t-j-1)!} \\
&\quad = 1 - \frac{g_{t-1}(x+t)g_{t-1}(y+1)}{(t-1)!g_{t-1}((x+2)(y+1))} \\
&\quad = 1 - \frac{g_{t-2}(x+t)g_{t-2}(y)}{(t-1)!g_{t-2}((x+2)(y+1)-1)} \\
&\quad = \frac{(t-1)!g_{t-2}((x+2)(y+1)-1) - g_{t-2}(x+t)g_{t-2}(y)}{(t-1)!g_{t-2}((x+2)(y+1)-1)}.
\end{aligned}$$

Let

$$G_t(x,y) = (t-1)!g_{t-2}((x+2)(y+1)-1) - g_{t-2}(x+t)g_{t-2}(y). \tag{9}$$

Then $G_t(x,y)$ is a polynomial in x and y of degree $t-2$. Now note that

$$G_t(-1,y) = (t-1)!g_{t-2}(y) - g_{t-2}(t-1)g_{t-2}(y) = 0$$

and

$$\begin{aligned} G_t(x,-2) &= (t-1)!g_{t-2}(-x-3) - g_{t-2}(x+t)g_{t-2}(-2) \\ &= (-1)^{t-2}(t-1)!g_{t-2}(x+t) - (-1)^{t-2}(t-1)!g_{t-2}(x+t) = 0. \end{aligned}$$

Consequently $(x+1)(y+2)$ is a factor of $G_t(x,y)$. Thus the above equation for u_0 becomes

$$\begin{aligned} &\binom{(x+2)(y+1)-(t-1)}{y+1}u_0 \\ &\quad = \frac{(x+1)(y+2)}{(t-1)!g_{t-2}((x+2)(y+1)-1)} \\ &\qquad \cdot \left\{ \frac{G_t(x,y)}{(x+1)(y+2)} - \sum_{j=1}^{t-3}\binom{t-1}{j}g_j(x+t)g_j(y+1)R_{t-j}(x,y) \right\} \end{aligned}$$

so that

$$u_0 = \binom{(x+2)(y+1)-(t-1)}{y+1}^{-1} \frac{(x+1)(y+2)R_t(x,y)}{(t-1)!g_{t-2}((x+2)(y+1)-1)},$$

where

$$R_t(x,y) = \tfrac{G_t(x,y)}{(x+1)(y+2)} - \textstyle\sum_{j=1}^{t-3}\binom{t-1}{j}g_j(x+t)g_j(y+1)R_{t-j}(x,y). \tag{10}$$

Now by induction each $R_{t-j}(x,y)$ is a polynomial in x and y of degree $t-j-3$ with integer coefficients, and clearly $G_t(x,y)/(x+1)(y+2)$ is a polynomial in x and y of degree $t-3$ with integer coefficients. Therefore $R_t(x,y)$, is a polynomial in x and y of degree (at most) $t-3$ with integer coefficients. We will see in Section 3 (Equation (12)) that u_0 has the alternate formulation

$$\begin{aligned} u_0 = &\binom{(x+2)(y+1)-(t-1)}{y+1}^{-1} \\ &\cdot \sum_{j=0}^{t-1}\frac{(-1)^j}{j!}\cdot\frac{g_j(x+t)g_j(y+j)}{g_j((x+2)(y+1)-(t-1)+j)}, \end{aligned}$$

from which it follows that the leading coefficient r_t (i.e. that of $x^{t-3}y^{t-3}$) in $R_t(x,y)$ is given by

$$r_t = (t-1)!\sum_{j=0}^{t-1}\frac{(-1)^j}{j!} > 0.$$

Hence, $R_t(x,y)$ is in fact of degree $t-3$. This completes the proof. $\square$

m	$S_m(x,y)$	P_m
3	1	0
4	$(xy-6)+(x+y)\cdot 0$	6
5	$(3(xy)^2-5(xy)+80)+(x+y)(5(xy)+12)+(x^2+y^2)\cdot 4$	0
6	$(11(xy)^3-51(xy)^2-106(xy)-1800)+(x+y)(27(xy)^2-156(xy)-540)$ $+(x^2+y^2)(22(xy)-180)+(x^3+y^3)\cdot 0$	9
7	$(53(xy)^4-132(xy)^3+1805(xy)^2+11604(xy)+60480)$ $+(x+y)(186(xy)^3-978(xy)^2+7170(xy)+26352)$ $+(x^2+y^2)(259(xy)^2-804(xy)+9432)$ $+(x^3+y^3)(174(xy)+432)+(x^4+y^4)\cdot 72$	0
8	$(309(xy)^5-250(xy)^4-7035(xy)^3-131410(xy)^2-942296(xy)-2822400)$ $+(x+y)(1390(xy)^4-11410(xy)^3+28210(xy)^2-472020(xy)-1572480)$ $+(x^2+y^2)(2555(xy)^3-21830(xy)^2+10360(xy)-614880)$ $+(x^3+y^3)(2330(xy)^2-18300(xy)-60480)$ $+(x^4+y^4)(976(xy)-10080)+(x^5+y^5)\cdot 0$	11
9	$(2119(xy)^6+4791(xy)^5-166145(xy)^4+1179255(xy)^3$ $+13574524(xy)^2+83964096(xy)+174182400)$ $+(x+y)(11655(xy)^5-117525(xy)^4+664245(xy)^3$ $+215850(xy)^2+40589040(xy)+116052480)$ $+(x^2+y^2)(27115(xy)^4-302295(xy)^3+1809790(xy)^2+2701800(xy)+49518720)$ $+(x^3+y^3)(33945(xy)^3-307050(xy)^2+1900920(xy)+7300800)$ $+(x^4+y^4)(24406(xy)^2-119016(xy)+1281600)$ $+(x^5+y^5)(10200(xy)+25920)+(x^6+y^6)\cdot 2880$	0

TABLE I. $S_m(x,y)$ **and** P_m **for** $3 \le m \le 9$

Remark 3. From Theorem 2.4, Conjecture 2. is equivalent to the assertion that for each $m \ge 3$ $R_m(x,y)$ is symmetric and all of its coefficients are multiples of $m-2$. Note that for the leading coefficient r_m in $R_m(x,y)$ we do in fact have

$$r_m \equiv (-1)^{m-2}(m-1)+(-1)^{m-1} \equiv 0 \pmod{m-2}.$$

We used the algorithm in Theorem 2.4, generating each $R_m(x,y)$ from its predecessors by the recurrence relation given in Equation (10) (where $G_m(x,y)$ is given by Equation (9), and of course $R_3(x,y)=1$), and so validated Conjecture 2. for all $4 \le t \le 9$, in each case demonstrating the existence of P_m and so determining $\mathcal{P}_t$. We used Mathematica to do the symbolic manipulations; we give our data in Table I.

The following summarizes the results of this section.

Theorem 2.5. *Let $4 \le t \le 9$. Then for every h,k with $(h-t)(k-t-1) \ge \mathcal{P}_t$, there exists a tight* ItBD $(X,H,\mathcal{B})$ *of type t-(v,h,k,λ) for some $\lambda = \lambda(t,h,k)$, having* $\mathrm{Sym}(H)\times\mathrm{Sym}(X\setminus H)$ *as an automorphism group, where $\mathcal{P}_4=\mathcal{P}_5=6$, $\mathcal{P}_6=\mathcal{P}_7=9$ and $\mathcal{P}_8=\mathcal{P}_9=11$.*

Proof. We need only note that $(x+1)(y+1) = (h-t+1)(k-t) > \mathcal{P}_t \geq t-1$, as required. $\square$

It is of interest to note that for each even m in Table I, $S_m(x,y) < 0$ when $xy = P_m - 1$. Thus P_m is in fact the smallest positive integer such that whenever $xy \geq P_m$ we have $S_m(x,y) \geq 0$. It is also of interest to note that for each odd m in Table I we have $S_m(x,y) \geq 0$ for all $x, y \geq 0$.

We conclude this section by posing the following companion to Conjecture 2..

Conjecture 2. The quantity $\mathcal{P}_t$ given by Equation (8) is well defined for all $t \geq 4$. Moreover, $\mathcal{P}_4 = \mathcal{P}_5 = 6$ and for $t \geq 6$,

$$\mathcal{P}_t = \begin{cases} t+3 & \textit{if } t \textit{ is even, and} \\ t+2 & \textit{if } t \textit{ is odd.} \end{cases}$$

$\square$

3. ASYMPTOTIC EXISTENCE

In this section we prove Theorem 1.5. We begin by determining an expression for u_0 in the matrix equation $M_t\vec{u} = J_t$ alternative to that proposed by Conjecture 2.. To do this we calculate the entries in the first row of M_t^{-1}. We will need the following preliminary result.

Lemma 3.1. *For any $y \geq 0$ and $j \geq 1$,*

$$\sum_{r=0}^{j} (-1)^r \binom{y+r}{r} \binom{y+1}{j-r} = 0.$$

Proof. We have

$$(1+x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k$$

and

$$(1-x)^{-n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k,$$

for any $n \geq 1$. Hence,

$$(1+x)^{y+1} = \sum_{k=0}^{\infty} \binom{y+1}{k} x^k$$

and

$$(1+x)^{-(y+1)} = \sum_{k=0}^{\infty} (-1)^k \binom{y+k}{k} x^k.$$

Therefore,

$$\sum_{j=0}^{\infty} (\sum_{r=0}^{j} (-1)^r \binom{y+r}{r} \binom{y+1}{j-r}) x^j = (1+x)^{-(y+1)}(1+x)^{y+1} = 1,$$

and the result follows. ☐

Theorem 3.2. *Let M_t be the t by t matrix whose $[i,j]$-th entry is*

$$M_t[i,j] = \binom{x+t-i}{j-i}\binom{(x+2)(y+1)-(t-1)+i}{y+1-(j-i)}$$

for $0 \le i,j \le t-1$, where $(x+1)(y+1) \ge t-1$. Then M_t is invertible and

$$M_t^{-1}[0,j] = \binom{(x+2)(y+1)-(t-1)}{y+1}^{-1} \cdot (-1)^j \binom{x+t}{j}\binom{y+j}{j}\binom{(x+2)(y+1)-(t-1)+j}{j}^{-1}$$

for $j = 0,1,\ldots,t-1$.

Proof. We proceed by induction on j, the case $j=0$ being obvious, i.e.

$$M_t^{-1}[0,0] = \binom{(x+2)(y+1)-(t-1)}{y+1}^{-1}.$$

Now let $1 \le j \le t-1$. Then the dot product of row 0 of M_t^{-1} with column j of M_t is 0, i.e.

$$\sum_{r=0}^{j} M_t^{-1}[0,r]M_t[r,j] = 0.$$

(Note that $M_t[r,j] = 0$ when $r > j$.) Let $a = \binom{(x+2)(y+1)-(t-1)}{y+1}^{-1}$. By induction we have

$$a\sum_{r=0}^{j-1}(-1)^r\binom{x+t}{r}\binom{y+r}{r}\binom{(x+2)(y+1)-(t-1)+r}{r}^{-1} \cdot \binom{x+t-r}{j-r}\binom{(x+2)(y+1)-(t-1)+r}{(y+1)-(j-r)} + M_t^{-1}[0,j]\binom{x+t-j}{j-j}\binom{(x+2)(y+1)-(t-1)+j}{y+1-(j-j)} = 0.$$

Multiplying both sides of the above equation by

$$\binom{(x+2)(y+1)-(t-1)}{y+1-j}^{-1},$$

we get

$$a\sum_{r=0}^{j-1}(-1)^r\binom{x+t}{r}\binom{y+r}{r}\binom{(x+2)(y+1)-(t-1)+r}{r}^{-1}$$
$$\cdot\binom{x+t-r}{j-r}\frac{g_r((x+2)(y+1)-(t-1)+r)}{g_r(y+1-(j-r))}$$
$$+M_t^{-1}[0,j]\frac{g_j((x+2)(y+1)-(t-1)+j)}{g_j(y+1)}=0.$$

Now

$$\binom{x+t}{r}\binom{x+t-r}{j-r}=\binom{x+t}{j}\binom{j}{r}$$

and

$$\binom{(x+2)(y+1)-(t-1)+r}{r}^{-1}\frac{g_r((x+2)(y+1)-(t-1)+r)}{g_r(y+1-(j-r))}$$
$$=\binom{y+1-(j-r)}{r}^{-1},$$

and so we have

$$a\binom{x+t}{j}\sum_{r=0}^{j-1}(-1)^r\binom{j}{r}\binom{y+r}{r}\binom{y+1-(j-r)}{r}^{-1}$$
$$+M_t^{-1}[0,j]\frac{g_j((x+2)(y+1)-(t-1)+j)}{g_j(y+1)}=0.$$

Thus,

$$M_t^{-1}[0,j]=-a\binom{x+t}{j}\binom{(x+2)(y+1)-(t-1)+j}{j}^{-1}$$
$$\cdot\sum_{r=0}^{j-1}(-1)^r\binom{j}{r}\binom{y+r}{r}\binom{y+1-(j-r)}{r}^{-1}\binom{y+1}{j}.$$

Now

$$\frac{g_j(y+1)}{g_r((y+1)-(j-r))}=g_{j-r}(y+1),$$

so that

$$\begin{aligned} M_t^{-1}[0,j] &= -a\binom{x+t}{j}\binom{(x+2)(y+1)-(t-1)+j}{j}^{-1}\sum_{r=0}^{j-1}(-1)^r\binom{y+r}{r}\binom{y+1}{j-r} \\ &= -a\binom{x+t}{j}\binom{(x+2)(y+1)-(t-1)+j}{j}^{-1}\left\{\sum_{r=0}^{j}(-1)^r\binom{y+r}{r}\binom{y+1}{j-r}\right. \\ &\qquad \left. -(-1)^j\binom{y+j}{j}\right\} \\ &= (-1)^j a\binom{x+t}{j}\binom{y+j}{j}\binom{(x+2)(y+1)-(t-1)+j}{j}^{-1} \\ &= (-1)^j\binom{x+t}{j}\binom{y+j}{j}\binom{(x+2)(y+1)-(t-1)+j}{j}^{-1}\binom{(x+2)(y+1)-(t-1)}{y+1}^{-1}, \end{aligned}$$

by Lemma 3.1. □

From the recursive nature of M_t (see Equation (6)), we have that

$$M_t^{-1} = \begin{bmatrix} b_0 & b_1 \ \cdots \ b_{t-1} \\ 0 & \\ \vdots & \begin{bmatrix} \ M_{t-1}^{-1} \ \end{bmatrix} \\ 0 & \end{bmatrix}, \tag{11}$$

where

$$b_j = \binom{(x+2)(y+1)-(t-1)}{y+1}^{-1} \cdot(-1)^j\binom{x+t}{j}\binom{y+j}{j}\binom{(x+2)(y+1)-(t-1)+j}{j}^{-1}$$

for $j = 0, 1, \ldots, t-1$. Hence, if $M_t\vec{u} = J_t$, where $\vec{u} = [u_0, u_1, \ldots, u_{t-1}]^T$ we have

$$u_0 = \binom{(x+2)(y+1)-(t-1)}{y+1}^{-1}\sum_{j=0}^{t-1}\frac{(-1)^j g_j(x+t)g_j(y+j)}{j!g_j((x+2)(y+1)-(t-1)+j)}. \tag{12}$$

Remark 4. We note that by Equation (12), Conjecture 2. is equivalent to the assertion that for each $t \geq 4$,

$$\sum_{j=0}^{t-1}\frac{(-1)^j g_j(x+t)g_j(y+j)}{j!g_j((x+2)(y+1)-(t-1)+j)} = \frac{(t-2)(x+1)(y+2)S_t(x,y)}{(t-1)!g_{t-2}((x+2)(y+1)-1)},$$

where $S_t(x,y)$ is a symmetric polynomial in x and y of degree $t-3$ with integer coefficients.

We will want conditions under which $u_0 \geq 0$, where u_0 is as given in Equation (12). Equivalently, we want to ensure that

$$\sum_{j=0}^{t-1}\frac{(-1)^j g_j(x+t)g_j(y+j)}{j!g_j((x+2)(y+1)-(t-1)+j)} \geq 0.$$

To this end, we will prove the following three results (Lemmas 3.3, 3.4 and 3.6).

Lemma 3.3. *Let $t \geq 4$. Then there exist positive integers N_1 and N_2 such that whenever $x \geq N_1$ and $y \geq N_2$, $u_0 \geq 0$.*

Proof. Note that each of $g_j(x+t)g_j(y+j)$ and $g_j((x+2)(y+1)-(t-1)+j)$ is a polynomial in x and y of degree j with leading coefficient 1, and so

$$\lim_{x,y\to\infty} \sum_{j=0}^{t-1} \frac{(-1)^j g_j(x+t)g_j(y+j)}{j!g_j((x+2)(y+1)-(t-1)+j)} = \sum_{j=0}^{t-1} \frac{(-1)^j}{j!} > 0,$$

as required. □

Lemma 3.4. *Let $t \geq 4$ and let $y = c \geq 1$. Then there exists $N(y) > 0$ such that whenever $x \geq N(y)$, $u_0 \geq 0$.*

Proof. Here $g_j(x+t)g_j(y+j) = g_j(x+t)g_j(c+j)$ is a polynomial in x of degree j with leading coefficient $g_j(c+j)$, while $g_j((x+2)(y+1)-(t-1)+j) = g_j((x+2)(c+1)-(t-1)+j)$ is a polynomial in x of degree j with leading coefficient $(c+1)^j$. Hence,

$$\lim_{x\to\infty} \sum_{j=0}^{t-1} \frac{(-1)^j g_j(x+t)g_j(y+j)}{j!g_j((x+2)(y+1)-(t-1)+j)}$$
$$= \sum_{j=0}^{t-1} \frac{(-1)^j g_j(c+j)}{j!(c+1)^j} = \sum_{j=0}^{t-1} \binom{c+j}{j}\left(\frac{-1}{c+1}\right)^j.$$

To see that this sum is positive for $t \geq 4$, we consider the ratios of successive pairs of terms, i.e. take the ratio of the $j = 2\alpha$ term with the absolute value of the $j = 2\alpha+1$ term, $\alpha = 0, 1, \ldots, \lfloor \frac{t-2}{2} \rfloor$. (Note that if t is odd this comparison does not include the last ($j = t-1$) term, which is fine as this term is positive.) We get

$$\begin{aligned}
\frac{\binom{c+2\alpha}{2\alpha}\left(\frac{1}{c+1}\right)^{2\alpha}}{\binom{c+2\alpha+1}{2\alpha+1}\left(\frac{1}{c+1}\right)^{2\alpha+1}} &= \frac{g_{2\alpha}(c+2\alpha)}{(2\alpha)!(c+1)^{2\alpha}} \cdot \frac{(2\alpha+1)!(c+1)^{2\alpha+1}}{g_{2\alpha+1}(c+2\alpha+1)} \\
&= \frac{(2\alpha+1)(c+1)}{c+2\alpha+1} \\
&= 1 + \frac{2\alpha c}{c+2\alpha+1},
\end{aligned}$$

which equals 1 when $\alpha = 0$ and because $c \geq 1$, is greater than 1 when $\alpha \geq 1$. Hence the sum, and so the limit, is > 0 as required. □

Remark 5. Note that the proof of Lemma 3.4 does not work for $y = c = 0$, because in this case

$$\sum_{j=0}^{t-1} \binom{c+j}{j}\left(\frac{-1}{c+1}\right)^j = \sum_{j=0}^{t-1}(-1)^j = \begin{cases} 0 & \text{when } t \text{ is even;} \\ 1 & \text{when } t \text{ is odd.} \end{cases}$$

This is consistent with the fact that for $t \geq 4$ there does not exist a tight ItBD of type t-(v,h,k,λ) for any h, λ when $k = t+1$ (i.e. $y = k-t-1 = 0$); see [3].

Before proceeding to our third limit result, we will require the following lemma.

Lemma 3.5. *For each $c \geq 1$ and each $t \geq 3$,*

$$(c+1)^t > \left(\frac{c+2}{c+1}\right)^c \binom{t+c}{c}.$$

Proof.

$$\begin{aligned}
&\left(\frac{c+2}{c+1}\right)^c \binom{t+c}{c} \\
&\quad = \left(\frac{c+2}{c+1}\right)^{c+1} \left(\frac{c+1}{c+2}\right) \binom{c+t}{t} \\
&\quad = \left(1+\frac{1}{c+1}\right)^{c+1} \left(\frac{c+1}{c+2}\right) \\
&\qquad \cdot \left(\frac{c+t}{t}\right)\left(\frac{c+t-1}{t-1}\right)\cdots\left(\frac{c+3}{3}\right)\left(\frac{c+2}{2}\right)\left(\frac{c+1}{1}\right) \\
&\quad < e\left(\frac{c+t}{t}\right)\left(\frac{c+t-1}{t-1}\right)\cdots\left(\frac{c+3}{3}\right)\left(\frac{c+1}{2}\right)\left(\frac{c+1}{1}\right) \\
&\quad < \left(\frac{c+t}{t}\right)\left(\frac{c+t-1}{t-1}\right)\cdots\left(\frac{c+4}{4}\right)\left(\frac{c+3}{2}\right)\left(\frac{c+1}{1}\right)\left(\frac{c+1}{1}\right), \\
&\qquad \text{because } e < 3 \text{ and } t \geq 3, \\
&\quad \leq (c+1)^t \\
&\qquad \text{because } \frac{c+t-i}{t-i} \leq c+1 \text{ and } \frac{c+3}{2} \leq c+1 \text{ when } c, (t-i) \geq 1.
\end{aligned}$$

□

Lemma 3.6. *Let $t \geq 4$ and let $x = c \geq 1$. Then there exists $N(x) > 0$ such that whenever $y \geq N(x)$, $u_0 \geq 0$.*

Proof. In this case, $g_j(x+t)g_j(y+j) = g_j(c+t)g_j(y+j)$ is a polynomial in y of degree j with leading coefficient $g_j(c+t)$, while $g_j((x+2)(y+1)-(t-1)+j) = g_j((c+2)(y+1)-(t-1)+j)$ is a polynomial in y of degree j with leading coefficient $(c+2)^j$. Hence,

$$\begin{aligned}
&\lim_{y\to\infty} \sum_{j=0}^{t-1} \frac{(-1)^j g_j(x+t)g_j(y+j)}{j!g_j((x+2)(y+1)-(t-1)+j)} \\
&\qquad = \sum_{j=0}^{t-1} \frac{(-1)^j g_j(c+t)}{j!(c+2)^j} = \sum_{j=0}^{t-1} \binom{c+t}{j}\left(\frac{-1}{c+2}\right)^j.
\end{aligned}$$

Now

$$\begin{aligned}
\sum_{j=0}^{t-1}\binom{c+t}{j}\left(\frac{-1}{c+2}\right)^j &= \sum_{j=0}^{t+c}\binom{t+c}{j}\left(\frac{-1}{c+2}\right)^j - \sum_{j=t}^{t+c}\binom{t+c}{j}\left(\frac{-1}{c+2}\right)^j \\
&= \left(1-\frac{1}{c+2}\right)^{t+c} - \sum_{j=t}^{t+c}\binom{t+c}{j}\left(\frac{-1}{c+2}\right)^j \\
&\qquad \text{(by the binomial theorem)} \\
&= \left(\frac{c+1}{c+2}\right)^{t+c} - \binom{t+c}{c}\left(\frac{-1}{c+2}\right)^t - \binom{t+c}{c-1}\left(\frac{-1}{c+2}\right)^{t+1} \\
&\qquad - \cdots - \binom{t+c}{1}\left(\frac{-1}{c+2}\right)^{t+c-1} - \left(\frac{-1}{c+2}\right)^{t+c} \\
&= \left(\frac{1}{c+2}\right)^{t+c}\Bigg\{(c+1)^{t+c} - (-1)^t\binom{t+c}{c}(c+2)^c \\
&\qquad - (-1)^{t+1}\binom{t+c}{c-1}(c+2)^{c-1} \\
&\qquad - \cdots - (-1)^{t+c-1}\binom{t+c}{1}(c+2) - (-1)^{t+c}\Bigg\}. \qquad (13)
\end{aligned}$$

We consider two cases.

(i) t **is odd.**

Consider the ratios of successive pairs of terms in the sum

$$\begin{aligned}
&\binom{t+c}{c}(c+2)^c - \binom{t+c}{c-1}(c+2)^{c-1} + \cdots \\
&\qquad +(-1)^{c-1}\binom{t+c}{1}(c+2) + (-1)^c \\
&\qquad\qquad = \sum_{k=0}^{c}(-1)^k\binom{t+c}{c-k}(c+2)^{c-k},
\end{aligned}$$

i.e. we take the ratio of the $k = 2\alpha$ term with the absolute value of the $k = 2\alpha + 1$ term, $\alpha = 0, 1, \ldots, \lfloor\frac{c-1}{2}\rfloor$. (Note that if c is even this comparison does not include the last ($k = c$) term, which is fine as this term is $1 > 0$.) We get

$$\begin{aligned}
\frac{\binom{t+c}{c-2\alpha}(c+2)^{c-2\alpha}}{\binom{t+c}{c-2\alpha-1}(c+2)^{c-2\alpha-1}} &= \frac{g_{c-2\alpha}(t+c)\cdot(c+2)^{c-2\alpha}(c-2\alpha-1)!}{(c-2\alpha)!g_{c-2\alpha-1}(t+c)\cdot(c+2)^{c-2\alpha-1}} \\
&= \frac{(c+2)(t+2\alpha+1)}{c-2\alpha} \\
&> t+2\alpha+1 \\
&> 1.
\end{aligned}$$

Hence the sum, and so the limit, is > 0 as required.

(ii) t **is even.**

By Lemma 3.5 we have

$$(c+1)^t > \left(\frac{c+2}{c+1}\right)^c \binom{t+c}{c},$$

because $c \geq 1$ and $t \geq 4$. Hence,

$$(c+1)^{t+c} > (c+2)^c \binom{t+c}{c},$$

which takes care of the first two terms of Equation (13). Now we consider the ratios of successive pairs of terms of the sum

$$\begin{aligned}\binom{t+c}{c-1}(c+2)^{c-1} - \binom{t+c}{c-2}(c+2)^{c-2} + \cdots + \\ (-1)^{c-2}\binom{t+c}{1}(c+2) + (-1)^{c-1} \\ = \sum_{k=1}^{c}(-1)^{k-1}\binom{t+c}{c-k}(c+2)^{c-k},\end{aligned}$$

i.e. we take the ratio of the $k = 2\alpha - 1$ term with the absolute value of the $k = 2\alpha$ term, $\alpha = 1, 2, \ldots, \lfloor \frac{c}{2} \rfloor$. (Note that if c is odd this comparison does not include the last ($k = c$) term, which is fine as this term is $1 > 0$.) We get

$$\begin{aligned}\frac{\binom{t+c}{c-2\alpha+1}(c+2)^{c-2\alpha+1}}{\binom{t+c}{c-2\alpha}(c+2)^{c-2\alpha}} &= \frac{g_{c-2\alpha+1}(t+c)\cdot(c+2)^{c-2\alpha+1}(c-2\alpha)!}{(c-2\alpha+1)!g_{c-2\alpha}(t+c)\cdot(c+2)^{c-2\alpha}} \\ &= \frac{(c+2)(t+2\alpha)}{c-2\alpha+1} \\ &> t+2\alpha \\ &> 1.\end{aligned}$$

Thus in this case as well, the sum, and hence the limit, is > 0.

This completes the proof. $\square$

Remark 6. Note that the proof of Lemma 3.6 does not work for $x = c = 0$, because in this case

$$\begin{aligned}\sum_{j=0}^{t-1}\binom{c+t}{j}\left(\frac{-1}{c+2}\right)^j &= \sum_{j=0}^{t-1}\binom{t}{j}\left(\frac{-1}{2}\right)^j \\ &= \left\{\sum_{j=0}^{t}\binom{t}{j}\left(\frac{-1}{2}\right)^j\right\} - \left(\frac{-1}{2}\right)^t \\ &= \left(\frac{1}{2}\right)^t - \left(\frac{-1}{2}\right)^t \\ &= \begin{cases} 0 & \text{when } t \text{ is even;} \\ \left(\frac{1}{2}\right)^{t-1} & \text{when } t \text{ is odd.}\end{cases}\end{aligned}$$

Thus it remains an open problem to construct examples of tight ItBDs of type t-(v, h, k, λ) with $h = t$ (i.e. $x = h - t = 0$) when $t \geq 4$. Such designs do exist for $t = 2$ and 3; see [3].

We are now ready to prove our asymptotic result (Theorem 1.5).

Theorem 3.7. *For each $t \geq 4$ there exists a constant $\mathcal{C}_t > 0$ such that whenever $(h-t)(k-t-1) \geq \mathcal{C}_t$ there exists a tight* ItBD $(X, H, \mathcal{B})$ *of type t-(v, h, k, λ) for some $\lambda = \lambda(t, h, k)$, having* $\mathrm{Sym}(H) \times \mathrm{Sym}(X \setminus H)$ *as an automorphism group.*

Proof. We proceed by induction on t. Theorem 2.5 handles $t = 4$ with $\mathcal{C}_t = 6$. Now let $t \geq 5$ and consider the matrix equation

$$M_t \vec{u} = J_t,$$

where M_t is as given in Equation (6). By induction, there exists a constant $\mathcal{C}_{t-1} > 0$ such that whenever $((h-1)-(t-1))((k-1)-(t-1)-1) \geq \mathcal{C}_{t-1}$ there is a non-negative rational vector $[u_1, u_2, \ldots, u_{t-1}]^T$ satisfying

$$M_{t-1}[u_1, u_2, \ldots, u_{t-1}]^T = J_{t-1},$$

i.e. $[u_1, u_2, \ldots, u_{t-1}]^T = M_{t-1}^{-1} J_{t-1}$. Now let $\vec{u} = [u_0, u_1, u_2, \ldots, u_{t-1}]^T$; then taking $x = h - t$ and $y = k - t - 1$, we have u_0 as given by Equation (12). Now let N_1, N_2, $N(y)$ and $N(x)$ be as given by Lemmas 3.3, 3.4 and 3.6, and take

$$\mathcal{C}_t = \max(\{t-4, \mathcal{C}_{t-1}\} \cup \{xN(x) : 1 \leq x < N_1\} \cup \{yN(y) : 1 \leq y < N_2\}).$$

Now suppose $xy \geq \mathcal{C}_t$. The conditions $xy \geq \mathcal{C}_{t-1} > 0$ and $xy \geq t-4$ guarantee that $u_i \geq 0$ for $i = 1, 2, \ldots, t-1$ and that $(x+1)(y+1) \geq t-1$ (so that $a_0 \neq 0$ in Equation (6)). Then we have one of three possibilities:

(i) $x \geq N_1$ and $y \geq N_2$. In this case we apply Lemma 3.3 to assert that $u_0 \geq 0$.
(ii) $1 \leq y < N_2$. As $xy \geq \mathcal{C}_t \geq yN(y)$, we have $x \geq N(y)$, whereupon we apply Lemma 3.4 to assert that $u_0 \geq 0$.
(iii) $1 \leq x < N_1$. As $xy \geq \mathcal{C}_t \geq xN(x)$ we have $y \geq N(x)$, whereupon we apply Lemma 3.6 to assert that $u_0 \geq 0$.

Thus in all cases $u_0 \geq 0$, and so $\vec{u}$ is a non-negative rational vector. Taking λ to be any positive integer making $\lambda\vec{u}$ integral establishes the result. □

Theorem 1.5 now follows as a corollary to Theorem 3.7.

If we consider C_t to be the smallest positive integer for which Theorem 1.5 holds, then Theorem 2.5 gives the following upper bounds on C_t for $4 \leq t \leq 9$: $C_4 \leq C_5 \leq 6$, $C_6 \leq C_7 \leq 9$ and $C_8 \leq C_9 \leq 11$. Note that if Conjecture is true, then we have the upper bounds

$$C_t \leq \begin{cases} t+3 & \text{if } t \text{ is even, and} \\ t+2 & \text{if } t \text{ is odd} \end{cases}$$

for every $t \geq 6$.

4. CONCLUSION AND OPEN PROBLEMS

The results in this article raise a number of interesting questions.

1. Can one prove or disprove Conjecture ? Failing this, can we at least show that $\mathcal{P}_t$ is well defined and determine an upper bound for it?
2. The index λ in the designs whose existence is asserted by Theorem 3.7 will generally be very large (see the examples constructed following Theorem 2.2). Can we construct tight designs with 'small' index? In particular, can we construct tight Steiner designs (i.e. with $\lambda = 1$)? (To do this one will almost certainly have to consider automorphism groups other than $\mathrm{Sym}(H) \times \mathrm{Sym}(X \setminus H)$.)
3. Theorem 1.5 implies that for each fixed $t \geq 4$ and $k \geq t+2$ we can construct a tight ItBD of type t-(v, h, k, λ) whenever h is 'large enough'. That this is not the case for $k = t+1$ has already been mentioned in the remark following Lemma 3.4. On the other hand, Theorem 1.5 also implies that for each fixed t and $h \geq t+1$ we can construct a tight ItBD of type t-(v, h, k, λ) whenever k is 'large enough', but the theorem does not cover the case $h = t$. (See the remark following Lemma 3.6.) This leaves open the question of whether or not there exists tight ItBDs of type t-(v, h, k, λ), when $h = t$, $t \geq 4$.
4. Related to Problem **2**, can one construct classes of tight designs which are simple (i.e. have no repeated blocks)? The examples constructed following Theorem 2.2 are of course not simple, as in each case the starter blocks form a multiset. Again, constructing simple tight designs would likely involve considering automorphism groups other than $\mathrm{Sym}(H) \times \mathrm{Sym}(X \setminus H)$.

How hard can it be?

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