On Steiner 3–wise balanced designs of order 17

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ABSTRACT

We determine all $S(3,\mathcal{K},17)'s$ which either; (i) have a block of size at least 6; or (ii) have an automorphism group order divisible by 17, 5, or 3; or (iii) contain a semi-biplane; or (iv) come from an $S(3,\mathcal{K},16)$ which is not an $S(3,4,16)$. There is an $S(3,\mathcal{K},17)$ with $|G|=n$ if and only if $n \in \{2^a3^b : 0 \leq a \leq 7, 0 \leq b \leq 1\} \cup \{18, 60, 144, 288, 320, 1920, 5760, 16320\}$. We also search the $S(3,\mathcal{K},17)'s$ listed in the appendix for subdesigns $S(2,\mathcal{K},17)$ and generate 22 nonisomorphic $S(3,\mathcal{K},18)'s$ by adding a new point to such a subdesign.

1. INTRODUCTION

A *t–wise balanced design* (tBD) of type t–$(v,\mathcal{K},\lambda)$ is a pair $(X,\mathcal{B})$ where X is a v–element set of points and $\mathcal{B}$ is a collection of subsets of X called blocks with the property that the size of every block is in $\mathcal{K}$ and every t–element subset of X is contained in exactly λ blocks. If $\mathcal{K}$ is a set of positive integers strictly between t and v then we say the tBD is *proper.* A t–$(v,\mathcal{K},\lambda)$ design is also denoted by $S_\lambda(t,\mathcal{K},v)$. If $|\mathcal{K}|=1$, then the tBD is called a t–(v,k,λ) design, where $\mathcal{K}=\{k\}$. When $t=2$ a tBD is called a *pairwise balanced design* and a nice survey of them is given in [3].

CCC 1063-8539/94/030117-13
Journal of Combinatorial Designs Vol. 4, No. 5, (1996)

If $\lambda = 1$, then we use the notation $S(t, \mathcal{K}, v)$. An $S(2, \mathcal{K}, v)$ is also called a *a linear space.*

2. ELEMENTARY PROPERTIES

By taking the derived incidence structure the following result is immediate:

Theorem 2.1. *A t-$(v, \mathcal{K}, \lambda)$ design implies the existence of a $(t-1)$-$(v-1, \mathcal{K}-1, \lambda)$ design where $\mathcal{K} - 1 = \{k_i - 1 : k_i \in \mathcal{K}\}$.*

The strategy used in [2] to find all $S(t, \mathcal{K}, v)$ designs for small v was to find (or know) all possible $S(t-1, \mathcal{K}, v-1)$ designs and then examine possible extensions. The $S(2, \mathcal{K}, 16)$ (linear spaces of order 16) were, in fact, completely classified in [1]. However, we were only able to use some of this information because of the large number of such linear spaces. Eventually we classified those $S(3, \mathcal{K}, 17)'s$ which contained certain substructures or which had certain automorphisms. The *type* of a point x in a linear space $(P, \mathcal{L})$ of order 16, was defined in [1] to be the multiset $\{|\ell| - 1 : x \in \ell \in \mathcal{L}\}$. The type is thus a partition of 15 into parts each at least two (since x appears exactly once with each point in $P \setminus \{x\}$). According to [1] there are ten nontrivial partitions that arise. They are assigned letters here for convenience:

A:555	F:53322	G:522222	H:4443	I: 44322
J:43332	K:432222	L:33333	M:333222	N:3222222

The type of the linear space is the multiset of types of all of its n points. To indicate the number of blocks of each of the possible sizes in a tBD or linear space we use exponential notation. For example $(5^3 4^5 3^{20})$ means that there are 3 blocks of size 5, 5 blocks of size 4 and 20 blocks of size 3. Using this notation the following theorem is given in [1].

Theorem 2.2. *There are 398 isomorphism classes of proper linear spaces on 16 points, of which*

(1) *two have type* `A=1,G=15` $(6^3 3^{25})$;
(2) *one has type* `F=5,G=1,M=10` $(6^1 4^{10} 3^{15})$;
(3) *one has type* `F=3,G=3,M=4,N=6` $(6^1 4^6 3^{23})$;
(4) *one has type* `H=1,J=12,L=3` $(5^3 4^{13} 3^4)$;
(5) *two have type* `H=1,K=12,L=1,N=2` $(5^3 4^5 3^{20})$;
(6) *one has type* `I=15,L=1` $(5^6 4^5 3^{10})$;
(7) *one has type* `I=3,J=3,K=6,M=3,N=1` $(5^3 4^7 3^{16})$;
(8) *two have type* `I=3,K=9,L=1,N=3` $(5^3 4^5 3^{20})$;
(9) *one has type* `I=3,K=9,M=4` $(5^3 4^6 3^{18})$;
(10) *two have type* `K=15,L=1` $(5^3 4^5 3^{20})$;
(11) *one has type* `L=16` (4^{20});
(12) *two have type* `L=6,M=10` $(4^{15} 3^{10})$;

(13) *one has type* `L=2,M=12,N=2` $(4^{12}3^{16})$;
(14) *three have type* `L=1,M=12,N=3` $(4^{11}3^{18})$;
(15) *five have type* `L=1,M=10,N=5` $(4^{10}3^{20})$;
(16) *146 have type* `L=1,N=15` $(4^{5}3^{30})$;
(17) *one has type* `M=16` $(4^{12}3^{16})$;
(18) *one has type* `M=14,N=2` $(4^{11}3^{18})$;
(19) *13 have type* `M=10,N=6` $(4^{9}3^{22})$;
(20) *49 have type* `M=6,N=10` $(4^{7}3^{26})$;
(21) *111 have type* `M=4,N=12` $(4^{6}3^{28})$;
(22) *28 have type* `M=2,N=14` $(4^{5}3^{30})$;
(23) *23 have type* `N=16` $(4^{4}3^{32})$; *and*

3. COMPUTER SEARCHES

We did several computer searches. For example, after obtaining several $S(3,\mathcal{K},17)$'s we tried to extend some to $S(4,\mathcal{K},18)$'s. To attempt an extension of one of $D_2, \ldots, D_{16}$ (see the Appendix and Table of Summary Data), we would have first ruled out blocks of size 7 or greater. We then consider the 18564 6-sets and 8568 5-sets in an 18-set. These 27132 sets can be specified by listing for each which 4-sets that they contain. If we take say D_6 and add a point to each block, we then fix the corresponding 116 sets in the 27132. Eliminating conflicting sets leaves, in the case of D_6, 3383 blocks (candidates) for the extension to choose from. Using an exhaustive backtrack search we find at most 72 additional blocks trying all possible extensions. This is a typical result.

A similar technique was used in the extensions of some linear spaces on 16 points where the problem is smaller. For possible extensions where sets larger than size 6 have been ruled out, there are 2380 4-sets, 6188 5-sets and 12376 6-sets in a 17-set, which contains 680 triples. In a search for $S(3,\mathcal{K},17)$ with $\mathcal{K}=\{4,5\}$ we have 8568 sets and we input the blocks of a linear space extended by a point and then backtrack as before.

The searches for systems with a prescribed automorphism are based on the same technique except that we use orbits of all sets for all possible set sizes.

4. BLOCK SIZE DISTRIBUTION

Theorem 4.1. *The size of a block in an* $S(3,\mathcal{K},17)$ *is at most* 6.

Proof. If x is a point the derived incidence structure with respect to x is a linear space. Thus, by Theorem 2.2, the only possible block sizes are 4, 5, 6 or 7. Let n_i be the number of blocks of size i, $i=4,5,6,7$. Counting triples in two ways we get $35n_7+20n_6+10n_5+4n_4=680$, hence $n_7 \equiv 0 \pmod 2$. Consequently, if there is a block of size 7, there are at least two blocks A and B of size 7. Let $x \in A$, then by Theorem 2.2 the derived incidence structure has type `A=1,G=15` or `F=5,G=1,M=10` or `F=3,G=3,M=4,N=6`. The first of these have 3 blocks of size 6 implying together with B that there are 4 blocks of size 7 in our $S(3,\mathcal{K},17)$. It is easy to see that this is impossible since they can pairwise intersect in at most two points. The remaining two types of linear spaces have only one block of size 6. Consequently A and B are disjoint. Let C be the remaining 3 points. Careful scrutiny of Theorem 2.2 shows that every other block has size 4 or 5. Let the type of a set Z be the 3-tuple $(|A \cap Z|, |B \cap Z|, |C \cap Z|)$. The only types of blocks containing a 3-tuple of type $(1,1,1)$ are: $(2,2,1)$, $(2,1,2)$, $(1,2,2)$, $(1,1,3)$, $(1,1,2)$, $(1,2,1)$, and $(2,1,1)$. These each contain an even number of type $(1,1,1)$ triples except for type $(1,1,3)$ which contains 3. There are 63 type $(1,1,1)$ triples, hence there is a block Z of type $(1,1,3)$. Let x be the point in the intersection of A and Z. Then the derived incidence structure with respect to x is a linear space space of order 16 containing a block of size 6 disjoint from a block of size 4. The data given in [1] shows that this is impossible. □

If $(X, \mathcal{B})$ is an $S(3, \mathcal{K}, 17)$, then for each $k \in \mathcal{K}$, $\mathcal{B}_k = \{B \in \mathcal{B} : |B| = k\}$ will denote the set of blocks of size k; and $X_k = \cup_{B \in \mathcal{B}_k} B$, the points in blocks of size k. The number of blocks of size k containing a subset A of points is denoted by $\deg_{\mathcal{B}_k}(A)$.

Theorem 4.2. *If an $S(3, \mathcal{K}, 17)$ design $(X, \mathcal{B})$ contains a block of size 6, then it is the unique design D_1 given in the appendix. Furthermore the blocks of size 6 form a 2-(16,6,2) design.*

Proof. Suppose $\mathcal{B}_6 \neq \emptyset$. Then there is a point x and a block B such that $x \in B \in \mathcal{B}_6$. Thus the derived incidence structure with respect to x is a linear space of order 16 appearing in Theorem 2.2 on lines (4) through (10). Consequently $\deg_{\mathcal{B}_6}(x) \in \{3, 6\}$ and the points in $X_6 \setminus \{x\}$ are points of type `H`, `I`, `J` or `K` and therefore $|X_6| \geq 13$. Counting incidence of points in X_6 and blocks in $\mathcal{B}_6$ in two ways we see that

$$6|\mathcal{B}_6| = \sum_{x \in X_6} \deg_{\mathcal{B}_6}(x) \geq 3|X_6| \geq 3 \cdot 13.$$

Therefore $|\mathcal{B}_6| \geq 7$.

Let α_i be the number of points of an $S(3, \mathcal{K}, 17)$ that have degree i in $\mathcal{B}_6$, $i \in \{0, 3, 6\}$. The following equations hold:

$$\left.\begin{array}{rcl} \alpha_0 + \alpha_3 + \alpha_6 & = & 17 \\ \alpha_3 + \alpha_6 & \geq & 13 \\ 3\alpha_3 + 6\alpha_6 & = & 6|\mathcal{B}_6| \\ \binom{3}{2}\alpha_3 + \binom{6}{2}\alpha_6 & \leq & 2\binom{|\mathcal{B}_6|}{2} \end{array}\right\} \qquad (1)$$

The 7 solutions to these equations are given in Table I

For Solution I consider the subdesign $(X_6, \mathcal{B}_6)$. It represents a collection of seven 6-element blocks on fourteen points that pairwise intersect in no more than 2 points and in which every point has degree 3. Interchanging the role of points and blocks gives the dual structure which is a collection of 14 triples in which every point has degree 3 and in which every pair is in at most 2 triples. An easy counting argument shows that every pair is in exactly 2 triples and consequently we have a 2-(7,3,2) design.

There are exactly 4 nonisomorphic 2-(7,3,2) designs that are the union of two Fano planes intersect ing in 0,1,3 or 7 blocks respectively, [5]. Set $T = X \setminus X_6$.

TABLE I. Solutions to equations (1).

	α_0	α_3	α_6	$\lvert\mathcal{B}_6\rvert$
I	3	14	0	7
II	1	14	2	9
III	0	14	3	10
IV	0	4	13	15
V	0	2	15	16
VI	1	0	16	16
VII	0	0	17	17

If $x, y \in X_6$ correspond to a pair of repeated blocks in the dual, then $y \cup T$ is a block in the derived linear space through x, see [1]. Hence $\{x, y\} \cup T$ is a block. Therefore there is at most one repeated block in the dual, else the triple T is covered more than once. Thus the two Fano planes comprising the 2-(7,3,2) design $(X_6, \mathcal{B}_6)$ intersect in 0 or 1 block.

Up to isomorphism the two possibilities are:

1. $\{0,1,2,8,9,12\}$, $\{0,1,3,6,7,11\}$, $\{0,2,3,4,5,10\}$, $\{1,4,5,11,12,13\}$, $\{2,7,9,10,11,13\}$, $\{3,6,8,10,12,13\}$, $\{4,5,6,7,8,9\}$

2. $\{0,1,3,6,9,12\}$, $\{0,1,2,7,8,11\}$, $\{0,2,3,4,5,10\}$, $\{1,4,6,10,11,13\}$, $\{3,5,8,11,12,13\}$, $\{2,7,9,10,12,13\}$, $\{4,5,6,7,8,9\}$

An exhaustive computer search showed that these two possibilities cannot extend to an $S(3, \mathcal{K}, 17)$ design.

Remark 1. Suppose $\alpha_6 \neq 0$ and let $x \in X$ have $\deg_{\mathcal{B}_6}(x) = 6$. Then according to [1] the dual of the derived design of $(X, \mathcal{B}_6)$ through x consists of the 15 pairs from a 6–element set. Consequently if $x \in X$ has degree 6 in $\mathcal{B}_6$, then $\deg_{\mathcal{B}_6}(x, y) = 2$ for all but one $y \in X \setminus \{x\}$, which has degree 0.

If $\alpha_6 = 2$, then by Remark it is easy to see that $|\mathcal{B}_6| \geq 10$ and thus Solution II is impossible. By similar reasoning if $\alpha_6 = 3$, then $|\mathcal{B}_6| > 10$, thus Solution III is also impossible.

We now show that Solution IV is impossible. Let a_1, a_2, a_3, a_4 denote the points of degree 3 in $\mathcal{B}_6$ and let $x_1, x_2, \ldots, x_{13}$ denote the points of degree 6. If $\deg_{\mathcal{B}_6}(x_i, x_j) = 0$ for some $i \neq j$, then by Remark , $\deg_{\mathcal{B}_6}(x_j, a_1) = \deg_{\mathcal{B}_6}(x_i, a_1) = 2$, contrary to $\deg_{\mathcal{B}_6}(a_1) = 3$. Hence $\deg_{\mathcal{B}_6}(x_i, x_j) = 2$ for all $i \neq j$ and for each x_i there is a unique a_j such that $\deg_{\mathcal{B}_6}(x_i, a_j) = 0$. Since there are 13 x_is, then some a_i say a_1 has $\deg_{\mathcal{B}_6}(x_i, a_1) = 0$ for at least 4 choices of i, say $i = 1, 2, 3, 4$. Let the points of the dual of $(X, \mathcal{B}_6)$ be labeled $1, 2, 3, \ldots, 15$ Then without loss we have as blocks $x_1 = \{1,2,3,4,5,6\}$, $x_2 = \{1,2,7,8,9,10\}$, $x_3 = \{3,4,7,8,11,12\}$, $x_4 = \{5,6,9,10,11,12\}$, and $a_1 = \{13,14,15\}$. Up to relabeling in the dual the block $a_2 = \{1,3,7\}$, but then $\deg_{\mathcal{B}_6}(x_4, a_2) = \deg_{\mathcal{B}_6}(x_4, a_1) = 0$, contrary to Remark .

In Solution V, there are 2 points a_1 and a_2 of degree 3 and 15 points $x_1, x_2, \ldots, x_{15}$ of degree 6 in $\mathcal{B}_6$. Similar to the above argument we again see that $\deg_{\mathcal{B}_6}(x_i, x_j) = 2$ for all $i \neq j$ and for each x_i there is a unique a_j such that $\deg_{\mathcal{B}_6}(x_i, a_j) = 0$. Hence counting in two ways we see modulo 2 that

$$0 \equiv \sum_{i=1}^{15} \deg_{\mathcal{B}_6}(a_1, x_i) = 3 \cdot 5 = 15$$

a contradiction.

For Solution VI consider the subdesign $(X_6, \mathcal{B}_6)$. It represents a collection of 16 6-element blocks on 16 points that pairwise intersect in no more than 2 points and in which every point has degree 6. Interchanging the role of points and blocks gives the dual structure which is easily seen to be a 2-(16,6,2) biplane. In particular any $x \in X_6$ has derived design isomorphic to the linear space with 6 blocks of size 5 (line (6) of Theorem 2.2). Using the additional 5 blocks of size 4 and the 10 blocks of size 3 in this derived design one can show, by an elementary but somewhat tedious argument, that the biplane is determined. (It is the one whose group has order 11,520, see [4].) An exhaustive computer search shows that this structure has a unique completion to an $S(3, \mathcal{K}, 17)$ design. It is given in the appendix as design D_1.

Finally for Solution VII, by Remark for each $x \in X$ there is a unique $y \in X \setminus \{x\}$ such that $\deg_{\mathcal{B}_6}(x, y) = 0$. This is impossible since $|X| = 17$. □

An important consequence of Theorem 4.2 together with the known uniqueness of the inversive plane $S(3, 5, 17)$ [4] is that any other $S(3, \mathcal{K}, 17)$ designs (other than D_1 and D_2 in the Appendix) must have $\mathcal{K} = \{4, 5\}$. Also, $\mathcal{K} \neq \{4\}$, since an $S(3, 4, v)$ must have $v \equiv 2$ or $4 \pmod 6$. **So in searches for the remainder of this paper we assume that $\mathcal{K} = \{4, 5\}$ and that $|\mathcal{B}_4|, |\mathcal{B}_5| > 1$.**

Suppose an $S(3, \{4, 5\}, 17)$ design $(X, \mathcal{B})$ is not the extension of an $S(2, 4, 16)$ (the affine geometry AG(2,4)). Then for each $x \in X$ we see from Theorem 2.2 that the derived design through x contains at least 10 blocks of size 3. Thus $\deg_{\mathcal{B}_4}(x) \geq 10$. Counting triples in two ways gives:

$$\begin{aligned} 680 &= 4|\mathcal{B}_4| + 10|\mathcal{B}_5| \qquad (2) \\ &= \sum_{x \in X} \deg_{\mathcal{B}_4}(x) + 10|\mathcal{B}_5| \\ &\geq 17(10) + 10|\mathcal{B}_5| \end{aligned}$$

Thus $|\mathcal{B}_5| \leq 50$, (equation (2) shows that $|\mathcal{B}_5|$ is even). Also since the derived design through x contains at most 32 blocks of size 3 we have

$$|\mathcal{B}_4| = \frac{1}{4} \sum_{x \in X} \deg_{\mathcal{B}_4}(x) \leq \frac{17 \cdot 32}{4} = 136.$$

As a prelude to our next Theorem we define the f-type (of the set of 4-blocks in a linear space with only 4-blocks and 3-blocks) to be the set of frequencies of the points among the 4-blocks. An f-type of, say F=10, $5^1 3^{10} 1^5$, means that there are 10 4-blocks and, among these blocks there is 1 point with frequency 5, 10 points with frequency 3, and 5 points with frequency 1. We list the f-types according to the number of 4-blocks F: F=4, 1^{16}; F=5, $5^1 1^{15}$ or $3^2 1^{14}$; F=6, $3^4 1^{12}$; F=7, $3^6 1^{10}$; F=8 does not occur; F=9, $3^{10} 1^6$; F=10, $5^1 3^{10} 1^5$; F=11, $5^1 3^{12} 1^3$ or $3^{14} 1^2$; F=12, $5^2 3^{12} 1^2$ or 3^{16}; F=15, $5^6 3^{10}$, F=20, 5^{16}.

Theorem 4.3. *An $S(3, \mathcal{K}, 17)$ has $|\mathcal{B}_5| \geq 18$.*

Proof. The maximum number of blocks in a derived linear space on 16 points is 36. Counting points in two ways yields: $4|\mathcal{B}_4| + 5|\mathcal{B}_5| \leq 17 \cdot 36 = 612$. Using equation (1), and since $|\mathcal{B}_5|$ is even, we get that $|\mathcal{B}_5| \geq 14$. Suppose now that $|\mathcal{B}_5| = 14$ and hence $|\mathcal{B}_4| = 135$. The average number of blocks in a derived linear space is $(5 \cdot 14 + 4 \cdot 135)/17 = 35\frac{15}{17}$. Thus at least 15 points have derived linear spaces with 36 blocks. Each linear space with 36 blocks has 4 disjoint 4-blocks. Assume (up to relabeling) that our $S(3, \{4,5\}, 17)$ has the following blocks: $\{0\ 1\ 2\ 3\ 16\}$, $\{4\ 5\ 6\ 7\ 16\}$, $\{8\ 9\ 10\ 11\ 16\}$, $\{12\ 13\ 14\ 15\ 16\}$. Up to relabeling we can assume that the linear spaces through the points 0,1,2,3 each have four 4-blocks. Easily this forces 3 more 5-blocks through each of 0,1,2,3 and these 12 blocks are distinct since otherwise a triple is covered twice. But then there are 3+3+3+3+4=16 5-blocks, a contradiction. Since $|\mathcal{B}_5|$ is even then $|\mathcal{B}_5| \geq 16$. So assume $|\mathcal{B}_5| = 16$ where $\mathcal{K} = \{4,5\}$. The average number of blocks in a derived linear space is $\frac{(5 \cdot 16 + 4 \cdot 130)}{17} = 35\frac{5}{17}$. So at least 5 points have derived linear spaces with F=4. We can assume that our $S(3, \mathcal{K}, 17)$ has the following blocks: $\{0\ 1\ 2\ 3\ 16\}$, $\{4\ 5\ 6\ 7\ 16\}$, $\{8\ 9\ 10\ 11\ 16\}$, $\{12\ 13\ 14\ 15\ 16\}$. Three cases arise from a second point whose linear space has F=4. This produces one of the following sets of 3 blocks: (1) $\{0\ 4\ 5\ 8\ 9\}$, $\{0\ 6\ 7\ 12\ 13\}$, $\{0\ 10\ 11\ 14\ 15\}$; (2) $\{0\ 4\ 5\ 8\ 9\}$, $\{0\ 6\ 10\ 12\ 13\}$, $\{0\ 7\ 11\ 14\ 15\}$; (3) $\{0\ 4\ 5\ 9\ 13\}$, $\{0\ 7\ 10\ 11\ 14\}$, $\{0\ 8\ 12\ 15\ 16\}$. In Case (1) the only points which can have derived linear space with F=4 are the points $\{0\ 1\ 2\ 3\ 16\}$. There were 15 nonisomorphic ways to select 3 more 5-blocks through point 1. Only one of these ways could be extended to a collection of 16 5-blocks whose derived linear space had the proper structure. This gave: $\{0\ 1\ 2\ 3\ 16\}$, $\{4\ 5\ 6\ 7\ 16\}$, $\{8\ 9\ 10\ 11\ 16\}$, $\{12\ 13\ 14\ 15\ 16\}$, $\{0\ 4\ 5\ 8\ 9\}$, $\{0\ 6\ 7\ 12\ 13\}$, $\{0\ 10\ 11\ 14\ 15\}$, $\{1\ 4\ 5\ 14\ 15\}$, $\{1\ 6\ 7\ 8\ 9\}$, $\{1\ 10\ 11\ 12\ 13\}$, $\{2\ 4\ 5\ 10\ 11\}$, $\{2\ 6\ 7\ 14\ 15\}$, $\{2\ 8\ 9\ 12\ 13\}$, $\{3\ 4\ 5\ 12\ 13\}$, $\{3\ 6\ 7\ 10\ 11\}$, $\{3\ 8\ 9\ 14\ 15\}$. An exhaustive computer search showed that these 16 5-blocks can extend only to the $S(3,5,17)$ which has 68 5-blocks. In each of Case (2) and Case (3) one can first show that (up to isomorphism) the derived linear space through point 1 has F=4. In Case (2) there were then 24 nonisomorphic ways (which did not include cases from Case (1) above) to get the 3 additional 5-blocks through point 1. In Case (3) there were 8 ways (avoiding cases from Case (1) and (2)) to get the 3 additional 5-blocks through point 1. None of these 32 cases could be completed to a set of 16 5-blocks whose derived linear spaces had the proper structure. Thus $|\mathcal{B}_5| \neq 16$ and so $|\mathcal{B}_5| \geq 18$. □

Conjecture 1. An $S(3, \mathcal{K}, 17)$ has $|\mathcal{B}_5| \geq 20$. Further, we believe that if $|\mathcal{B}_5| = 20$, then $\mathcal{B}_5$ is equivalent to the affine geometry AG(2,4). □

It is of interest to know the possible block sizes in a potential $S(4, \mathcal{K}, 18)$. Recall that $S(4,5,18)$ and $S(4,6,18)$ do not exist. The first violates the necessary conditions and the second was disposed of in [7].

Theorem 4.4. *Design D_1 does not extend to an $S(4, \mathcal{K}, 18)$. Thus any $S(4, \mathcal{K}, 18)$ has $\mathcal{K} = \{5,6\}$.*

Proof. Suppose D_1 extends to an $S(4, \mathcal{K}, 18)$ whose point set is $Y = \{0,1,...,17\}$. Now D_1 is the only $S(3, \mathcal{K}, 17)$ with blocks of size larger than 5 and such blocks comprise a $2-(16,6,2)$. Assume the points used in the $2-(16,6,2)$ are $\{1,2,...,16\}$ where 0 is the additional point in D_1. Note that if a point of Y is contained in a block of size 7 then it must be contained in exactly 16 such 7-blocks. Easily, point 0 will be contained in 7-blocks so that all points of Y are in 7-blocks. If b_7 is the number of 7-blocks in our $S(4, \mathcal{K}, 18)$ then counting points in two ways yields $7 \cdot b_7 = 18 \cdot 16$, a contradiction. □

5. AUTOMORPHISMS

Theorem 5.1. *Let p be a prime dividing $|G|$ where G is an automorphism group of a proper $S(3, \mathcal{K}, 17)$. Then p =2, 3, 5 or 17.*

Proof. Since G acts on 17 points the possible prime divisors of $|G|$ are 2, 3, 5, 7, 11, 13 or 17.

If $p = 17$, then an elementary backtracking search shows that there is a unique $S(3, \mathcal{K}, 17)$, namely the $S(3, 5, 17)$ design D_2 in the appendix. The automorphism group of this design has order $2^6 \cdot 3 \cdot 5 \cdot 17$. Note that the primes 2, 3, 5, and 17 do arise.

Let $p = 11$ or 13 and let $g = (1, 2, 3, \ldots, p)(p + 1)(p + 2) \ldots (17)$ be an automorphism of order p. Set $X = \{1, 2, \ldots, p\}$ and $Y = \{p + 1, \ldots, 17\}$. If a block B contains 3 points from Y, then B is a subset of Y. We still need blocks C where C has exactly 2 points from Y. Since $|C| \geq 4$, then C has at least 2 points from X. This leads to 2 distinct blocks containing the same 3–element set, a contradiction.

An element of order 7 cannot fix 10 points by using an argument similar to the above. So let $g = (1, 2, \ldots, 7)(8, 9, \ldots, 14)(15)(16)(17)$, then some block B contains the three fixed points 15, 16 and 17. Since B has at least 4 points we get an easy contradiction. □

An easy computer search establishes the following.

Theorem 5.2. *There is a unique $S(3, \mathcal{K}, 17)$ that has an automorphism of order 17. It is the Steiner system $S(3, 5, 17)$ and appears as design D_2 in the appendix.*

Theorem 5.3. *There are exactly 7 proper $S(3, \mathcal{K}, 17)'s$ with automorphism of order 5.*

Proof. Let g be an automorphism of a proper $S(3, \mathcal{K}, 17)$ where $|g| = 5$. It is easy to show that g fixes 2 points. A complete computer search was conducted and yielded exactly 6 nonisomorphic designs (in addition to D_1). The new associated group sizes were 16320, 5760, 320, 320, 60, and 60. These designs are listed in the appendix. Note that 5^2 does not divide the order of the automorphism groups. □

Suppose an element of order 3 fixed a set F of at least 8 points. Easily, 3 points from F force a block entirely contained in F. But then there is a subdesign $S(3, \mathcal{K}, |F|)$ which can be replaced by a single block consisting of precisely the points of F. But this is not possible since any $S(3, \mathcal{K}, 17)$ cannot have blocks of size larger than 6. Furthermore we have from Theorem 4.2, that design D_1 is the unique $S(3, \mathcal{K}, 17)$ containing a block of size 6. Thus assuming that there are no blocks of size 6 we see that an element of order 3 must fix either 2 or 5 points. An exhaustive computer search establishes the following:

Theorem 5.4. *There are exactly 28 proper $S(3,\mathcal{K},17)'s$ with an automorphism of order 3 with 2 fixed points and there are exactly 75 proper $S(3,\mathcal{K},17)'s$ with an automorphism of order 3 with 5 fixed points.*

TABLE II. Group sizes of the 28 designs with an element of order 3 with 2 fixed points.

Group order	18	48	60	96	144	192	288	384	1920	5760	16320
Number of $S(3,\mathcal{K},17)$'s	1	6	2	8	3	1	2	2	1	1	1

TABLE III. Group sizes of the 75 designs with an element of order 3 with 5 fixed points.

Group order	3	6	12	18	24	48	96	144	192	288	384	5760
Number of $S(3,\mathcal{K},17)$'s	3	10	3	1	5	29	15	3	2	2	1	1

There are 7 designs, namely those with group orders divisible by 9, which have automorphisms of order 3 of both types (2 and 5 fixed points).

Theorem 5.5. *If g is an order 2 automorphism of an $S(3,\mathcal{K},17)$, then g has 1 or 5 fixed points.*

Proof. Let g be an order 2 automorphism of an $S(3,\mathcal{K},17)$ design $(X,\mathcal{B})$ fixing f points. If a is a fixed point, then the derived design structure with respect to a is a linear space on $X\setminus\{a\}$ that has an automorphism of order 2 fixing $f-1$ points. In [1] the linear spaces are determined and their automorphism groups may be calculated. We found that the automorphisms of order 2 in these groups either fix 0, 2, 4 or 6 points. Hence $f\in\{1,3,5,7\}$.

An $S(3,\mathcal{K},17)$ design having an order 2 automorphism fixing 1 point is D_1 and one with an order 2 automorphism that fixes 5 points is D_2.

Suppose g is an order 2 automorphism of an $S(3,\mathcal{K},17)$ design$(X,\mathcal{B})$ fixing 3 points. Then without loss we may assume that

$$g=(a)(b)(c)(0,1)(2,3)(4,5)(6,7)(8,9)(10,11)(12,13).$$

Let $X_j=\{2j,2j+1\}$, $j=0,1,2,\ldots,6$. There are 22 triples fixed by g and any block containing a fixed 3-subset must be fixed by g. There are thus 4 types of fixed blocks:

1. $X_{j_1}\cup X_{j_2}$, $j_1\neq j_2$;
2. $X_{j_1}\cup\{x,y\}$, $x,y\in\{a,b,c\}$, $x\neq y$;
3. $X_{j_1}\cup X_{j_2}\cup\{x\}$, $j_1\neq j_2$, $x\in\{a,b,c\}$; and
4. $X_{j_1}\cup\{a,b,c\}$.

Let N_i, $i=1,2,3,4$, be the number of blocks in $\mathcal{B}$ of type i. Then counting the number of fixed triples consisting of a fixed point and a 2-cycle in two ways we have

$$2N_2+2N_3+3N_4=21 \tag{3}$$

There are 658 triples that are not fixed by g. They are paired into 329 orbits of length 2. These orbits must also be covered and the only baseblocks with an odd number of full orbits contain either 3 or 2 fixed points. These are thus fixed blocks and consequently $N_{\mathbf{2}} + N_{\mathbf{4}} \equiv 1 \pmod 2$, but $N_{\mathbf{4}} = 1$ since any such 5-set contains the triple $\{a, b, c\}$. So $N_{\mathbf{2}}$ is even. The resulting 5 solutions to equation (3) are given in Table IV.

TABLE IV. The 5 solutions to equation (3).

Solution:	1	2	3	4	5
$N_{\mathbf{2}}$:	0	2	4	6	8
$N_{\mathbf{3}}$:	9	7	5	3	1
$N_{\mathbf{4}}$:	1	1	1	1	1

Let $B^\star = \{a, b, c, 0, 1\}$ be the unique type **4** block. The type **3** blocks induce a graph Γ on the vertices $X_1, \ldots, X_6$. The pair $\{X_i, X_j\}$ is an edge labeled u if $X_i \cup X_j \cup \{u\}$ is a block, $u \in \{a, b, c\}$. We label the vertex X_i by $\{u_1, u_2\}$ if $X_i \cup \{u_1, u_2\}$ is a block otherwise we label it by the empty set. It is now easy to see that the labels incident to a vertex together with the vertex label is a partition of the fixed points. Hence the degree of every vertex of Γ is 1, 2 or 3, $N_{\mathbf{3}}$ is the number of edges and $N_{\mathbf{2}}$ is the number of vertices of degree 1. Thus $N_{\mathbf{2}} \leq 6$ and it is easy to see that there are 6 graphs satisfying these conditions. The first 5 can be labeled uniquely (up to isomorphism) the last has three labelings. They are given in Table V. An exhaustive computer search showed that none of these graphs extends to an $S(3, \{4, 5\}, 17)$.

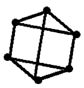
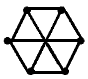
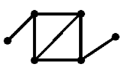
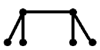
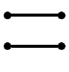
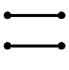
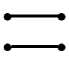
Suppose g is an order 2 automorphism of an $S(3, \mathcal{K}, 17)$ design $(X, \mathcal{B})$ fixing 7 points. Without loss

$$g = (0,1)(2,3)(4,5)(6,7)(8,9)(10)(11)(12)(13)(14)(15)(16).$$

The derived design with respect to 16 is a linear space with an order 2 automorphism fixing the 6 points 10, 11, 12, 13, 14 and 15. Careful examination of the automorphism groups of the 398 proper linear spaces on 16 points yields 7 such linear spaces. Applying Theorems 4.1 and 4.2 and the fact that the order 2 automorphisms of D_1 each fix exactly one point we need only consider linear spaces having blocks of size 3 and 4 only. Among the 7 only *four* of them are of this type. They are given in Table VI. Consider just the blocks of size 5. In each of the four linear spaces they are the same and they contain exactly 3 blocks of size four with two fixed points. Thus $\{B \cap \{10, 11, 12, \ldots, 16\} : B \in \mathcal{B}_5\}$ is an $S(2, 3, 7)$ the Fano plane. Consequently there are 7 blocks of size 5 that contain 3 fixed points. But any block containing three fixed points must be fixed by g and so the other two points are a 2-cycle. There are then 5 2-cycles to be distributed into the 7 blocks of size 5 that each contain exactly 3 fixed points. It is impossible to do this without covering a triple twice since the fixed points in these 7 blocks form a Fano plan. □

Theorem 5.6. *There is an* $S(3, \mathcal{K}, 17)$ *with* $|G| = n$ *if and only if* $n \in \{2^a 3^b : 0 \leq a \leq 7, 0 \leq b \leq 1\} \cup \{18, 60, 144, 288, 320, 1920, 5760, 16320\}$.

TABLE V. The possible configurations of fixed blocks under an automorphism of order 2 that fixes 3 points

	{a 2 3 4 5}, {a 6 7 8 9}, {a 10 11 12 13}, {b 2 3 6 7}, {b 4 5 10 11}, {b 8 9 12 13}, {c 2 3 12 13}, {c 4 5 6 7}, {c 8 9 10 11}, {a b c 0 1}
	{a 2 3 4 5}, {a 6 7 8 9}, {a 10 11 12 13}, {b 2 3 8 9}, {b 4 5 10 11}, {b 6 7 12 13}, {c 2 3 12 13}, {c 4 5 6 7}, {c 8 9 10 11}, {a b c 0 1}
	{a 2 3 4 5}, {a 6 7 8 9}, {a 10 11 12 13}, {b 2 3 6 7 }, {b 4 5 8 9}, {c 2 3 8 9}, {c 4 5 6 7}, {b c 10 11}, {b c 12 13}, {a b c 0 1}
	{a 2 3 4 5}, {a 6 7 8 9}, {a 10 11 12 13}, {b 2 3 6 7}, {b 4 5 10 11 }, {c 2 3 10 11}, {c 4 5 6 7}, {b c 8 9 }, {b c 12 13}, {a b c 0 1}
	{a 2 3 4 5}, {a 6 7 8 9}, {b 2 3 10 11}, {b 6 7 12 13}, {c 2 3 6 7}, {a c 10 11}, {a c 12 13}, {b c 4 5}, {b c 8 9 }, {a b c 0 1}
	{a 2 3 4 5}, {a 6 7 8 9}, {a 10 11 12 13}, {b c 2 3 }, {b c 4 5}, {b c 6 7}, {b c 8 9}, {b c 10 11}, {b c 12 13}, {a b c 0 1}
	{a 2 3 4 5}, {a 6 7 8 9}, {b 10 11 12 13}, {b c 2 3 }, {b c 4 5}, {b c 6 7}, {b c 8 9}, {a c 10 11}, {a c 12 13}, {a b c 0 1}
	{a 2 3 4 5}, {b 6 7 8 9}, {c 10 11 12 13}, {b c 2 3}, {b c 4 5}, {a c 6 7}, {a c 8 9}, {a b 10 11}, {a b 12 13}, {a b c 0 1}

Proof. By previous theorems we know all $S(3,\mathcal{K},17)'s$ for which either 3, 5, or 17 divides $|G|$. If H is a 2-sylow subgroup of G, then the lengths of the point orbits of H must be powers of 2 and these lengths must sum to the odd number 17. Thus H must fix at least one point, say x. The automorphism group of the derived linear space through x contains H. An inventory of the groups of the 398 nontrivial linear spaces on 16 points gives groups of order 1, 2, 3, 4, 5, 6, 8, 12, 16, 18, 20, 24, 32, 40, 48, 60, 72, 96, 120, 600, 360, 5760. Easily $|H| \leq 2^7$. So the only additional sizes for $|G|$ that are not divisible by 3, 5, or 17 are $\{2^a : 0 \leq a \leq 7\}$. Hence

$$|G| \in \mathcal{N} = \{1,2,3,4,6,8,12,16,18,24,32,48,60,64,96,128, \\ 144,192,288,320,384,1920,5760,16320\}.$$

For each $n \in \mathcal{N}$ we provide, in the Appendix, at least one example of an $S(3,\mathcal{K},17)$ whose automorphism group G has $|G| = n$. $\square$

TABLE VI. The four nonisomorphic linear spaces with block sizes 3 and 4 only that have an automorphism of order 2 that fixes 6 points.

1. {1 2 4 10}, {0 3 5 10}, {6 7 10 15}, {1 3 8 15}, {0 1 11 12}, {2 3 13 14}, {0 2 9 15}, {8 9 10}, {10 11 13}, {10 12 14}, {1 5 6}, {1 7 13}, {1 9 14}, {2 5 11}, {2 6 8}, {2 7 12}, {3 4 11}, {0 4 7}, {4 5 15}, {4 6 14}, {4 8 12}, {4 9 13}, {3 6 12}, {3 7 9}, {0 6 13}, {0 8 14}, {5 7 14}, {5 8 13}, {5 9 12}, {11 14 15}, {12 13 15}, {6 9 11}, {7 8 11}.

2. {1 2 4 10}, {0 3 5 10}, {6 7 10 15}, {1 3 8 15}, {0 1 11 12}, {2 3 13 14}, {0 2 9 15}, {8 9 10}, {10 11 13}, {10 12 14}, {1 5 6}, {1 7 14}, {1 9 13}, {2 5 11}, {2 6 8}, {2 7 12}, {3 4 11}, {0 4 7}, {4 5 15}, {4 6 13}, {4 8 12}, {4 9 14}, {3 6 12}, {3 7 9}, {0 6 14}, {0 8 13}, {5 7 13}, {5 8 14}, {5 9 12}, {11 14 15}, {12 13 15}, {6 9 11}, {7 8 11}.

3. {1 2 4 10}, {0 3 5 10}, {6 7 10 15}, {1 3 8 15}, {0 1 11 12}, {2 3 13 14}, {0 2 9 15}, {8 9 10}, {10 11 13}, {10 12 14}, {1 5 6}, {1 7 13}, {1 9 14}, {2 5 11}, {2 6 8}, {2 7 12}, {3 4 11}, {0 4 7}, {4 5 15}, {4 6 14}, {4 8 13}, {4 9 12}, {3 6 12}, {3 7 9}, {0 6 13}, {0 8 14}, {5 7 14}, {5 8 12}, {5 9 13}, {11 14 15}, {12 13 15}, {6 9 11}, {7 8 11}.

4. {1 2 4 10}, {0 3 5 10}, {6 7 10 15}, {1 3 8 15}, {0 1 11 12}, {2 3 13 14}, {0 2 9 15}, {8 9 10}, {10 11 13}, {10 12 14}, {1 5 6}, {1 7 14}, {1 9 13}, {2 5 11}, {2 6 8}, {2 7 12}, {3 4 11}, {0 4 7}, {4 5 15}, {4 6 13}, {4 8 14}, {4 9 12}, {3 6 12}, {3 7 9}, {0 6 14}, {0 8 13}, {5 7 13}, {5 8 12}, {5 9 14}, {11 14 15}, {12 13 15}, {6 9 11}, {7 8 11}.

6. $S(3,\mathcal{K},17)$'S WITH INTERESTING SUBDESIGNS; SOME $S(3,\mathcal{K},18)$'S

In [2] there are $S(3,\mathcal{K},16)'s$ which contain semi-biplanes (see [6]). Briefly, a *semi-biplane* ($sbp(v,k)$) is a collection of v blocks of size k on a v-set, such that any unordered pair of points occurs in either 0 or 2 blocks. A *biplane* $bp(v,k)$ is a semi-biplane in which every pair of points occurs in exactly 2 blocks. Thus a $bp(v,k)$ is a $2-(v,k,2)$ design. For $v \leq 16$ it is known (see [6]) that there is a unique biplane bp(11,5), there are 3 biplanes bp(16,6), a unique sbp(12,5), and a unique sbp(16,5). Among the 3 biplanes on 16 points one is *special*. The blocks of this design can be thought of as the 16 subgraphs of $K_{4,4}$ isomorphic to two disjoint $K_{1,3}$'s, the points are the 16 edges of $K_{4,4}$. We found:

Theorem 4.2′. *The only $S(3,\mathcal{K},17)$ which contains a bp(16,6) is D_1 and the biplane it contains is the special one.*

Proof. This is just a rephrasing of Theorem 4.2 in the language of biplanes. □

Theorem 6.1. *There are exactly fourteen proper $S(3,\mathcal{K},17)$'s which contain an sbp(16,5).*

Proof. A complete search was done for such proper $S(3,\mathcal{K},17)'s$. The nonisomorphic solutions are precisely the designs in the appendix with $n_5 = 68, 52$ or 36. □

By a complete search we found the following two results:

Theorem 6.2. *There are no $S(3,\mathcal{K},17)'s$ which contain an sbp(14,5).*

Theorem 6.3. *The only two $S(3,\mathcal{K},17)'s$ which contain an sbp(12,5) are D_2 (the $S(3,5,17)$) and D_{16} (see appendix).*

If an $S(3,\mathcal{K},v)$ contains an $S(2,\mathcal{K},v)$ as a subdesign it can be extended to an $S(3,\mathcal{K},v+1)$ by adding a new point to each of the blocks of the $S(2,\mathcal{K},v)$. We call this process *subdesign extension*. In [2] the authors determined all $S(3,\mathcal{K},16)'s$ with at least two block sizes. One such design D_5 (in [2]) is of type $4^{60}6^{16}$ and contains both a $bp(16,6)$ and an $AG(2,4)$. Note that the *affine geometry* $AG(2,4)$ is an $S(2,4,16)$ design and is unique up to isomorphism. Adding a point to the blocks of the $AG(2,4)$ of this $S(3,\mathcal{K},16)$ produces design D_1 in the appendix.

All $S(3,\mathcal{K},16)$'s with $|\mathcal{K}|>1$ appear in [2] and these were searched for sub $AG(2,4)'s$. We get:

Theorem 6.4. *The only $S(3,\mathcal{K},17)'s$ which are extensions of $AG(2,4)$, but do not come directly from an $S(3,4,16)$, are the 15 designs $D_1, D_2, ..., D_{15}$.*

An $S(2,\mathcal{K},17)$ can be found as a subdesign of D_i for $i \in \{1\}\cup\{6,...,11\}\cup\{17,...,33\}$. In 3 cases it can be found in two nonisomorphic ways and in 3 cases it can be found in three nonisomorphic ways. Each $S(2,\mathcal{K},17)$ is isomorphic to the $S(2,\{4,5\},17)$ obtained by deleting the points forming a line in the projective plane of order 5 from the points and blocks of the plane. The resulting 22 designs found by this subdesign extension are given in the appendix. The structure of these $S(2,\{4,5\},17)$'s shows that further subdesign extension is impossible. Thus none of the 33 designs with $v=18$ can contain an $S(2,\mathcal{K},18)$.

7. ARE THERE ANY $S(4,\mathcal{K},18)$'S

In the last column of the summary table we have indicated some cases that are known to not extend to an $S(4,\mathcal{K},18)$. As a consequence of the nonextendability of the $S(3,5,17)$ design D_2, we have:

Theorem 7.1. *If an $S(4,5,17)$ exists, then it cannot contain an $S(3,5,17)$.*

Proof. The unique $S(3,5,17)$ design is D_2. If an $S(4,5,17)$ contained an $S(3,5,17)$ as a subdesign, then adding an 18th point to each block of the subdesign $S(3,5,17)$ would produce an $S(4,\mathcal{K},18)$ which would be an extension of D_2, a contradiction. □

In the following set of 24 5-blocks the first 20 are equivalent to an AG(2,4) and the additional (unique up to isomorphism) 4 5-blocks contain 1, 2, 17: {0 1 2 3 4 17}, {0 1 5 6 13 17}, {0 1 7 9 16 17}, {0 1 8 10 11 17}, {0 1 12 14 15 17}, {0 2 5 7 11 17}, {0 2 6 10 12 17}, {0 2 8 9 15 17}, {0 2 13 14 16 17}, {0 3 5 8 14 17}, {0 3 6 7 15 17}, {0 3 9 10 13 17}, {0 3 11 12 16 17}, {0 4 5 9 12 17}, {0 4 6 8 16 17}, {0 4 7 10 14 17}, {0 4 11 13 15 17}, {0 5 10 15 16 17}, {0 6 9 11 14 17}, {0 7 8 12 13 17}, plus

TABLE VII. Summary data.

	$k=4$	$k=5$	$k=6$	2	3	5	17	$\|G\|$	bp's and sbp's	Extends AG(2,4)	Subdes. exts.	Extends to a 4BD
D_1	40	20	16	2^7	3	5	1	1920	special bp(16,6)	yes	1	no
D_2	0	68	0	2^6	3	5	17	16320	sbp(16,5), sbp(12,5)	yes	0	no
D_3	40	52	0	2^6	1	5	1	320	sbp(16,5)	yes	0	no
D_4	40	52	0	2^6	1	1	1	64	sbp(16,5)	yes	0	no
D_5	40	52	0	2^5	1	1	1	32	sbp(16,5)	yes	0	no
D_6	80	36	0	2^6	1	5	1	320	sbp(16,5)	yes	1	no
D_7	80	36	0	2^6	1	1	1	64	sbp(16,5)	yes	1	no
D_8	80	36	0	2^5	1	1	1	32	sbp(16,5)	yes	1	no
D_9	80	36	0	2^4	1	1	1	16	sbp(16,5)	yes	1	no
D_{10}	80	36	0	2^4	1	1	1	16	sbp(16,5)	yes	1	no
D_{11}	80	36	0	2^4	1	1	1	16	sbp(16,5)	yes	1	no
D_{12}	80	36	0	2^4	1	1	1	16	sbp(16,5)	yes	0	no
D_{13}	80	36	0	2^3	1	1	1	8	sbp(16,5)	yes	0	no
D_{14}	80	36	0	2^3	1	1	1	8	sbp(16,5)	yes	0	no
D_{15}	80	36	0	2^2	1	1	1	4	sbp(16,5)	yes	0	no
D_{16}	100	28	0	2^2	3	5	1	60	sbp(12,5)	no	0	no
D_{17}	120	20	0	2^7	3^2	5	1	5760		yes	1	?
D_{18}	120	20	0	2^7	3	1	1	384		yes	3	?
D_{19}	120	20	0	2^5	3^2	1	1	288		yes	1	?
D_{20}	120	20	0	2^6	3	1	1	192		yes	2	?
D_{21}	120	20	0	2^4	3^2	1	1	144		yes	1	?
D_{22}	120	20	0	2^7	1	1	1	128		yes	3	?
D_{23}	120	20	0	2^5	3	1	1	96		yes	1	?
D_{24}	120	20	0	2^2	3	5	1	60		yes	3	?
D_{25}	120	20	0	2^4	3	1	1	48		yes	0	?
D_{26}	120	20	0	2^3	3	1	1	24		yes	0	?
D_{27}	120	20	0	2	3^2	1	1	18		yes	0	?
D_{28}	120	20	0	2^2	3	1	1	12		yes	0	?

TABLE VII. (continued)

	$k=4$	$k=5$	$k=6$	2	3	5	17	$\lvert G\rvert$	bp's and sbp's	Extends AG(2,4)	Subdes. exts.	Extends to a 4BD
D_{29}	120	20	0	2	3	1	1	6		yes	0	?
D_{30}	120	20	0	1	3	1	1	3		yes	0	?
D_{31}	120	20	0	2	1	1	1	2		yes	0	?
D_{32}	120	20	0	2	1	1	1	2		yes	0	?
D_{33}	120	20	0	1	1	1	1	1		yes	0	?

{1 2 5 8 16 17}, {1 2 6 7 14 17}, {1 2 9 11 12 17}, {1 2 10 13 15 17}. A complete search showed that these 24 blocks do not extend to an $S(4, \mathcal{K}, 18)$.

Theorem 7.2. *No $S(4, \mathcal{K}, 18)$ can have two or more derived point extensions of AG(2,4).*

We are prepared to make the following:

Conjecture 1. *There is no $S(4, \mathcal{K}, 18)$.* □

We also conjecture:

Conjecture 2. *An $S(3, \mathcal{K}, 17)$ is one of $D_1, \ldots, D_{16}$, or it extends an $S(3, 4, 16)$ containing an $AG(2, 4)$.* □

ACKNOWLEDGMENTS

The authors would like to thank Ghislaine Heathcote for providing the data on linear spaces of order 16. Her encouragement during this project was also greatly appreciated. The research of the first author was supported by NSA grant MDA904-93-H-3049 and by the Center for Communication and Information Science at the University of Nebraska. The research of the third author was supported by NSERC of Canada, Grant No. OGP008651.

APPENDIX

This appendix provides generating information for specific designs. A design of type $4^{40}5^{20}6^{16}$ means there are 40 4-blocks, 20 5-blocks and 16 6-blocks. A given subgroup H (G will be the full automorphism group of the design) will act on the listed base blocks to generate the design. Any additional generators to produce G are mentioned, along with $|G|$. When there is a subdesign $S(2, \mathcal{K}, 17)$, which then gives an $S(3, \mathcal{K}, 18)$, we list one such $S(2, \mathcal{K}, 17)$ for each nonisomorphic $S(3, \mathcal{K}, 18)$.

$\mathbf{D}_1$: The unique $S(3, \mathcal{K}, 17)$ of type $4^{40}5^{20}6^{16}$.

$H = \langle(1\ 2\ 3\ 4\ 5\ 6\ 7\ 8)(9\ 10\ 11\ 12\ 13\ 14\ 15\ 16)\rangle$, $|H| = 8$.

Base blocks: {1 2 3 6 12 14}, {1 3 9 10 13 16}, {0 1 2 7 9}, {0 1 5 10 14}, {0 1 12 13 15}, {1 2 4 16}, {1 2 10 15}, {1 3 5 7}, {1 4 13 14}, {1 5 12 16}, {1 9 11 12}, {9 11 13 15}.

$G = \langle$H, (1 3 13 16 10 9)(2 4 6)(5 14 15 12 7 11)$\rangle$, $|G| = 1920$.

This is the unique $S(3, \mathcal{K}, 17)$ with a block of size 6. The group G is 2-transitive group on 16 points and fixes 0. The base blocks of size 6 form the special biplane $bp(16, 6)$, whose group has order 11520.

Sub S(2,$\mathcal{K}$,17): Use H with base blocks: {0 1 5 10 14}, {1 2 4 16}, {1 9 11 12}.

$\mathbf{D}_2$: The unique S(3,5,17) of type 5^{68}.

$H = \langle$(0 7 14 1 8 10 2 9 11 3 5 12 4 6 13)$\rangle$, $|H| = 15$.

Base blocks: {0 1 2 3 4}, {0 1 5 6 13}, {0 1 7 9 16}, {0 1 12 14 15}, {0 2 5 7 11}, {0 5 10 15 16}.

$G = \langle$ H, (0 1 2 8 6 15 3 5 13 10 7 9 12 4 16 11 14), (1 2 4 3)(6 7 9 8)(11 12 14 13)(15 16)$\rangle$, $|G| = 16320$.

$\mathbf{D}_3$: An $S(3, \mathcal{K}, 17)$ of type $4^{40}5^{52}$.

$H = \langle$(0 1 2 3 4)(5 6 7 8 9)(10 11 12 13 14), (0 4)(1 3)(5 9)(6 8)(10 14)(11 13)$\rangle$, $|H| = 10$.

Base blocks: {0 1 2 3 4}, {0 1 5 6 13}, {0 1 7 9 16}, {0 1 8 10 11}, {0 1 12 14 15}, {0 2 5 7 11}, {0 2 13 14 16}, {0 5 10 15 16}, {0 6 9 11 14}, {5 6 7 8 9}, {5 6 12 14 16}, {5 7 13 14 15}, {0 2 6 15}, {0 2 8 12}, {0 6 7 12}, {0 7 8 15}, {0 10 12 13}, {5 10 11 14}.

$G = \langle$ H, (0 1 2 7 5 12 3 14)(4 6 11 10 13 9 8 15)$\rangle$, $|G| = 320$.

$\mathbf{D}_4$: An $S(3, \mathcal{K}, 17)$ of type $4^{40}5^{52}$.

$H = \langle$(0 2 12 9 7 14 15 5)(1 10 8 13 4 6 11 3), (0 8)(1 15)(2 3)(4 12)(5 6)(7 11)(9 10)(13 14)$\rangle$, $|H| = 16$.

Base blocks: {0 1 2 3 4}, {0 1 10 13 16}, {0 2 5 8 9}, {0 7 8 11 16}, {0 1 8 12}, {0 1 9 14}, {0 6 9 11}.

$G = \langle$ H, (0 2)(1 5)(3 8)(4 9)(6 15)(7 14)(10 12)(11 13)$\rangle$, $|G| = 64$.

$\mathbf{D}_5$: An $S(3, \mathcal{K}, 17)$ of type $4^{40}5^{52}$.

$H = \langle$(0 11 7 8)(2 5 13 6)(3 10 14 9)(12 15), (1 12 4 15)(2 10 3 5)(6 13 9 14)(8 11)$\rangle$, $|H| = 16$.

Base blocks: {0 1 2 3 4}, {0 1 5 6 7}, {0 1 10 13 16}, {0 2 5 8 9}, {0 7 8 11 16}, {1 2 5 14 15}, {1 4 12 15 16}, {2 3 13 14 16}, {0 1 8 12}, {0 1 9 14}, {0 6 9 11}, {1 3 12 14}, {2 3 5 6}.

$G = \langle$ H, (0 1)(2 5)(3 6)(4 7)(8 15)(9 14)(10 13)(11 12)$\rangle$, $|G| = 32$.

$\mathbf{D}_6$: An $S(3, \mathcal{K}, 17)$ of type $4^{80}5^{36}$.

$H = \langle$(0 1 2 3 4)(5 6 7 8 9)(10 11 12 13 14), (0 4)(1 3)(5 9)(6 8)(10 14)(11 13)$\rangle$, $|H| = 10$.

Base blocks: {0 1 7 9 16}, {0 2 6 10 12}, {0 2 8 9 15}, {0 2 13 14 16}, {0 5 10 15 16}, {0 7 8 12 13}, {5 6 12 14 16}, {10 11 12 13 14}, {0 1 2 11}, {0 1 3 8}, {0 1 5 12}, {0 1 13 15}, {0 5 6 9}, {0 5 7 14}, {0 6 11 13}, {0 7 10 11}, {0 11 14 15}, {5 6 8 13}, {5 7 11 15}, {5 12 13 15}.

$G = \langle$ H, (0 1 2 7 5 12 3 14)(4 6 11 10 13 9 8 15)$\rangle$, $|G| = 320$.

Sub S(2,$\mathcal{K}$,17): Use $K = \langle$(0 1 2 7 5 12 3 14)(4 6 11 10 13 9 8 15)$\rangle$ with base blocks: {0 5 10 15 16}, {0 1 3 8}, {0 6 11 13}.

$\mathbf{D}_7$: An $S(3, \mathcal{K}, 17)$ of type $4^{80}5^{36}$.

$H = \langle$(0 3 2 15 14 11 7 10)(1 12 5 13 9 6 4 8), (0 1)(2 5)(3 6)(4 7)(8 15)(9 14)(10 13)(11 12)$\rangle$, $|H| = 16$.

Base blocks: {0 1 2 3 4}, {0 1 9 14 16}, {0 2 6 15 16}, {0 1 8 13}, {0 1 11 12}, {0 2 7 14}, {0 2 10 12}, {0 2 11 13}, {0 3 9 11}, {0 4 5 14}.

$G = \langle$ H,(0 1 14 9)(3 15 12 8)(4 5)(6 13 11 10)$\rangle$, $|G| = 64$

Sub S(2,$\mathcal{K}$,17): Use H with base blocks: {0 1 9 14 16}, {0 2 10 12}.

$\mathbf{D}_8$: An $S(3, \mathcal{K}, 17)$ of type $4^{80}5^{36}$.

$H = \langle$(0 2 14 7)(1 5 9 4)(3 15 11 10)(6 8 12 13), (0 3 1 6)(2 13 5 10)(4 15 7 8)(9 12 14 11)$\rangle$, $|H| = 16$.

Base blocks: {0 1 2 3 4}, {0 1 9 14 16}, {0 2 6 15 16}, {0 4 8 12 16}, {0 1 8 15}, {0 1 10 13}, {0 1 11 12}, {0 2 7 14}, {0 2 10 12}, {0 2 11 13}, {0 3 5 15}, {0 3 9 11}, {0 4 5 14}, {0 4 11 15}, {0 8 13 14}, {0 9 13 15}, {0 10 14 15}.

$G = \langle$ H, (2 7)(3 6)(4 5)(8 10)(11 12)(13 15)$\rangle$, $|G| = 32$.

Sub S(2,$\mathcal{K}$,17): Use H with base blocks: {0 1 9 14 16}, {0 2 10 12}, {0 3 5 15}.

$\mathbf{D}_9$: An $S(3,\mathcal{K},17)$ of type $4^{80}5^{36}$.

$H = \langle(2\ 7)(3\ 6)(4\ 5)(8\ 10)(11\ 12)(13\ 15),\ (0\ 1)(2\ 4)(5\ 7)(8\ 13)(9\ 14)(10\ 15),$
$(0\ 2\ 9\ 5)(1\ 4\ 14\ 7)(3\ 12\ 11\ 6)(10\ 15)\rangle$, $|H| = 16$.

Base blocks: {0 1 2 3 4}, {0 1 9 14 16}, {0 2 5 8 9}, {0 2 6 15 16}, {0 3 6 8 10}, {3 6 11 12 16}, {8 10 13 15 16}, {0 1 8 13}, {0 1 11 12}, {0 2 7 14}, {0 2 10 12}, {0 2 11 13}, {0 3 9 12}, {0 3 11 14}, {0 8 14 15}, {0 9 13 15}, {3 8 11 13}.

$G = H$, $|G| = 16$.

Sub S(2,$\mathcal{K}$,17): Use H with base blocks: {0 1 9 14 16}, {0 2 10 12}.

$\mathbf{D}_{10}$: An $S(3,\mathcal{K},17)$ of type $4^{80}5^{36}$.

$H = \langle(0\ 11\ 1\ 12)(2\ 8\ 5\ 15)(3\ 9\ 6\ 14)(4\ 10\ 7\ 13),\ (0\ 3\ 1\ 6)(2\ 13\ 5\ 10)(4\ 15\ 7\ 8)(9\ 12\ 14\ 11),$
$(0\ 1)(2\ 4)(5\ 7)(8\ 13)(9\ 14)(10\ 15)\rangle$, $|H| = 16$.

Base blocks: {0 1 2 3 4}, {0 1 9 14 16}, {0 2 5 8 9} {0 2 6 15 16}, {0 4 8 12 16}, {2 4 5 7 16}, {0 1 8 15}, {0 1 11 12}, {0 2 7 14}, {0 2 10 12}, {0 2 11 13}, {0 3 5 15}, {0 3 9 12}, {0 3 11 14}, {0 4 5 14}, {0 4 11 15}, {0 8 13 14}, {0 9 13 15}, {2 4 8 13}, {2 5 10 13}.

$G = H$, $|G| = 16$.

Sub S(2,$\mathcal{K}$,17): Use H with base blocks: {0 1 9 14 16}, {2 4 5 7 16}, {0 2 10 12}, {0 3 5 15}.

$\mathbf{D}_{11}$: An $S(3,\mathcal{K},17)$ of type $4^{80}5^{36}$.

$H = \langle(2\ 7)(3\ 6)(4\ 5)(8\ 10)(11\ 12)(13\ 15),\ (0\ 1)(2\ 4)(5\ 7)(8\ 13)(9\ 14)(10\ 15),$
$(0\ 3)(1\ 6)(2\ 8)(4\ 10)(5\ 15)(7\ 13)(9\ 12)(11\ 14),\ (0\ 9)(1\ 14)(2\ 5)(3\ 11)(4\ 7)(6\ 12)\rangle$, $|H| = 16$.

Base blocks: {0 1 2 3 4}, {0 1 9 14 16}, {0 2 5 8 9}, {0 2 6 15 16}, {0 4 8 12 16}, {2 4 5 7 16}, {0 1 8 13}, {0 1 11 12}, {0 2 7 14}, {0 2 10 12}, {0 2 11 13}, {0 3 5 15}, {0 3 9 12}, {0 3 11 14}, {0 4 5 14}, {0 4 11 15}, {0 8 14 15}, {0 9 13 15}, {2 4 8 13}, {2 5 10 13}.

$G = H$, $|G| = 16$.

Sub S(2,$\mathcal{K}$,17): Use H with base blocks: {0 1 9 14 16}, {2 4 5 7 16}, {0 2 10 12}, {0 3 5 15}, {2 9 11 15}.

$\mathbf{D}_{12}$: An $S(3,\mathcal{K},17)$ of type $4^{80}5^{36}$.

$H = \langle(0\ 1)(2\ 5)(3\ 6)(4\ 7)(8\ 15)(9\ 14)(10\ 13)(11\ 12),$
$(0\ 3)(1\ 6)(2\ 8)(4\ 10)(5\ 15)(7\ 13)(9\ 12)(11\ 14),\ (0\ 2)(1\ 5)(3\ 8)(4\ 9)(6\ 15)(7\ 14)(10\ 12)(11\ 13),$
$(0\ 4)(1\ 7)(2\ 9)(3\ 10)(5\ 14)(6\ 13)(8\ 12)(11\ 15)\rangle$, $|H| = 16$.

Base blocks: {0 1 2 3 4}, {0 1 9 14 16}, {0 2 6 15 16}, {0 3 7 13 16}, {0 4 8 12 16}, {0 5 10 11 16}, {0 1 8 15}, {0 1 10 12}, {0 2 7 14}, {0 3 5 15}, {0 3 9 11}, {0 4 5 14}, {0 4 6 13}, {0 4 11 15}, {0 5 12 13}, {0 7 8 11}, {0 7 12 15}, {0 8 13 14}, {0 9 13 15}, {0 10 14 15}.

$G = H$, $|G| = 16$.

$\mathbf{D}_{13}$: An $S(3,\mathcal{K},17)$ of type $4^{80}5^{36}$.

$H = \langle(0\ 1)(2\ 5)(3\ 6)(4\ 7)(8\ 15)(9\ 14)(10\ 13)(11\ 12),$
$(0\ 3)(1\ 6)(2\ 8)(4\ 10)(5\ 15)(7\ 13)(9\ 12)(11\ 14),\ (0\ 9)(1\ 14)(2\ 4)(3\ 12)(5\ 7)(6\ 11)(8\ 10)(13\ 15)\rangle,$
$|H| = 8$.

Base blocks: {0 1 2 3 4}, {0 1 9 14 16}, {0 2 5 8 9}, {0 2 6 15 16}, {0 3 7 13 16}, {0 4 8 12 16}, {0 5 10 11 16}, {2 4 5 7 16}, {0 1 8 15}, {0 1 10 13}, {0 1 11 12}, {0 2 7 12}, {0 2 10 14}, {0 2 11 13}, {0 3 5 15}, {0 3 9 12}, {0 3 11 14}, {0 4 5 14}, {0 4 6 13}, {0 4 11 15}, {0 5 12 13}, {0 6 9 11}, {0 6 12 14}, {0 7 8 11}, {0 8 13 14}, {0 9 13 15}, {2 4 8 13}, {2 5 10 13}.

$G = H$, $|G| = 8$.

$\mathbf{D}_{14}$: An $S(3,\mathcal{K},17)$ of type $4^{80}5^{36}$.

$H = \langle(0\ 1)(2\ 5)(3\ 6)(4\ 7)(8\ 15)(9\ 14)(10\ 13)(11\ 12),$
$(0\ 3)(1\ 6)(2\ 8)(4\ 10)(5\ 15)(7\ 13)(9\ 12)(11\ 14),\ (0\ 9)(1\ 14)(2\ 4)(3\ 12)(5\ 7)(6\ 11)(8\ 10)(13\ 15)\rangle,$
$|H| = 8$.

Base blocks: {0 1 2 3 4}, {0 1 9 14 16}, {0 2 5 8 9}, {0 2 6 15 16}, {0 3 7 13 16}, {0 4 8 12 16}, {0 5 10 11 16}, {2 4 5 7 16}, {0 1 8 11}, {0 1 10 13}, {0 2 7 14}, {0 2 10 12}, {0 2 11 13}, {0 3 5 15}, {0 3 9 12}, {0 3 11 14}, {0 4 5 14}, {0 4 6 13}, {0 4 11 15}, {0 5 12 13}, {0 6 9 11}, {0 6 12 14}, {0 7 8 15}, {0 8 13 14}, {0 9 13 15}, {0 10 14 15}, {2 4 8 13}.

$G = H$, $|G| = 8$.

D$_{15}$: An $S(3,\mathcal{K},17)$ of type $4^{80}5^{36}$.

$H = \langle(0\ 1)(2\ 5)(3\ 6)(4\ 7)(8\ 15)(9\ 14)(10\ 13)(11\ 12),$
$(0\ 11)(1\ 12)(2\ 13)(3\ 14)(4\ 15)(5\ 10)(6\ 9)(7\ 8)\rangle$, $|H| = 4$.

Base blocks: {0 1 2 3 4}, {0 1 9 14 16}, {0 2 5 8 9}, {0 2 6 15 16}, {0 3 6 8 10}, {0 3 7 13 16}, {0 4 7 9 10}, {0 4 8 12 16}, {0 5 10 11 16}, {2 3 9 10 16}, {2 4 5 7 16}, {3 4 14 15 16}, {0 1 8 11}, {0 1 10 13}, {0 2 7 14}, {0 2 10 12}, {0 2 11 13}, {0 3 5 15}, {0 3 9 12}, {0 3 11 14}, {0 4 5 14}, {0 4 6 13}, {0 4 11 15}, {0 5 12 13}, {0 6 9 11}, {0 6 12 14}, {0 7 8 15}, {0 8 13 14}, {0 9 13 15}, {0 10 14 15}, {2 3 5 6}, {2 3 13 14}, {2 4 8 13}, {2 4 10 15}, {2 5 10 13}, {2 6 9 13}, {2 6 10 14}, {3 4 6 7}, {3 4 8 9}, {3 6 9 14}, {3 7 8 14}, {3 7 8 14}, {3 7 9 15}.

$G = H$, $|G| = 4$.

D$_{16}$: An $S(3,\mathcal{K},17)$ of type $4^{100}5^{28}$.

$H = \langle(0\ 1\ 2\ 3\ 4)(5\ 6\ 7\ 8\ 9)(12\ 16\ 15\ 13\ 14),\ (0\ 5)(1\ 9)(2\ 8)(3\ 7)(4\ 6)(10\ 11)(12\ 13)(15\ 16)\rangle$, $|H| = 10$.

Base blocks: {0 1 2 3 4}, {0 1 5 6 15}, {0 1 7 9 10}, {0 2 5 7 12}, {0 5 10 11 14}, {12 13 14 15 16}, {0 1 8 12}, {0 1 11 13}, {0 1 14 16}, {0 2 6 16}, {0 2 10 15}, {0 2 13 14}, {0 6 12 14}, {0 7 13 15}, {0 8 14 15}, {0 9 12 15}, {0 10 13 16}, {0 11 12 16}.

$G = \langle$ H, $(1\ 4\ 8\ 11\ 7)(2\ 6\ 9\ 3\ 10)(12\ 15\ 16\ 13\ 14)\rangle$, $|G| = 60$.

D$_{17}$: An $S(3,\mathcal{K},17)$ of type $4^{120}5^{20}$.

$H = \langle(0\ 1\ 8\ 4\ 12\ 10\ 11\ 3\ 14\ 7\ 5\ 6\ 13\ 9\ 2)\rangle$, $|H| = 15$.

Base blocks: {0 1 7 9 16}, {0 5 10 15 16}, {0 1 2 11}, {0 1 5 12}, {0 1 13 15}, {0 1 4 10}, {0 1 6 14}, {0 3 6 10}, {0 3 7 13}, {0 3 9 15}.

$G = \langle$ H, $(0\ 14\ 3\ 2\ 1\ 6\ 8\ 11)(4\ 7\ 12\ 13\ 10\ 9\ 5\ 15)\rangle$, $|G| = 5760$.

Sub S(2,$\mathcal{K}$,17): Use $K = \langle(0\ 1\ 2\ 7\ 5\ 12\ 3\ 14)(4\ 6\ 11\ 10\ 13\ 9\ 8\ 15)\rangle$, with base blocks: {0 5 10 15 16}, {0 1 3 8}, {0 6 11 13}.

D$_{18}$: An $S(3,\mathcal{K},17)$ of type $4^{120}5^{20}$.

$H = \langle(0\ 1\ 2)(3\ 4\ 5)(6\ 7\ 8)(9\ 10\ 11)(12\ 13\ 14),\ (0\ 1\ 4\ 12)(2\ 13\ 9\ 3)(5\ 10\ 11\ 14)(6\ 7\ 8\ 16)\rangle, |H| = 24$.

Base blocks: {0 1 3 6 15}, {0 4 10 14 15}, {0 7 11 13 15}, {6 7 8 15 16}, {0 1 2 16}, {0 1 4 12}, {0 1 5 14}, {0 1 7 9}, {0 1 8 13}, {0 1 10 11}, {0 4 5 11}, {0 4 6 16}, {0 5 6 9}, {0 6 7 10}, {0 8 10 16}.

$G = \langle$ H, $(0\ 3\ 8\ 1\ 10\ 9\ 16\ 5)(2\ 6\ 11\ 14\ 13\ 7\ 12\ 4)\rangle$, $|G| = 384$. **Three sub S(2,$\mathcal{K}$,17)'s:** Use H for the first two subdesigns.

Base blocks of first subdesign: {0 7 11 13 15}, {0 1 4 12}, {0 5 6 9}, {0 8 10 16}.

Base blocks of second subdesign: {0 4 10 14 15}, {6 7 8 15 16}, {0 1 8 13}, {0 5 6 9}. Use $K = \langle(0\ 1\ 2)(3\ 4\ 5)(6\ 7\ 8)(9\ 10\ 11)(12\ 13\ 14),\ (0\ 11\ 13)(1\ 3\ 10)(2\ 14\ 5)(4\ 12\ 9)(6\ 8\ 16)\rangle$, $|K| = 12$, for the third subdesign.

Base blocks of the third subdesign: {0 4 10 14 15}, {6 7 8 15 16}, {0 1 7 9}, {0 3 8 12}.

D$_{19}$: An $S(3,\mathcal{K},17)$ of type $4^{120}5^{20}$.

$H = \langle(0\ 1\ 2)(3\ 4\ 5)(6\ 7\ 8)(9\ 10\ 11)(12\ 13\ 14),\ (0\ 14\ 10\ 4)(1\ 2\ 5\ 13)(3\ 11\ 9\ 12)(6\ 16\ 7\ 8)\rangle, |H| = 24$.

Base blocks: {0 1 3 6 15}, {0 4 10 14 15}, {0 7 11 13 15}, {6 7 8 15 16}, {0 1 2 16}, {0 1 4 11}, {0 1 5 14}, {0 1 7 9}, {0 1 8 13}, {0 4 6 8}, {0 4 7 16}, {0 5 6 9}, {0 6 7 10}, {0 8 10 16}.

$G = \langle$ H, $(0\ 2\ 1\ 10\ 3\ 12)(4\ 13\ 11\ 14\ 9\ 5)(7\ 16\ 8)\rangle$, $|G| = 288$.

Sub S(2,$\mathcal{K}$,17): Use H, with base blocks: {0 4 10 14 15}, {6 7 8 15 16}, {0 1 8 13}, {0 5 6 9}.

D$_{20}$: An $S(3,\mathcal{K},17)$ of type $4^{120}5^{20}$.

$H = \langle(0\ 1\ 2)(3\ 4\ 5)(6\ 7\ 8)(9\ 10\ 11)(12\ 13\ 14),\ (2\ 16)(4\ 13)(5\ 9)(7\ 10)(8\ 12)(11\ 14)\rangle, |H| = 24$.

Base blocks: {0 1 3 6 15}, {0 4 10 14 15}, {3 7 12 14 15}, {0 1 2 16}, {0 1 4 8}, {0 1 5 10}, {0 1 11 14}, {0 3 4 13}, {0 3 5 14}, {0 3 8 12}, {0 4 5 11}, {0 5 6 9}, {3 4 6 8}, {3 4 7 11}, {3 4 9 14}, {3 6 11 14}.

$G = \langle$ H,$(1\ 6)(2\ 9)(5\ 16)(7\ 11)(8\ 12)(10\ 14)\rangle$, $|G| = 192$.

Two subdesigns: Use $K = \langle(0\ 2\ 11\ 6)(1\ 8\ 14\ 9)(3\ 5\ 16\ 4)(7\ 10\ 13\ 12),$
$(0\ 6)(1\ 3)(2\ 11)(4\ 8)(5\ 9)(7\ 10)(12\ 13)(14\ 16)\rangle$ where $|K| = 8$.

Base blocks of first subdesign: $\{0\ 9\ 12\ 15\ 16\}$, $\{0\ 1\ 5\ 10\}$, $\{0\ 2\ 6\ 11\}$, $\{0\ 3\ 4\ 13\}$, $\{1\ 3\ 14\ 16\}$, $\{7\ 10\ 12\ 13\}$.

Base blocks of second subdesign: $\{0\ 7\ 11\ 13\ 15\}$, $\{1\ 8\ 9\ 14\ 15\}$, $\{0\ 1\ 5\ 10\}$, $\{0\ 2\ 4\ 9\}$, $\{0\ 3\ 8\ 12\}$, $\{1\ 7\ 12\ 16\}$.

$\mathbf{D}_{21}$: An $S(3,\mathcal{K},17)$ of type $4^{120}5^{20}$.

$H = \langle(0\ 14\ 12\ 11\ 9\ 2)(1\ 6\ 13\ 8\ 10\ 7)(3\ 5\ 4), (0\ 10\ 1\ 11\ 6\ 8)(2\ 14\ 13\ 12\ 9\ 7)(4\ 16\ 5)\rangle, |H| = 24.$

Base blocks: $\{0\ 1\ 3\ 6\ 15\}$, $\{0\ 2\ 5\ 8\ 15\}$, $\{0\ 7\ 11\ 13\ 15\}$, $\{3\ 4\ 5\ 15\ 16\}$, $\{0\ 1\ 2\ 16\}$, $\{0\ 1\ 5\ 10\}$, $\{0\ 1\ 7\ 9\}$, $\{0\ 1\ 8\ 13\}$, $\{0\ 1\ 11\ 14\}$, $\{0\ 2\ 10\ 13\}$, $\{0\ 3\ 4\ 11\}$, $\{0\ 3\ 5\ 13\}$, $\{0\ 3\ 7\ 16\}$.

$G = \langle$ H, $(0\ 7\ 13)(2\ 10\ 6)(3\ 4\ 16)(8\ 9\ 14)\rangle$, $|G| = 144$.

Sub S(2,$\mathcal{K}$,17): Use $K = \langle(0\ 1\ 2)(3\ 4\ 5)(6\ 7\ 8)(9\ 10\ 11)(12\ 13\ 14),$ $(0\ 9\ 12)(1\ 6\ 11)(2\ 13\ 8)(3\ 4\ 5)(7\ 14\ 10)\rangle$, $|K| = 12$, with base blocks: $\{0\ 7\ 11\ 13\ 15\}$, $\{3\ 4\ 5\ 15\ 16\}$, $\{0\ 1\ 5\ 10\}$, $\{0\ 6\ 14\ 16\}$.

$\mathbf{D}_{22}$: An $S(3,\mathcal{K},17)$ of type $4^{120}5^{20}$.

$H = \langle(0\ 8\ 5\ 14\ 2\ 9\ 6\ 15)(1\ 10\ 7\ 16\ 3\ 11\ 4\ 13), (0\ 2)(4\ 7)(8\ 15)(9\ 14)(10\ 16)(11\ 13)\rangle . |H| = 16.$

Base blocks: $\{0\ 1\ 2\ 3\ 12\}$, $\{0\ 4\ 9\ 12\ 15\}$, $\{0\ 7\ 11\ 12\ 16\}$, $\{0\ 1\ 4\ 5\}$, $\{0\ 1\ 6\ 7\}$, $\{0\ 1\ 8\ 11\}$, $\{0\ 1\ 13\ 14\}$, $\{0\ 2\ 4\ 7\}$, $\{0\ 2\ 5\ 6\}$, $\{0\ 2\ 8\ 9\}$, $\{0\ 2\ 10\ 11\}$, $\{0\ 4\ 8\ 14\}$, $\{0\ 4\ 10\ 16\}$, $\{0\ 5\ 8\ 16\}$, $\{0\ 7\ 10\ 13\}$, $\{1\ 3\ 4\ 7\}$, $\{1\ 3\ 10\ 11\}$.

$G = \langle$ H, $(0\ 7\ 3\ 6)(1\ 5\ 2\ 4)(8\ 16\ 10\ 14)(9\ 13\ 11\ 15)\ \rangle$, $|G| = 128$.

Three subdesigns: Use $K = \langle(0\ 3)(1\ 2)(4\ 5)(6\ 7)(8\ 10)(9\ 11)(13\ 15)(14\ 16),$ $(0\ 2)(1\ 3)(4\ 7)(5\ 6)(8\ 9)(10\ 11)(13\ 16)(14\ 15), (0\ 7)(1\ 5)(2\ 4)(3\ 6)(8\ 15)(9\ 14)(10\ 13)(11\ 16),$ $(0\ 10)(1\ 9)(2\ 11)(3\ 8)(4\ 16)(5\ 14)(6\ 15)(7\ 13)\rangle$ where $|K| = 16$.

Base blocks of first subdesign: $\{0\ 1\ 2\ 3\ 12\}$, $\{0\ 4\ 8\ 14\}$, $\{0\ 5\ 11\ 15\}$, $\{0\ 6\ 9\ 16\}$, $\{0\ 7\ 10\ 13\}$.

Base blocks of second subdesign: $\{0\ 1\ 2\ 3\ 12\}$, $\{0\ 4\ 10\ 16\}$, $\{0\ 5\ 9\ 13\}$, $\{0\ 6\ 11\ 14\}$, $\{0\ 7\ 8\ 15\}$.

Base blocks of third subdesign: $\{0\ 4\ 9\ 12\ 15\}$, $\{0\ 1\ 8\ 11\}$, $\{0\ 2\ 5\ 6\}$, $\{0\ 3\ 14\ 16\}$, $\{0\ 7\ 10\ 13\}$.

$\mathbf{D}_{23}$: An $S(3,\mathcal{K},17)$ of type $4^{120}5^{20}$.

$H = \langle(1\ 2\ 13)(4\ 5\ 16)(7\ 8\ 14)(10\ 11\ 15), (0\ 10\ 13\ 15)(1\ 9\ 2\ 11)(5\ 16)(6\ 14\ 7\ 8)\rangle, |H| = 24.$

Base blocks: $\{0\ 1\ 2\ 12\ 13\}$, $\{0\ 3\ 6\ 9\ 12\}$, $\{0\ 4\ 11\ 12\ 14\}$, $\{3\ 4\ 5\ 12\ 16\}$, $\{6\ 7\ 8\ 12\ 14\}$, $\{0\ 1\ 3\ 5\}$, $\{0\ 2\ 4\ 5\}$, $\{0\ 1\ 6\ 8\}$, $\{0\ 1\ 7\ 14\}$, $\{0\ 1\ 9\ 11\}$, $\{0\ 3\ 7\ 10\}$, $\{0\ 4\ 6\ 10\}$, $\{0\ 4\ 7\ 9\}$, $\{3\ 4\ 6\ 7\}$, $\{4\ 5\ 7\ 8\}$.

$G = \langle$ H, $(0\ 9\ 1\ 11)(2\ 10\ 13\ 15)(3\ 5\ 16\ 4)(6\ 14)\rangle$, $|G| = 96$.

One subdesign: Use $K = \langle(0\ 10\ 13\ 15)(1\ 9\ 2\ 11)(5\ 16)(6\ 14\ 7\ 8),$ $(0\ 9)(1\ 10)(2\ 15)(5\ 16)(8\ 14)(11\ 13), (3\ 4)(5\ 16)(6\ 14)(7\ 8)(9\ 11)(10\ 15)\ \rangle$ where $|K| = 16$.

Base blocks of subdesign: $\{0\ 1\ 2\ 12\ 13\}$, $\{3\ 4\ 5\ 12\ 16\}$, $\{6\ 7\ 8\ 12\ 14\}$, $\{0\ 3\ 8\ 11\}$, $\{0\ 5\ 10\ 14\}$.

$\mathbf{D}_{24}$: An $S(3,\mathcal{K},17)$ of type $4^{120}5^{20}$.

$H = \langle(0\ 13\ 10)(1\ 2\ 9)(3\ 6\ 11)(4\ 12\ 5)(7\ 8\ 14), (0\ 13\ 10)(1\ 4\ 11)(2\ 14\ 5)(3\ 8\ 9)(6\ 12\ 7)\rangle,$ $|H| = 12.$

Base blocks: $\{0\ 1\ 6\ 9\ 16\}$, $\{0\ 10\ 13\ 15\ 16\}$, $\{1\ 3\ 5\ 7\ 16\}$, $\{1\ 8\ 12\ 15\ 16\}$, $\{0\ 1\ 2\ 14\}$, $\{0\ 1\ 3\ 13\}$, $\{0\ 1\ 4\ 12\}$, $\{0\ 1\ 5\ 15\}$, $\{0\ 1\ 7\ 10\}$, $\{0\ 1\ 8\ 11\}$, $\{0\ 2\ 4\ 10\}$, $\{0\ 2\ 6\ 15\}$, $\{0\ 9\ 14\ 15\}$, $\{1\ 2\ 4\ 7\}$, $\{1\ 4\ 11\ 15\}$, $\{1\ 6\ 14\ 15\}$, $\{1\ 2\ 3\ 8\}$, $\{1\ 2\ 5\ 6\}$, $\{1\ 2\ 9\ 15\}$, $\{1\ 6\ 14\ 15\}$.

$G = \langle$ H, $(0\ 12\ 5\ 3\ 9)(1\ 2\ 6\ 7\ 10)(4\ 13\ 8\ 14\ 11)\rangle = \mathrm{PSL}_2(5)$ acting on the 15 unordered pairs of the projective line. The points 15 and 16 are fixed. $|G| = 60$.

Three sub S(2,$\mathcal{K}$,17)'s: Use H.

Base blocks of first subdesign: $\{1\ 3\ 5\ 7\ 16\}$, $\{0\ 10\ 13\ 15\ 16\}$, $\{0\ 1\ 2\ 14\}$, $\{1\ 4\ 11\ 15\}$.

Base blocks of second subdesign: $\{1\ 3\ 5\ 7\ 16\}$, $\{0\ 10\ 13\ 15\ 16\}$, $\{0\ 1\ 4\ 12\}$, $\{1\ 2\ 9\ 15\}$.

Base blocks of third subdesign: $\{1\ 3\ 5\ 7\ 16\}$, $\{0\ 10\ 13\ 15\ 16\}$, $\{0\ 1\ 8\ 11\}$, $\{1\ 6\ 14\ 15\}$.

$\mathbf{D}_{25}$: An $S(3,\mathcal{K},17)$ of type $4^{120}5^{20}$.

$H = \langle(0\,17)(1\,5\,13\,4\,2\,16)(6\,9)(7\,11\,14\,10\,8\,15), (0\,4)(1\,3)(2\,16)(5\,13)(6\,10)(7\,9)(8\,15)(11\,14)\rangle,$ $|H| = 24.$

Base blocks: $\{0\ 1\ 2\ 12\ 13\}$, $\{0\ 3\ 6\ 9\ 12\}$, $\{0\ 4\ 11\ 12\ 14\}$, $\{6\ 7\ 8\ 12\ 14\}$, $\{0\ 1\ 3\ 7\}$, $\{0\ 1\ 5\ 16\}$, $\{0\ 1\ 8\ 9\}$, $\{0\ 1\ 11\ 15\}$, $\{0\ 4\ 7\ 8\}$, $\{0\ 6\ 7\ 10\}$, $\{6\ 7\ 11\ 15\}$.

$G = \langle$ H, $(0\ 7)(1\ 6)(2\ 14)(3\ 10)(4\ 9)(5\ 15)(8\ 13)(11\ 16)\rangle$, $|G| = 48$.

$\mathbf{D}_{26}$: An $S(3,\mathcal{K},17)$ of type $4^{120}5^{20}$.

$H = \langle (0\ 1\ 2)(3\ 4\ 5)(6\ 7\ 8)(9\ 10\ 11),\ (0\ 14\ 5\ 3\ 15\ 2)(1\ 4)(6\ 13\ 11\ 9\ 16\ 8)(7\ 10)\rangle$, $|H| = 24$.

Base blocks: {0 1 2 12 13}, {0 3 6 9 12}, {0 8 10 12 16}, {0 1 3 10}, {0 1 5 16}, {0 1 6 7}, {0 7 8 9}, {0 7 10 11}.

$G = H$, $|G| = 24$.

$\mathbf{D}_{27}$: An $S(3,\mathcal{K},17)$ of type $4^{120}5^{20}$.

$H = \langle\ (0\ 1\ 2)(3\ 4\ 5)(6\ 7\ 8)(9\ 10\ 11)(12\ 13\ 14),\ (1\ 2)(3\ 5)(6\ 8)(9\ 12)(10\ 14)(11\ 13),$ $(0\ 14\ 10)(1\ 12\ 11)(2\ 13\ 9)(6\ 7\ 8)\rangle$, $|H = 18$.

Base blocks: {0 1 3 6 15}, {0 4 10 14 15}, {0 7 11 13 15}, {0 9 12 15 16}, {3 4 5 15 16}, {6 7 8 15 16}, {0 1 2 16}, {0 1 4 11}, {0 1 7 8}, {0 1 9 10}, {0 3 4 9}, {0 3 5 7}, {0 3 8 12}, {0 3 11 16}, {0 4 6 8}, {0 4 7 16}, {0 6 10 13}, {0 6 14 16}.

$G = H$, $|G| = 18$.

$\mathbf{D}_{28}$: An $S(3,\mathcal{K},17)$ of type $4^{120}5^{20}$.

$H = \langle (0\ 5\ 1\ 3\ 2\ 4)(6\ 11\ 7\ 9\ 8\ 10)(13\ 16)(14\ 15),\ (0\ 6)(1\ 7)(2\ 8)(3\ 9)(4\ 10)(5\ 11)(13\ 14)(15\ 16)\rangle$, $|H| = 12$.

Base blocks: {012 12 13}, {0369 12}, {0 4 11 12 14}, {12 13 14 15 16}, {0 1 3 7}, {0 1 4 6}, {0 1 5 16}, {0 1 8 9}, {0 1 10 14}, {0 1 11 15}, {0 3 13 16}, {0 3 14 15}, {0 4 9 15}, {0 6 13 14}, {0 6 15 16}, {0 7 14 16}.

$G = H$, $|G| = 12$.

$\mathbf{D}_{29}$: An $S(3,\mathcal{K},17)$ of type $4^{120}5^{20}$.

$H = \langle (0\ 1\ 2)(3\ 4\ 5)(6\ 7\ 8)(9\ 10\ 11),\ (0\ 9)(1\ 10)(2\ 11)(3\ 6)(4\ 7)(5\ 8)(13\ 15)(14\ 16)\rangle$, $|H| = 6$.

Base blocks: {012 12 13}, {0369 12}, {04 11 12 14}, {057 12 15}, {345 12 16}, {12 13 14 15 16}, {0134}, {015 16}, {0167}, {0189}, {01 10 14}, {01 11 15}, {037 10}, {03 13 16}, {03 14 15}, {045 13}, {046 10}, {0479}, {048 15}, {058 14}, {06 13 14}, {06 15 16}, {078 13}, {07 14 16}, {09 13 15}, {3467}, {348 14}, {36 13 15}.

$G = H$, $|G| = 6$.

$\mathbf{D}_{30}$: An $S(3,\mathcal{K},17)$ of type $4^{120}5^{20}$.

$H = \langle (0\ 1\ 2)(3\ 4\ 5)(6\ 7\ 8)(9\ 10\ 11)\rangle$, $|H| = 3$.

Base blocks: {0 1 2 12 13}, {0 3 6 9 12}, {0 4 11 12 14}, {0 5 7 12 15}, {0 8 10 12 16}, {3 4 5 12 16}, {3 7 11 12 13}, {6 7 8 12 14}, {9 10 11 12 15}, {12 13 14 15 16}, {0134}, {015 16}, {0167}, {0189}, {01 10 14}, {01 11 15}, {037 10}, {038 11}, {03 13 16}, {03 14 15}, {045 13}, {046 10}, {0479}, {048 15}, {056 11}, {058 14}, {059 10}, {06 13 14}, {06 15 16}, {078 13}, {07 14 16}, {09 11 16}, {09 13 15}, {0 10 11 13}, {3467}, {348 14}, {349 10}, {34 11 15}, {36 10 16}, {36 13 15}, {378 16}, {38 10 13}, {39 13 14}, {39 15 16}, {3 11 14 16}, {679 15}, {67 10 11}, {69 11 14}, {69 13 16}, {6 10 14 15}.

$G = H$, $|G| = 3$.

$\mathbf{D}_{31}$: An $S(3,\mathcal{K},17)$ of type $4^{120}5^{20}$.

$H = \langle (0\ 4)(2\ 13)(3\ 5)(6\ 9)(10\ 14)(11\ 15)\rangle$, $|H| = 2$.

Base blocks: {0 1 9 14 16}, {0 2 6 15 16}, {0 3 7 13 16}, {0 4 8 12 16}, {0 5 10 11 16}, {1 2 12 13 16}, {1 3 5 8 16}, {1 7 11 15 16}, {2 3 9 10 16}, {2 8 11 14 16}, {3 6 11 12 16}, {6 7 8 9 16}, {7 10 12 14 16}, {0 1 2 3}, {0 1 4 7}, {0 1 5 6}, {0 1 8 11}, {0 1 10 13}, {0 1 12 15}, {0 2 4 13}, {0 2 5 8}, {0 2 7 14}, {0 2 9 11}, {0 2 10 12}, {0 3 4 11}, {0 3 5 14}, {0 3 6 8}, {0 3 9 12}, {0 3 10 15}, {0 4 6 9}, {0 4 10 14}, {0 5 7 9}, {0 5 12 13}, {0 6 7 10}, {0 6 11 13}, {0 6 12 14}, {0 7 8 15}, {0 7 11 12}, {0 8 9 10}, {0 8 13 14}, {0 9 13 15}, {0 11 14 15}, {1 2 5 15}, {1 2 6 8}, {1 2 7 9}, {1 2 10 11}, {1 3 6 15}, {1 3 7 14}, {1 3 10 12}, {1 6 9 12}, {1 6 11 14}, {1 7 8 12}, {1 8 10 14}, {2 3 5 6}, {2 3 7 11}, {2 3 8 15}, {2 3 13 14}, {2 5 9 14}, {2 5 11 12}, {2 6 7 12}, {2 6 9 13}, {2 6 10 14}, {2 7 8 13}, {2 7 10 15}, {2 8 9 12}, {2 11 13 15}, {2 12 14 15}, {3 5 7 12}, {3 5 11 15}, {3 6 9 14}, {3 7 8 10}, {3 7 9 15}, {3 8 9 11}, {3 8 12 14}, {3 10 11 14}, {6 7 14 15}, {6 8 10 11}, {6 9 11 15}, {6 10 12 15}, {8 11 12 15}.

$G = H$, $|G| = 2$.

$\mathbf{D}_{32}$: An $S(3,\mathcal{K},17)$ of type $4^{120}5^{20}$.

$H = \langle (0\ 12)(1\ 11)(2\ 10)(3\ 9)(4\ 8)(5\ 13)(6\ 14)(7\ 15)\rangle$, $|H| = 2$.

Base blocks: {0 1 9 14 16}, {0 2 6 15 16}, {0 3 7 13 16}, {0 4 8 12 16}, {0 5 10 11 16}, {1 3 5 8 16}, {1 4 6 10 16}, {1 7 11 15 16}, {2 3 9 10 16}, {2 4 5 7 16}, {3 4 14 15 16}, {5 6 13 14 16}, {0 1 2 3}, {0 1 4 7}, {0 1 5 6}, {0 1 8 11}, {0 1 10 13}, {0 1 12 15}, {0 2 4 13}, {0 2 5 8}, {0 2 7 14}, {0 2 9 11}, {0 2 10 12}, {0 3 4 11}, {0 3 5 14}, {0 3 6 8}, {0 3 9 12}, {0 3 10 15}, {0 4 5 15}, {0 4 6 9}, {0 4 10 14}, {0 5 7 9}, {0 5 12 13}, {0 6 7 10}, {0 6 11 13}, {0 6 12 14}, {0 7 8 15}, {0 8 9 10}, {0 8 13 14}, {0 9 13 15}, {0 11 14 15}, {1 2 4 14}, {1 2 5 15}, {1 2 6 9}, {1 2 7 8}, {1 2 10 11}, {1 3 4 9}, {1 3 6 15}, {1 3 7 14}, {1 3 11 13}, {1 4 5 13}, {1 4 8 15}, {1 5 7 10}, {1 6 8 13}, {1 6 11 14}, {1 7 9 13}, {1 8 10 14}, {1 9 10 15}, {1 13 14 15}, {2 3 5 6}, {2 3 8 15}, {2 3 13 14}, {2 4 8 10}, {2 4 9 15}, {2 5 9 14}, {2 5 10 13}, {2 6 7 13}, {2 6 10 14}, {2 7 10 15}, {2 8 9 13}, {3 4 6 7}, {3 4 8 13}, {3 5 9 13}, {3 6 9 14}, {3 7 9 15}, {4 6 8 14}, {4 6 13 15}, {4 7 13 14}, {5 7 13 15}, {6 7 14 15}.

$G = H$, $|G| = 2$.

D_{33}: An $S(3, \mathcal{K}, 17)$ of type $4^{120}5^{20}$. $|G| = 1$.

Blocks: {0 1 9 14 16}, {0 2 6 15 16}, {0 3 7 13 16}, {0 4 8 12 16}, {0 5 10 11 16}, {1 2 12 13 16}, {1 3 5 8 16}, {1 4 6 10 16}, {1 7 11 15 16}, {2 3 9 10 16}, {2 4 5 7 16}, {2 8 11 14 16}, {3 4 14 15 16}, {3 6 11 12 16}, {4 9 11 13 16}, {5 6 13 14 16}, {5 9 12 15 16}, {6 7 8 9 16}, {7 10 12 14 16}, {8 10 13 15 16}, {0 1 2 3}, {0 1 4 7}, {0 1 5 6}, {0 1 8 11}, {0 1 10 13}, {0 1 12 15}, {0 2 4 13}, {0 2 5 8}, {0 2 7 14}, {0 2 9 11}, {0 2 10 12}, {0 3 4 11}, {0 3 5 14}, {0 3 6 8}, {0 3 9 12}, {0 3 10 15}, {0 4 5 15}, {0 4 6 9}, {0 4 10 14}, {0 5 7 9}, {0 5 12 13}, {0 6 7 10}, {0 6 11 13}, {0 6 12 14}, {0 7 8 15}, {0 7 11 12}, {0 8 9 10}, {0 8 13 14}, {0 9 13 15}, {0 11 14 15}, {1 2 4 14}, {1 2 5 15}, {1 2 6 8}, {1 2 7 9}, {1 2 10 11}, {1 3 4 9}, {1 3 6 15}, {1 3 7 14}, {1 3 10 12}, {1 3 11 13}, {1 4 5 13}, {1 4 8 15}, {1 4 11 12}, {1 5 7 10}, {1 5 9 11}, {1 5 12 14}, {1 6 7 13}, {1 6 9 12}, {1 6 11 14}, {1 7 8 12}, {1 8 9 13}, {1 8 10 14}, {1 9 10 15}, {1 13 14 15}, {2 3 4 12}, {2 3 5 6}, {2 3 7 11}, {2 3 8 15}, {2 3 13 14}, {2 4 6 11}, {2 4 8 10}, {2 4 9 15}, {2 5 9 14}, {2 5 10 13}, {2 5 11 12}, {2 6 7 12}, {2 6 9 13}, {2 6 10 14}, {2 7 8 13}, {2 7 10 15}, {2 8 9 12}, {2 11 13 15}, {2 12 14 15}, {3 4 5 10}, {3 4 6 7}, {3 4 8 13}, {3 5 7 12}, {3 5 9 13}, {3 5 11 15}, {3 6 9 14}, {3 6 10 13}, {3 7 8 10}, {3 7 9 15}, {3 8 9 11}, {3 8 12 14}, {3 10 11 14}, {3 12 13 15}, {4 5 6 12}, {4 5 8 9}, {4 5 11 14}, {4 6 8 14}, {4 6 13 15}, {4 7 8 11}, {4 7 9 10}, {4 7 12 15}, {4 7 13 14}, {4 9 12 14}, {4 10 11 15}, {4 10 12 13}, {5 6 7 11}, {5 6 8 15}, {5 6 9 10}, {5 7 8 14}, {5 7 13 15}, {5 8 10 12}, {5 8 11 13}, {5 10 14 15}, {6 7 14 15}, {6 8 10 11}, {6 8 12 13}, {6 9 11 15}, {6 10 12 15}, {7 9 11 14}, {7 9 12 13}, {7 10 11 13}, {8 9 14 15}, {8 11 12 15}, {9 10 11 12}, {9 10 13 14}, {11 12 13 14}.

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Received February 14, 1996
Accepted June 24, 1996