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On PBIBD Designs Based on Triangular Schemes

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Abstract. We settle all but four of the remaining open small parameter situations for partially balanced incomplete block designs with 2 associate classes (PBIBD(2)) that can be based on Triangular schemes.

Keywords: partially balanced incomplete block designs, association schemes, triangular designs.

1. Introduction

A *strongly regular graph* with parameters (v,d,μ_1,μ_2) is an ordinary graph, regular of degree d, $0 < d < v-1$, such that any two vertices have μ_1 common neighbors when they are adjacent and have μ_2 neighbors when they are nonadjacent. A strongly regular graph is also called a *two class association scheme*. Two vertices are *first associates* if they are adjacent; otherwise, they are *second associates*. In this article we concentrate on the strongly regular graph $T(n)$ known as the *triangular graph*. The vertices of this graph are the $v = \binom{n}{2}$ pairs of an n–element set, with two pairs being adjacent if they have an element in common. Thus, in the triangular graph, two pairs are first associates if they are incident edges of the complete graph, K_n. Given a strongly regular graph we can obtain a a *partially balanced incomplete block design with two associate classes* PBIBD(2) with parameters

$$(v,b,r,k,(\lambda_1,\lambda_2))$$

if the v vertices can be arranged into b blocks of size k such that every vertex occurs in r blocks, and if two vertices are adjacent, then they occur in λ_1 blocks; otherwise they occur in λ_2 blocks. A *triangular design* is a PBIBD(2) in which the strongly regular graph is a triangular graph. Thus a triangular design has $v = \binom{n}{2}$ vertices.

Table 1: Status of remaining parameter situations for Triangular designs, $k, r \leq 10$

Series	v	b	r	k	λ_1	λ_2	ρ_1	ρ_2	Status
1	15	27	9	5	3	2	9	5	$\exists$ Section 2.1
2	15	30	10	5	2	4	2	10	$\exists$ Section 2.2
3	21	42	10	5	1	3	1	11	$\nexists$ Corollary 3.2
4	21	42	10	5	3	1	15	5	$\exists$ Section 2.3
5	21	21	6	6	2	1	8	3	$\exists$ Section 2.4
6	21	28	8	6	3	1	13	3	$\exists$ Section 2.5
7	21	35	10	6	2	3	4	9	$\exists$ [1]
8	21	35	10	6	3	2	11	6	Open
9	21	30	10	7	2	4	0	10	$\exists$ Section 2.6
10	21	30	10	7	4	2	14	4	Open
11	21	30	10	7	5	1	21	1	$\nexists$ Corollary 3.3
12	21	21	10	10	4	5	2	7	Open
13	36	63	7	4	0	1	1	8	$\exists$ [3]
14	45	63	7	5	0	1	0	8	$\exists$ [3]
15	45	45	9	9	1	2	1	9	$\exists$ [3]
16	55	99	9	5	0	1	1	10	$\nexists$ Corollary 4.4
17	55	55	10	10	3	1	23	5	Open
18	66	99	9	6	0	1	0	10	$\nexists$ [7]

A comprehensive table of triangular designs and a review of some construction methods can be found in [3] and a general method for constructing triangular designs can be found in [5]. [4] and [8] report that there are 18 unsettled parameter situations when $r, k \leq 10$. We list in Table 1 these parameter situations, (for more details on the eigenvalues ρ_i, see Section 4), and give their current status, including the new designs exhibited in this article and the situations which we rule out.

2. Constructions

Let X be an n–element set and consider the triangular graph $T(n)$ on the pairs in X. If G is a subgroup of $\mathrm{SYM}(X)$, then G acts on the pairs in X in a natural way: $g(\{x,y\}) = \{g(x), g(y)\}$. This action preserves incidence and so any such subgroup is an automorphism group of $T(n)$. Let $\Delta^1_1, \Delta^1_2, \ldots, \Delta^1_{N_1}$ be the orbits under G of incident (first associate) pairs and let $\Delta^2_1, \Delta^2_2, \ldots, \Delta^2_{N_2}$ be the orbits under G of non-incident (second associate) pairs. Also, for fixed k, $0 < k < \binom{n}{2}$, let $\Gamma_1, \Gamma_2, \ldots \Gamma_M$ be the orbits of k–edge subgraphs of $T(n)$. We define the incidence matrix A to be the $N_1 + N_2$ by m matrix whose $[\Delta^c_i, \Gamma_j]$–entry is

$$A[\Delta^h_{i_h}, \Gamma_j] = \{K \in \Gamma_j : K \subseteq T\},$$

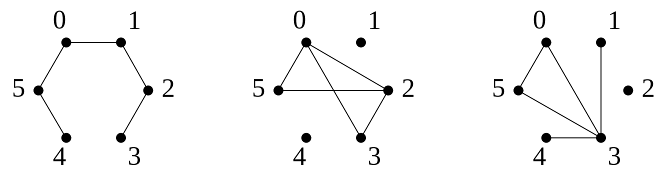
where T is any fixed representative in $\Delta^h_{i_h}$, $h \in \{1,2\}$, $1 \leq i_h \leq N_h$ and $1 \leq j \leq m$. Then there is a triangular design with parameters $(v, b, r, k, (\lambda_1, \lambda_2))$ if and only if there is a non-negative integer solution $\vec{u}$ to the matrix equation

$$A\vec{u} = \begin{bmatrix} \lambda_1 J_{N_1} \\ \lambda_2 J_{N_2} \end{bmatrix}.$$

We used the algorithms described in [6] to construct and solve these matrices for various parameter situations and various groups G. In [5], the authors use a similar method; however, the only automorphism group they consider is the full symmetric group. The results of our effort are given below.

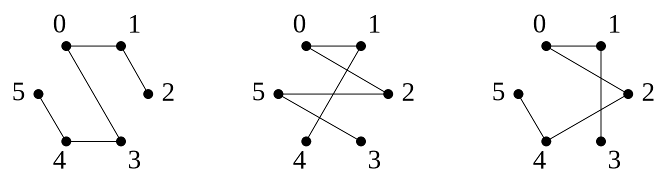
2.1.

Parameters:	$(15, 27, 9, 5, (3, 2))$
Group generators:	$\langle(0, 1, 2), (3, 4, 5)\rangle$
Group order:	9
Base blocks:	



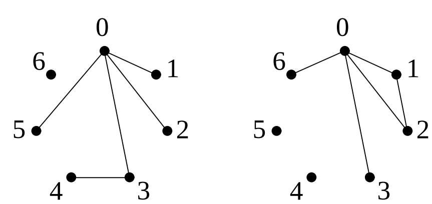
2.2.

Parameters:	$(15, 30, 10, 5, (2, 4))$
Group generators:	$\langle(0, 1, 2, 3, 4, 5), (0)(1, 5)(2, 4)(3)\rangle$
Group order:	12
Base blocks:	



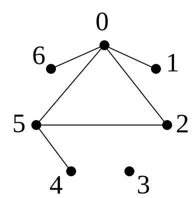
2.3.

Parameters:	$(21, 42, 10, 5, (3, 1))$
Group generators:	$\langle(0, 1, 2, 3, 4, 5, 6), (0)(1, 2, 4)(3, 6, 5)\rangle$
Group order:	21
Base blocks:	



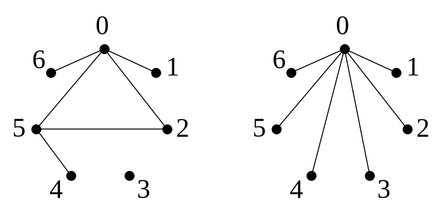
2.4.

Parameters: $(21,21,6,6,(2,1))$
Group generators: $\langle(0,1,2,3,4,5,6),(0)(1,2,4)(3,6,5)\rangle$
Group order: 21
Base blocks:



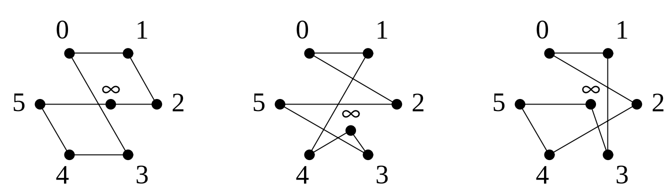
2.5.

Parameters: $(21,21,6,6,(3,1))$
Group generators: $\langle(0,1,2,3,4,5,6),(0)(1,2,4)(3,6,5)\rangle$
Group order: 21
Base blocks:



2.6.

Parameters: $(21,30,10,7,(2,4))$
Group generators: $\langle(0,1,2,3,4,5)(\infty),(0)(1,5)(2,4)(3)(\infty)\rangle$
Group order: 12
Base blocks:



3. Destructions

This section really represents a generalization of a technique used by Chang et al. in their Theorem 1 and its corollaries [4]. Although we don't give the details here, it is worth noting that one of their exhaustive searches, (the first in [4, Table 11]), could have been avoided in a manner similar to Corollary 3.2.

To show that a given parameter situation $(\binom{n}{2}, b, r, k, (\lambda_1, \lambda_2))$ is impossible, we consider any subgraph Γ of K_n. If $D_\Gamma = [d_0, d_1, \ldots, d_n]$ is the degree sequence of Γ, then Γ contains exactly $\alpha = \alpha(\Gamma) = \sum_{i=0}^{n} \binom{d_i}{2}$ pairs of incident edges. That is, Γ contains α first associate pairs and $\beta = \binom{m}{2} - \alpha$ second associate pairs; where $m = \frac{1}{2}\sum_{i=0}^{n} d_i$ is the number of edges of Γ. Let N_i be the number of blocks of a PBIBD(2) with the given parameters that intersect Γ in i edges. Counting blocks, edges, and pairs of edges, we obtain the matrix equation

$$M\vec{N} = \vec{R}, \tag{3.1}$$

where

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 2 & 3 & \cdots & n \\ 0 & 0 & 1 & 3 & \cdots & \binom{n}{2} \end{bmatrix} \tag{3.2}$$

$$\vec{N} = [N_0, N_1, N_2, N_3, \ldots, N_n]^\mathsf{T} \tag{3.3}$$

and

$$\vec{R} = [b, mr, \lambda_1\alpha + \lambda_2\beta]^\mathsf{T}. \tag{3.4}$$

If we can find $\vec{U}^\mathsf{T} = [u_0, u_1, u_2]^\mathsf{T}$ such that $\vec{U}^\mathsf{T} M \geq 0$, and $z = \vec{U}^\mathsf{T}\vec{R} < 0$, then no PBIBD(2) with parameters $(\binom{n}{2}, b, r, k, (\lambda_1, \lambda_2))$ can exist. Observe that the i–th entry of $\vec{U}^\mathsf{T} M$ is

$$f(i) = u_0 + u_1 i + u_2 \binom{i}{2},$$

a quadratic in the variable i.

Thus, one way to ensure that $\vec{U}^\mathsf{T} M \geq 0$ is to require that $u_2 \geq 0$ and that $f(i) = f(i+1) = 0$ for some integer i, $0 \leq i \leq n-1$. Then $f(x) \geq 0$ for all $x \leq i$ and $x \geq i+1$. We may further assume that $u_2 = 1$, because dividing by a positive u_2 will not change the sign of z.

The columns of M are pairwise linearly independent. Thus there is a unique choice of u_0, u_1, u_2, such that $u_2 = 1$ and $f(i) = f(i+1) = 0$ for a given integer i. This choice is

$$\vec{U}^\mathsf{T} = [\binom{i+1}{2}, -i, 1].$$

Consequently, we have established the following:

Theorem 3.1. *Let Γ be an subgraph of K_n with m edges and α pairs of incident edges. Then, necessary conditions for the existence of a Triangular* PBIBD(2) *with parameters*

$$(\binom{n}{2}, b, r, k, (\lambda_1, \lambda_2))$$

include

$$b\binom{i+1}{2} - imr + \lambda_1\alpha + \lambda_2\beta \geq 0$$

for each integer $i = 0, 1, 2, \ldots, n-1$. $(\alpha + \beta = \binom{m}{2}.)$

Two of the subgraphs Γ that seem to be the the most useful in the context of Theorem 3.1 are the star $K_{1,n}$ and the n–cycle C_n. Using these graphs we have the following corollaries.

Corollary 3.2. *The necessary conditions for the existence of a Triangular* PBIBD(2) *with parameters*

$$(\binom{n}{2}, b, r, k, (\lambda_1, \lambda_2))$$

include

$$b\binom{i+1}{2} - i(n-1)r + \lambda_1\binom{n-1}{2} \geq 0$$

for each integer $i = 0, 1, 2, \ldots, n-1$.

Using Corollary 3.2 with $i = 1$ shows that no triangular PBIBD(2) can have the parameters $(21, 42, 10, 5, (1, 3))$.

Corollary 3.3. *The necessary conditions for the existence of a Triangular* PBIBD(2) *with parameters*

$$(\binom{n}{2}, b, r, k, (\lambda_1, \lambda_2))$$

include

$$b\binom{i+1}{2} - inr + \lambda_1 n + \lambda_2(\binom{n}{2} - n) \geq 0$$

for each integer $i = 0, 1, 2, \ldots, n-1$.

Using Corollary 3.3 with $i = 2$ shows that no triangular PBIBD(2) can have the parameters $(21, 30, 10, 7, (5, 1))$.

In the opening paragraph of this section, we described our technique as a generalization of Chang et al.'s method. This is possibly inaccurate, as their relevant theorems are stated without proof, and we were unable to reproduce several of their theorems. In spite of this, we can confirm that their lists of possible designs are complete within the range they consider; however, in our enumeration, we applied positivity of eigenvalues as a criterion, so our intermediate lists differ markedly from theirs.

Given the paucity of theoretical criteria for non-existence of any designs, the other aspect of this method that certainly deserves mention is the question of its generality. It is clear that the only designs it might exclude are those where λ_1 differs markedly from λ_2. Unfortunately, we were unable to find a general series of designs that this method excludes, but that are not also excluded by the eigenvalue criterion.

4. Embedding

In the more familiar group divisible association scheme, if $rk - v\lambda_2 = 0$, (i.e., the "semi-regular" case), then we may remove one of the groups and still get a PBIBD(2) with the group divisible association scheme, also still in the semi-regular case. We also note that if N is the incidence matrix of a PBIBD(2) with the group divisible association scheme, then $rk - v\lambda_2$ is one of the eigenvalues of NN^T.

In the triangular association scheme, $T(n)$, the eigenvalues of NN^T are rk, $\rho_1 = r + (n-4)\lambda_1 - (n-3)\lambda_2$ and $\rho_2 = r - 2\lambda_1 + \lambda_2$ with multiplicities of 1, $n-1$, and $n(n-3)/2$. There are some similarities of the case $\rho_1 = 0$ with the semi-regular case. First, we note that for a triangular PBIBD(2) we have:

$$\begin{aligned} bk &= vr & (4.5)\\ r(k-1) &= 2(n-2)\lambda_1 + \binom{n-2}{2}\lambda_2. & (4.6) \end{aligned}$$

Under the assumption that $\rho_1 = 0$, we also have:

$$(n-3)\lambda_2 = r + (n-4)\lambda_1,$$

and equation (4.6) gives us

$$(n-2)\lambda_1 = r(c-1) \tag{4.7}$$

where $c = 2k/n$. Also, equation (4.5) can be rewritten as $b = (n-1)r/c$.

Now, as in Corollary 3.2, we consider a star $K_{1,n}$, and the vector

$$\vec{U}^{\mathsf{T}} = [a^2/2, 1/2 - a, 1]^{\mathsf{T}}.$$

From equation (3.2), the i-th element of $\vec{U}^{\mathsf{T}}M$ is $(i-a)^2/2$ (where we number from zero), and, evaluating at $a = c$ and using $m = n-1$, $\alpha = \binom{n-1}{2}$ and $\beta = 0$, together with equation (4.7), we also have $z = \vec{U}^{\mathsf{T}}\vec{R} = 0$. This implies that every block must contain c edges of $K_{1,n}$; consequently, deleting this star again gives us a PBIBD(2), only now with the triangular association scheme $T(n-1)$. This establishes Theorem 4.1.

Theorem 4.1. *Given a triangular* PBIBD(2) *with parameters*

$$\left(\binom{n}{2}, b, r, k, (\lambda_1, \lambda_2)\right)$$

with $c = 2k/n$ integer and a zero eigenvalue, ρ_1, of multiplicity $n-1$, then

(1) *every block is a c–regular subgraph of K_n, and*

(2) *deleting any $K_{1,n-1}$ yields a triangular* PBIBD(2) *with parameters*

$$\left(\binom{n-1}{2}, b, r, k-c, (\lambda_1, \lambda_2)\right)$$

with an eigenvalue of $\lambda_2 - \lambda_1$ of multiplicity $n-2$.

Observe that the Series 9 in Table 1 fits the criterion of Theorem 4.1. Here $c = 2$ and indeed all of its blocks are 2–regular subgraphs. Furthermore deleting the star centered at the point ∞ we get Series 2.

So now the natural question is when can we reverse this process, i.e., given a triangular PBIBD(2) with parameters

$$\left(\binom{n}{2}, b, r, k, (\lambda_1, \lambda_2)\right)$$

with an eigenvalue of $\lambda_2 - \lambda_1$ of multiplicity $n-1$ and $c = 2k/(n-1)$ integer, when can we add c points to each block to get a triangular PBIBD(2) with parameters

$$\left(\binom{n+1}{2}, b, r, k+c, (\lambda_1, \lambda_2)\right)$$

with an eigenvalue of zero of multiplicity n.

By an approach similar to that presented above, again considering a star, we can easily show that its edges only occur together in sets of size $c-1$ or size c, and that there are r blocks with size $c-1$. So, we take the set of r blocks with intersection of size $c-1$ with the edges of the star at $\{i\}$ as the set to which we add the edge $(i, n+1)$. This determines the embedding.

Since we know how often an edge of a star must pair with the remaining edges of the star, it is easy to show that each edge is a member of λ_1 of the $c-1$ sets; similarly for edges not in the star, each must lie in the same block as a $c-1$ set λ_2 times. We next must show that every block receives c new points. Since any block contains $2k = (n-1)c$ edges from the n stars, and this is some mix of $c-1$-sets and c-sets, clearly this mix is c of $c-1$-sets and $n-c$ of c-sets, hence each block will receive c new edges in our embedding. Thus every block when considered as a subgraph of K_n has exactly c vertices of degree $c-1$ and $n-c$ vertices of degree c. The final problem is to show that every pair of new points occurs together λ_1 times. That is if we consider for each block the set of vertices of degree $c-1$, then these b sets must form a 2–(n, c, λ_1) design (with repeated blocks). We state this as our next theorem.

Theorem 4.2. *Given a triangular* PBIBD(2) *with parameters*

$$\left(\binom{n}{2}, b, r, k, (\lambda_1, \lambda_2)\right)$$

with an eigenvalue of $\lambda_2 - \lambda_1$ *of multiplicity* $n-1$ *and* $c = 2k/(n-1)$ *integer, then*

(1) *every block is a subgraph of* K_n*, that has exactly* c *vertices of degree* $c-1$ *and* $n-c$ *vertices of degree* c*;*

(2) *if the sets of vertices of degree* $c-1$ *form a* $(n, b, (n-1)\lambda_1/2, c, \lambda_1)$*–BIBD, then we may extend the given triangular* PBIBD(2) *to a triangular* PBIBD(2) *with parameters*

$$\left(\binom{n+1}{2}, b, r, k+c, (\lambda_1, \lambda_2)\right)$$

Observe that Series 2 in Table 1 fits the criterion of Theorem 4.2 in which the BIBD of vertices of degree $c-1=1$ are all the pairs each repeated twice, a $(6,30,10,2,2)$–BIBD. Hence we may extend this design to obtain Series 9.

In the special case in which $\lambda_1 = 0$, it is straightforward to show that $c=1$. The required $(n,b,r,1,0)$–BIBD trivially exists so the conditions of Theorem 4.2 hold. We record this useful result as Theorem 4.3.

Theorem 4.3. *Given a triangular* PBIBD(2) *with parameters*

$$\left(\binom{n}{2}, n(n-2)\lambda_2, (n-2)\lambda_2, (n-1)/2, (0,\lambda_2)\right),$$

we may extend this to a triangular PBIBD(2) *with parameters*

$$\left(\binom{n+1}{2}, n(n-2)\lambda_2, (n-2)\lambda_2, (n+1)/2, (0,\lambda_2)\right).$$

Now we can exploit the result of Lam et al. [7] to resolve another open case.

Corollary 4.4. *A triangular* PBIBD(2) *with parameters*

$$(55,99,9,5,(0,1))$$

does not exist.

It might be noted that another of Chang et al.'s exhaustive searches, (the second in [4, Table 11]) could have been avoided, as it would extend to the design in last of their exhaustive searches – although in that last search, they also excluded a PBIBD(2) based on the 3 pseudo-triangular association schemes on 28 points, in addition to the triangular scheme $T(8)$. Batten and Beutelspacher [2, pp. 46–47] have confirmed this search for the triangular $T(8)$ scheme.

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