Steiner Graphical t–Wise Balanced Designs of Type n^r

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Abstract

In this article we completely determine all proper Steiner graphical t–wise balanced designs whose automorphsims contain the group $S_n \, wr \, S_r$ (wreath product) acting on the set of $n^2\binom{r}{2}$ edges of the complete r–partite graph, $K_{n,n,\ldots,n}$.

1 Introduction

A *t–wise balanced design* (tBD) of type t–$(v, \mathcal{K}, \lambda)$ is a pair $(X, \mathcal{B})$ where X is a v–element set of points and $\mathcal{B}$ is a collection of subsets of X, called blocks, with the property that the size of every block is in $\mathcal{K}$ and every t–element subset of X is contained in exactly λ blocks. If $\mathcal{K}$ is a set of positive integers strictly between t and v, then we say that the tBD is *proper*. If $\lambda = 1$, then the design is called a *Steiner* design. All designs in this article are proper Steiner designs.

A fundamental question in the theory of combinatorial designs is to ask what designs can be obtained with a particular automorphism group. In this article we completely determine all proper Steiner tBDs whose automorphisms contain the group $S_n \, wr \, S_r$ (wreath product) acting on the set of $n^2\binom{r}{2}$ edges of the complete r–partite graph, $K_{n,n,\ldots,n}$.

Denote by S_n the symmetric group on $\{1, 2, 3, \ldots, n\}$. If A is a subgroup of S_n and B is a subgroup of S_r, then the *wreath product A wr B* is the semidirect product of $N = \underbrace{A \times A \times \cdots \times A}_{r}$ by B in which the action of B on N is given by

$$(\alpha_1, \alpha_2, \ldots, \alpha_r)^\beta = (\alpha_{\beta(1)}, \alpha_{\beta(2)}, \ldots, \alpha_{\beta(r)});$$

where $\alpha_1, \ldots, \alpha_r \in A$ and $\beta \in B$. Thus an element of A wr B is a pair

$$((\alpha_1, \alpha_2, \ldots, \alpha_r), \beta)$$

and the product of two elements is given by

$$((\alpha_1, \alpha_2, \ldots, \alpha_r), \beta)\,((\gamma_1, \gamma_2, \ldots, \gamma_r), \delta) = \big((\alpha_1\gamma_{\beta(1)}, \alpha_2\gamma_{\beta(2)}, \ldots, \alpha_r\gamma_{\beta(r)}), \beta\delta\big)\,.$$

Let $X = \{1, 2, \ldots, n\}$ and $Y = \{1, 2, \ldots, r\}$, then the wreath product $A\,wr\,B$ acts on $X \times Y$ in a natural way:

$$((\alpha_1, \alpha_2, \ldots, \alpha_r), \beta)\,[i, j] = [\alpha_{\beta(j)}(i), \beta(j)].$$

It is an easy exercise to see that $S_n\,wr\,S_r$ is the automorphism group of $K_{\underbrace{n, n, \ldots, n}_{r}}$.

A t–wise balanced design $(X, \mathcal{B})$ with parameters t–$(v, \mathcal{K}, \lambda)$ is said to be *graphical of type* $n_1^{r_1} n_2^{r_2} \cdots n_s^{r_s}$ if X is the set of v edges of the complete r–partite graph $\Gamma = K_{n_1^{r_1}, n_2^{r_2}, \ldots, n_s^{r_s}}$ and whenever B is a block and α is an automorphism of Γ, then $\alpha(B)$ is also a block. That is, $\mathcal{B}$ is a union of orbits under the action of $\mathrm{Aut}(\Gamma)$ on the edges of Γ. The notation of type $n_1^{r_1} n_2^{r_2} \cdots n_s^{r_s}$ is adopted from that used for group divisible designs (see [3]), since Γ is a group divisible design of this type with block size 2.

Graphical tBDs of type 1^r, for $r \geq 2$, were called *graphical designs* in [2]. In particular, a graphical design $(X, \mathcal{B})$ of type 1^r has parameters t–$(\binom{r}{2}, \mathcal{K}, \lambda)$ and has the symmetric group S_r as the automorphism group. So X is the set of $v = \binom{r}{2}$ labeled edges of the undirected complete graph K_r and if $B \in \mathcal{B}$, then all subgraphs of K_r isomorphic to B are also in $\mathcal{B}$. All graphical designs with $\lambda = 1, 2$ were determined in [2].

Graphical tBDs of type $m^1 n^1$ $(m \leq n)$ were called *bigraphical* in [5, 8]. A bigraphical design has parameters t–$(m \cdot n, \mathcal{K}, \lambda)$ and has $S_m \times S_n$ as the automorphism group provided $m \neq n$ and $S_n\,wr\,S_2$ otherwise. Here, X is the set of all labeled edges of the complete bipartite graph $K_{m,n}$. In [5, 8] all bigraphical designs with $\lambda = 1, 2$ were determined. In the sequence of lemmas that follow we establish the following Theorem.

Main Theorem: *The only proper Steiner graphical tBDs of type n^r are those given in Tables I, II, and III.*

In an effort to reduce the size of this manuscript the proofs of many of the lemmas have been abridged. Unabridged versions can be found in the first authors Ph.D. thesis [7].

Before proceeding we mention a few definitions and notational conventions. The partite sets of $K_{n,n,\ldots,n}$ will be displayed vertically and will be denoted by $\mathrm{A} = \{a_1, a_2, \ldots, a_n\}$, $\mathrm{B} = \{b_1, b_2, \ldots, b_n\}$, $\mathrm{C} = \{c_1, c_2, \ldots, c_n\}, \ldots$. Occasionally it will be more convenient to denote the partite sets as $\mathrm{A}_1 = \{a_{11}, a_{21}, \ldots, a_{n1}\}$, $\mathrm{A}_2 = \{a_{12}, a_{22}, \ldots, a_{n2}\}, \ldots$. The automorphism that interchanges the two partite sets A and B, namely $a_i \leftrightarrow b_i$, for $1 \leq i \leq n$, is called a *swap* and its application to the graph is called *swapping*. *Cycling* the partite sets A, B, C we mean to apply the permutation $(a_1 b_1 c_1)(a_2 b_2 c_2) \cdots (a_n b_n c_n)$. An automorphism that interchanges two vertices within a partite set is called a *switch* and its application to the graph is called *switching*. Needless to say, these operations: swapping, cycling, and switching generate the automorphism group of $K_{n,n,\ldots,n}$. If B is a block in a graphical tBD of type n^r, then we say B is *complete* whenever $B = K_{n,n,\ldots,n}$. Since $|B| < v$, we reach a contradiction whenever covering a given t–element set with a block B, forces B to be complete. We denote the complete ℓ–partite graph on the partite sets $\mathrm{A}_1, \mathrm{A}_2, \ldots, \mathrm{A}_\ell$ by $K(\mathrm{A}_1, \mathrm{A}_2, \ldots, \mathrm{A}_\ell)$. The notation $G \dot\cup H$ denotes the disjoint union of the subgraphs G and H. Furthermore, in the figures that follow, a rectangle is drawn around a portion of a graph to indicate that the subgraph is complete in this region.

We proceed by stating two frequently used lemmas. The first was noted by Denniston in [4].

Lemma 1.1 *If $\lambda = 1$ and T is a t–element subset of the block B that is fixed by the automorphism α, then α also fixes B.*

Let T_0 be a subset of edges of $K_{n,n,\ldots,n}$. A sequence $\alpha_1, \alpha_2, \ldots, \alpha_k$ of automorphisms in $S_n\, wr\, S_r$ is said to *t–generate* T_k from T_0 if $|T_{i-1} \cap \alpha_i(T_{i-1})| \geq t$ and $T_i = T_{i-1} \cup \alpha_i(T_{i-1})$ for $1 \leq i \leq k$. Applying Lemma 1.1 it is easy to establish Lemma 1.2.

Lemma 1.2 *Let $T_0 \subseteq B$ where B is a block of a Steiner graphical tBD of type n^r. If a sequence of automorphisms t–generates T_k from T_0, then $T_k \subseteq B$.*

If $T_{i-1} \subseteq B$ where α_i t–generates T_i and $T_{i-1} \neq T_i$, then we say that α_i *produces additional edges* in B from T_{i-1}.

The next theorem is a block size bound given in [6] which we will frequently use.

Theorem 1.3 *Let $(X, \mathcal{B})$ be a proper, nontrivial tBD with $t \geq 2$ and $\lambda = 1$. Then $t+1 \leq k \leq \frac{v+t-3}{2}$ for all $k \in \mathcal{K}$.*

Tables I, II, and III display all Steiner graphical tBDs of type n^r with $S_n\, wr\, S_r$ as the full automorphism group where $n \geq 1$ and $r \geq 2$. The designs in Table I are the Steiner graphical tBDs of type 1^r found in [2]. The Designs in Table II are the Steiner graphical tBDs of type n^2 found in [5]. We will show that the designs in Table III are the only other Steiner graphical tBDs of type n^r for $n \geq 2$ and $r \geq 3$. Table IV shows all other Steiner bigraphical tBDs of type m^1n^1, $m \neq n$.

Table I. The Steiner graphical tBDs of type 1^r

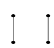
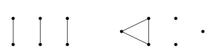
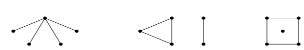
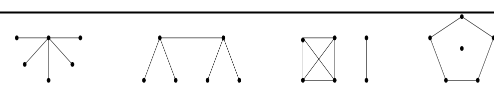
	Parameters	Representation
D_1:	$n = 1$ and $r = 4$ 1–$(6, 2, 1)$	
D_2:	$n = 1$ and $r = 6$ 2–$(15, 3, 1)$	
D_3:	$n = 1$ and $r = 6$ 2–$(15, \{3, 5\}, 1)$	
D_4:	$n = 1$ and $r = 5$ 3–$(10, 4, 1)$	
D_5:	$n = 1$ and $r = 6$ 4–$(15, \{5, 7\}, 1)$	

Table II. The Steiner graphical tBDs of type n^2

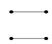
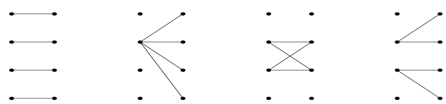
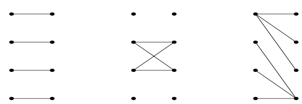
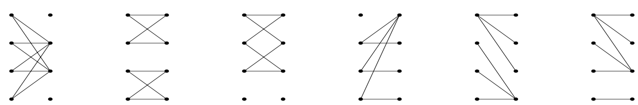
	Parameters	Representation
D_6:	$n \geq 2$ and $r = 2$ $1\text{–}(n^2, n, 1)$	$K_{1,n}$
D_7:	$n = 2$ and $r = 2$ $1\text{–}(4, 2, 1)$	
D_8:	$n = 3$ and $r = 2$ $2\text{–}(9, 3, 1)$	
D_9:	$n = 4$ and $r = 2$ $3\text{–}(16, 4, 1)$	
D_{10}:	$n = 4$ and $r = 2$ $3\text{–}(16, \{4, 6\}, 1)$	
D_{11}:	$n = 4$ and $r = 2$ $5\text{–}(16, \{6, 8\}, 1)$	

Table III. The Steiner graphical tBDs of type n^r, $n \geq 2$, $r \geq 2$

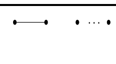
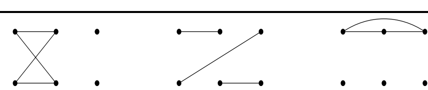
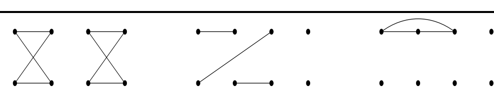
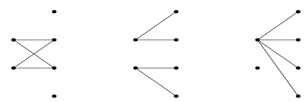
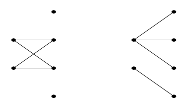
	Parameters	Representation
D_{12}:	$n \geq 2$ and $r = 4$ $1\text{–}(6n^2, 2n^2, 1)$	$K_{n,n} \dot\cup K_{n,n}$
D_{13}:	$n \geq 2$ and $r \geq 3$ $1\text{–}(n^2\binom{r}{2}, n^2, 1)$	$K_{n,n}$
D_{14}:	$n = 2$ and $r \geq 3$ $1\text{–}(4\binom{r}{2}, 2, 1)$	
D_{15}:	$n = 2$ and $r = 3$ $2\text{–}(12, \{3, 4\}, 1)$	
D_{16}:	$n = 2$ and $r = 4$ $2\text{–}(24, \{3, 8\}, 1)$	

Table IV. The Steiner graphical tBDs of type m^1n^1, $m \neq n$

	Parameters	Representation
D_{17}:	$2 \leq m < n$ $1\text{–}(mn, n, 1)$	$K_{1,n}$
D_{18}:	$m = 2, n = 4$ $3\text{–}(8, 4, 1)$	
D_{19}:	$m = 2, n = 4$ $3\text{–}(8, 4, 1)$	

2 $t = 1$ and 2

Lemma 2.1 *Suppose $r \geq 3$ and $t \leq n^2$. Let B be a block in a Steiner graphical tBD of type n^r. If $K(\mathrm{A}_i, \mathrm{A}_j, \mathrm{A}_k) \subseteq B$, for distinct partite sets A_i,A_j, and A_k, then B is complete.*

Proof: Without loss of generality assume $i, j, k = 1, 2, 3$. Expand the $K(\mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3)$ in B to the right by swapping partite sets A_3 and A_4 and then swapping partite sets A_2 and A_4. Note that each swap fixes $n^2 \geq t$ edges. Thus by Lemma 1.1, B contains $K(\mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_4)$ and then B contains $K(\mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3, \mathrm{A}_4)$. These operations can be continued until B is complete. ■

Theorem 2.2 *The only proper Steiner graphical 1–wise balanced designs of type n^r are in Tables I, II, and III.*

Proof: Suppose $(X, \mathcal{B})$ is a Steiner graphical 1–wise balanced design of type n^r with parameters $1\text{–}(n^2\binom{r}{2}, \mathcal{K}, 1)$. For $n = 1$, $r \geq 2$ and $n \geq 2$, $r = 2$, the designs have been characterized in [2] and [5], respectively. So we may assume that $r \geq 3$. Label the partite sets as $\mathrm{A}, \mathrm{B}, \mathrm{C}, \ldots$. Let $T = \{a_1b_1\}$ be the graph consisting of exactly one edge and let $B \in \mathcal{B}$ be the block containing T. Now $B \neq T$ because $t < |B| < v$, so B must contain at least one additional edge, e. Up to symmetry, there are five cases for e to consider. Each case is settled using Lemmas 1.1 and 1.2 with the appropriate, swapping, switching and cycling operations. Case 1: If $e = a_1b_2$, then $B = K(\mathrm{A}, \mathrm{B})$; hence we obtain the design D_{13}. Case 2: If $e = b_1c_1$, then we can force B to be complete. Case 3: If $e = a_2b_2$ and $n \geq 3$, then we can force $a_1b_2 \in B$ and hence we are in case 1. If $n = 2$, then we obtain the design D_{14}. Case 4: If $e = b_2c_2$, then we can force B to be complete. Case 5: If $e = c_1d_1$ and $r \geq 5$, then we can force B to be complete. If $r = 4$, then $B = K(\mathrm{A}, \mathrm{B}) \dot{\cup} K(\mathrm{C}, \mathrm{D}) = K_{n,n} \dot{\cup} K_{n,n}$ and the design is D_{12}. ■

Lemma 2.3 *In a Steiner graphical $2BD$ of type n^r, if $r \geq 3$, then $n = 2$.*

Proof: Suppose $n \geq 3$ and consider the 2–element set T in Figure 1. Let B be the block containing T. As $B \neq T$, B contains an additional edge, e. If $e = b_2c_1$, then there exists a sequence of permutations which 2–generates $K(\mathrm{A}, \mathrm{B}, \mathrm{C}) \subseteq B$ from $T \cup \{b_2c_1\}$. Thus by Lemma 2.1, B is complete. By symmetry, if $e = a_2b_1$ or $e = b_2c_3$, then B is complete. So, if $e \neq a_1c_2$, then at least one of the transpositions $(a_1a_2), (c_1c_2)$, or (c_2c_3) will fix e and an edge of T. Hence this transposition fixes $t = 2$ edges of B. Applying this transposition to the edges in $T \cup \{e\}$, we see that one of the edges b_2c_1, a_2b_1, or b_2c_3 is forced to be in B. Thus, when $e \neq a_1c_2$, we can easily force B to be complete. Now suppose $e = a_1c_2$. Apply the transpositions (b_2b_3) and (a_1a_2), each of which fixes $t = 2$ edges, to obtain $a_2b_1 \in B$; hence B is complete. So regardless of what the additional edge is, B is forced to be complete. Therefore $n = 2$. ■

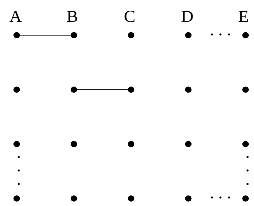


Figure 1:

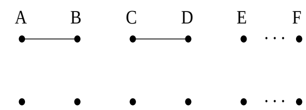


Figure 2:

Lemma 2.4 *There does not exist a Steiner graphical 2BD of type n^r for $r \geq 5$.*

Proof: By Lemma 2.3 we know that $n = 2$. Now consider the 2–element set T in Figure 2 and let B be the block containing T. As $B \neq T$, B contains an additional edge, e. Suppose $e \in K(\mathrm{A}, \mathrm{B})$. There exists a sequence of permutations which 2-generates $K(\mathrm{C}, \mathrm{D}, \mathrm{E})$ from $T \cup \{e\}$. Thus by Lemma 1.2, $K(\mathrm{C}, \mathrm{D}, \mathrm{E}) \subseteq B$ and by Lemma 2.1, B is complete. By symmetry, if $e \in K(\mathrm{C}, \mathrm{D})$, then $K(\mathrm{A}, \mathrm{B}, \mathrm{E}) \subseteq B$; hence B is complete. Now suppose $e \notin K(\mathrm{A}, \mathrm{B}) \dot{\cup} K(\mathrm{C}, \mathrm{D})$. At least one of the transpositions $(a_1a_2), (b_1b_2), (c_1c_2)$, or (d_1d_2) will fix e and an edge of T. Hence this transposition fixes $t = 2$ edges of B and forces one of the edges a_2b_1, a_1b_2, c_2d_1, or c_1d_2 to be in B. Thus by the above arguments, we can conclude that B is complete. Therefore, there cannot exist a Steiner graphical 2BD of type n^r when $r \geq 5$. ■

Theorem 2.5 *The only proper Steiner graphical 2-wise balanced designs of type n^r are listed in Tables I, II, and III.*

Proof: For $n = 1, r \geq 2$ and $n \geq 2, r = 2$, the designs have been determined in [2] and [5], respectively. So we may assume that $n \geq 2$ and $r \geq 3$. By Lemma 2.3 and Lemma 2.4, we only need to consider two cases: (i) $n = 2$, $r = 3$ and (ii) $n = 2$, $r = 4$. In case (i), consider Figures 3 and 4. One can easily show that the 2–element set T_1 forces the block B_1, T_2 forces the block B_2, and T_3 forces the block B_3. Thus we have the design D_{15}. In case (ii), the 2–element sets T_1, T_2, and T_3 in Figure 5 force the blocks B_1, B_2, and B_3 in Figure 6. Hence we have the design D_{16}. ■

3 $t \geq 3$

In the sequence of lemmas that follow we will show there do not exist any Steiner graphical tBDs of type n^r for $t \geq 3$ where $n \geq 2$ and $r \geq 3$. The concept of derived design will be frequently used. The *derived design* of a t–$(v, \mathcal{K}, \lambda)$ design $(X, \mathcal{B})$ with respect to $S \subseteq X$ is $(X', \mathcal{B}')$ where $X' = X \setminus S$ and $\mathcal{B}$= $\{B \setminus S : S \subseteq B \in \mathcal{B}\}$. If $|S| = j \leq t$, then the derived design is a $(t - j)$–$(v - j, \mathcal{K}', \lambda)$ design where $\mathcal{K}'$= $\{k - j : k \in \mathcal{K}\}$.

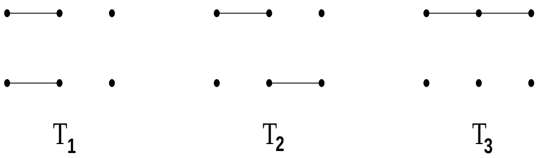


Figure 3:

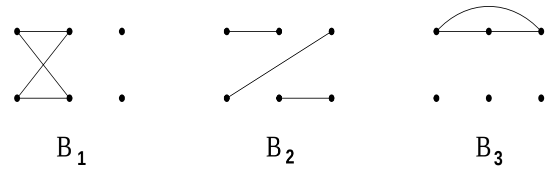


Figure 4:

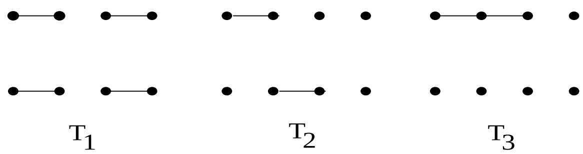


Figure 5:

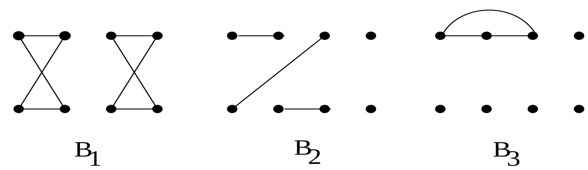


Figure 6:

Lemma 3.1 *Let $(X, \mathcal{B})$ be a Steiner graphical tBD of type n^r where $n \geq 2$ and $r \geq 3$. Write $t = \binom{x}{2} + y$ where $0 \leq y < x$ and let $B \in \mathcal{B}$.*

(1) *If $x \geq 4, n = 2$, and Figure 7 $\subseteq B$, then B is complete.*

(2) *If $x \geq 3, n \geq 3$, and Figure 8 $\subseteq B$, then B is complete.*

Proof: Recall that the rectangle drawn around the portion of the graph denotes that the graph is complete in this region.

1. Expand the rectangle to the right by swapping partite sets until we have the complete graph on $(x+1)$ partite sets contained in the block B. Note that swapping partite sets fixes $4\binom{x-1}{2}$ edges and simple calculations show that $4\binom{x-1}{2} \geq \binom{x}{2} + (x-1) \geq t$ whenever $x \geq 4$. Expansion is continued until B is complete.

2. Label the partite sets as $\mathrm{A}_1, \mathrm{A}_2, \ldots$. Expand the rectangle down one row by applying the permutations $\sigma_i = (a_{2i} a_{3i})$ for $1 \leq i \leq x$. Note that each σ_i displaces $2(x-1)$ edges and hence fixes $4\binom{x}{2} - 2(x-1)$ edges. Simple calculations show that $4\binom{x}{2} - 2(x-1) \geq \binom{x}{2} + (x-1) \geq t$ whenever $x \geq 2$. Continue to expand down until $K(\mathrm{A}_1, \mathrm{A}_2, \ldots, \mathrm{A}_x) \subseteq B$. If $r > x$, then expand

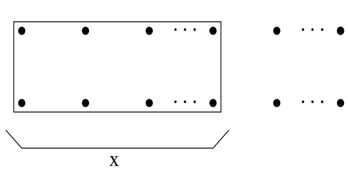


Figure 7:

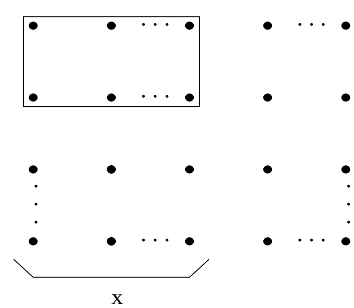


Figure 8:

the rectangle to the right by swapping partite sets. A simple swap fixes $n^2\binom{x-1}{2} \geq 9\binom{x-1}{2}$ edges as $n \geq 3$. Again, simple calculations show that $9\binom{x-1}{2} \geq \binom{x}{2} + (x-1) \geq t$ whenever $x \geq 3$. Expansion to the right is continued until B is complete.

■

Lemma 3.2 *If $t < n^2$ and if B is a block of a Steiner graphical tBD of type n^r ($n \geq 2$ and $r \geq 3$) with exactly t edges between any two partite sets, then B contains either the t–element set in Figure 9 where $t = n$, or the t–element set in Figure 10 where $t = n(n-1)$.*

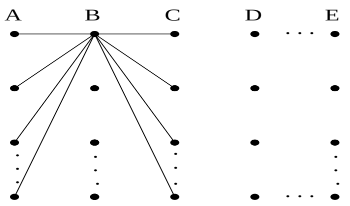


Figure 9: $t = n$

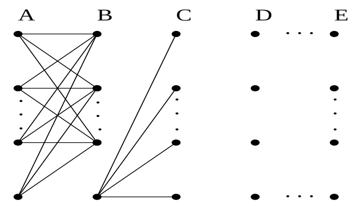


Figure 10: $t = n(n-1)$

Proof: Let $t < n^2$ and suppose B is a block with exactly t edges between the partite sets A and B. So $T = B \cap K(\text{A}, \text{B})$ is a t–element set. Then there exists an edge $e \in B \setminus T$. Now e cannot be between the partite sets A and B as then we would have $t+1$ edges between the partite sets A and B. So we may assume that, e either lies between the partite sets A and C, between the partite sets B and C, or between the partite sets C and D. If e is between the partite sets C and D, then switch inside C and D to force $K(\text{C}, \text{D}) \subseteq B$. This is a contradiction, for now switching in A and B will fix a t–element set contained in $K(\text{C}, \text{D})$ and will force an additional edge between the partite sets A and B. By interchanging the roles of the partite sets A and B, if necessary, we may assume the edge e lies between the partite sets B and C. Switching inside the partite set C guarantees at least $t+n$ edges in B.

Consider the permutation $\pi = (a_i a_j)$ for all $i \neq j$. Define $N_T(a_i) = \{b \in \text{B} : a_i b \in T\}$. Suppose $N_T(a_i) \neq N_T(a_j)$ for some $i \neq j$. Let $I = N_T(a_i) \cap N_T(a_j)$. Now π displaces $|N_T(a_i)| + |N_T(a_j)| - 2|I|$ edges in B. However, $|N_T(a_i)| + |N_T(a_j)| - 2|I| \leq |N_T(a_i)| + |N_T(a_j)| - |I| = |N_T(a_i) \cup N_T(a_j)| \leq n$. Thus π displaces at most n edges of T and hence fixes a set of at least t edges of B. Hence π produces an additional edge between the partite sets A and B while fixing a t–element set in B, a contradiction. Therefore, $N_T(a_i) = N_T(a_j)$ for all $i \neq j$.

If $|N_T(a_i)| = 1$, then B contains either the edge set given in Figure 9 or Figure 11. If $|N_T(a_i)| = n-1$, then B contains either the edge set given in Figure 10 or Figure 12. Otherwise, B contains Figure 13 or Figure 14.

In Figure 11, the permutation $(b_1 b_2)$ fixes $n = t$ edges in B and hence forces additional edges between the partite sets A and B, a contradiction. In Figure 12, the permutation $(b_2 b_n)$ displaces n edges and hence fixes at least $t + n - n = t$ edges. Thus this permutation forces additional edges between the partite sets A and B, a contradiction. In Figure 13 or Figure 14, note that $1 < |N_T(a_j)| < n-1$ and the permutation $(b_j b_{j-1})$ displaces n edges and fixes a t–element set. Thus this permutation produces additional edges between the partite sets A and B, a contradiction. Therefore B must contain either the edge set in Figure 9 or the edge set in Figure 10. ■

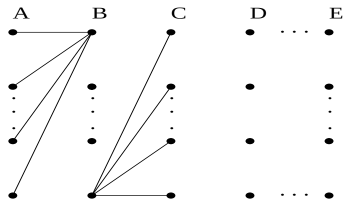


Figure 11:

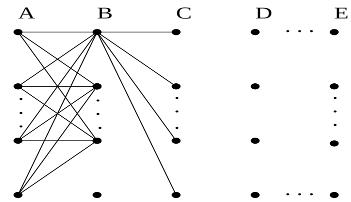


Figure 12:

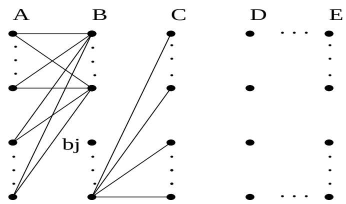


Figure 13: $1 < |N_T(a_j)| < n-1$

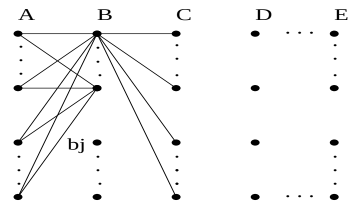


Figure 14: $1 < |N_T(a_j)| < n-1$

Lemma 3.3 *Let $(X, \mathcal{B})$ be a Steiner graphical t-wise balanced design of type n^r where $n \geq 2, r \geq 3$, and $t \leq n^2$. Define*

$$\mathcal{B}_{\mathrm{AB}} := \{Y \cap K(\mathrm{A}, \mathrm{B}) : Y \in \mathcal{B}, |Y \cap K(\mathrm{A}, \mathrm{B})| \geq t\}.$$

If $t \neq n$, or $n(n-1)$, then either $\mathcal{B}_{\mathrm{AB}}$ is one of the bigraphical t-wise balanced designs in [5] or $\mathcal{B}_{\mathrm{AB}}$ is a t-$(n^2, n^2, 1)$.

Proof: Clearly $\mathcal{B}_{\mathrm{AB}}$ is a t–$(n^2, \mathcal{K}, 1)$ design. However $\mathcal{B}_{\mathrm{AB}}$ may not be proper. Now $\mathcal{B}_{\mathrm{AB}}$ is not proper if and only if $|Y \cap K(\mathrm{A}, \mathrm{B})| = t$ or n^2 for some $Y \in \mathcal{B}$.

If $|Y \cap K(\mathrm{A}, \mathrm{B})| = t < n^2$, then by Lemma 3.2 we have $t = n$ or $t = n(n-1)$.

If $|Y \cap K(\mathrm{A}, \mathrm{B})| = n^2$, then $\mathcal{B}_{\mathrm{AB}}$ is a t-$(n^2, n^2, 1)$ design. ■

3.1 $t \geq 3$ and $n = 2$

We assume for this section that $(X, \mathcal{B})$ is a Steiner graphical tBD of type 2^r where $t \geq 3$ and $r \geq 3$. Note that the Steiner graphical tBDs of type 2^2 were determined in [5] and are included in Tables I, II, and III.

For some small values of t, we will show that there do not exist any Steiner graphical tBDs of type n^r for $n \geq 2$ and $r \geq 3$. This is a stronger statement than we need now, but we will use the full strength in Section 3.2 when we consider the cases for $n \geq 3$. For the majority of Section 3.1, we will assume $n = 2$ and write $t = \binom{x}{2} + y$ where $0 \leq y < x$ and $x \geq 3$. We will show that there do not exist any Steiner graphical tBDs of type 2^r $(r \geq 3)$ by considering two cases: (1) $r \geq x$ and (2) $r < x$. When $r \geq x$, we first establish a series of lemmas that rule out designs with small values of t. Following this, we have a sequence of lemmas for values of $t = \binom{x}{2} + y$ for most values of x and y. Finally, Theorem 3.22 establishes that there do not exist any Steiner graphical tBDs of type 2^r where $r \geq x$ and $t \geq 3$. The second part of Section 3.1 will discuss $r < x$.

Lemma 3.4 *There do not exist any proper Steiner graphical $3BDs$ of type n^r for $n \geq 2$ and $r \geq 3$.*

Proof: We first claim that $n = 2$. Indeed, suppose $n \geq 3$ and consider the 3–element set T in Figure 15. Let B be the block containing T. As $B \neq T$, B contains an additional edge, e. If $e = b_2c_1$, then B can be shown to be complete. By symmetry, if $e = b_2c_3$, then B is complete. So if e is not incident to c_2, then applying one of the permutations (c_1c_2) or (c_2c_3) displaces only one edge of T and hence fixes 3 edges of B. Furthermore, this permutation forces, by Lemma 1.1, either b_2c_1 or b_2c_3 to be in B; hence B is complete. Suppose e is incident to c_2. Then, up to symmetry, there are five cases to consider: a_1c_2, a_3c_2, b_1c_2, b_3c_2, c_2d_1. In each case it is possible to apply permutations to obtain $b_2c_1 \in B$. Thus, by the above arguments, B is complete. Therefore $n = 2$.

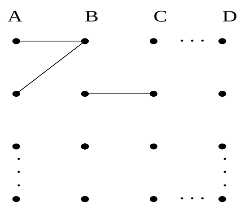


Figure 15:

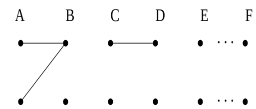


Figure 16:

Now consider the 3–element set T in Figure 16 and suppose $r \geq 5$. Let B be the block containing T. Since $B \neq T$, B contains an additional edge, e. If $e \in K(\text{C}, \text{D})$, then there exists a sequence of permutations which 3–generates $K(\text{C}, \text{D}, \text{E})$ from $T \cup \{e\}$. Hence by Lemma 2.1, B is complete. For all other possibilities for e, applying one of the permutations (c_1c_2) or (d_1d_2) displaces a single edge and hence fixes $t = 3$ edges of B. Furthermore, this permutation forces an additional edge in B between the partite sets C and D. Hence B is forced to be complete. Therefore, if there exists a Steiner graphical 3BD of type n^r, then $n = 2$ and $r = 3$ or $r = 4$.

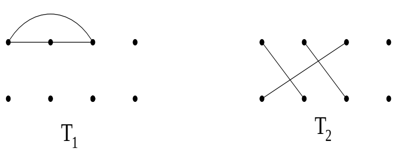


Figure 17:

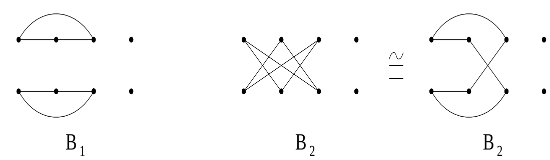


Figure 18:

The proofs that there does not exist a Steiner graphical 3BD of type 2^3 or of type 2^4 are essentially the same. Consider the 3–element sets T_1 and T_2 in Figure 17 and let B_1 and B_2 be the blocks containing T_1 and T_2, respectively. Note that the figures are shown for $r = 4$. If $r = 3$, then only consider the first three partite sets. As $B_1 \neq T_1$, B_1 contains an additional edge, e. There are, up to symmetry, four possibilities for e: a_2b_2, a_1b_2, c_1d_1, c_2d_2. In the first case, B_1 in Figure 18 is forced and in the other cases B_1 is forced to be complete. Similarly, as B_2 in Figure 18 is forced. However, $|B_1 \cap B_2| > 3 = t$, which contradicts the fact that $\lambda = 1$. Therefore there do not exist any Steiner graphical 3BDs of type n^r for $n \geq 2$ and $r \geq 3$. ■

Lemma 3.5 *There do not exist any Steiner graphical 4BDs of type n^r for $n \geq 2$ and $r \geq 3$.*

Proof: We first show that $n \geq 3$ by supposing $n = 2$. Consider the 4–element set T in Figure 19 and let B be the block containing T. As $B \neq T$, B contains an additional edge, e. Up to symmetry, there are five cases for e to consider: a_1b_2, a_1c_1, c_1d_1, b_1d_1, d_1e_1. For each case there exists a sequence of permutations which forces $K(\mathrm{A},\mathrm{B},\mathrm{C}) \subseteq B$ and hence by Lemma 2.1, B is complete. Thus if there exists a Steiner graphical 4BD of type n^r, then $n \geq 3$ and $r \geq 3$.

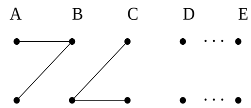


Figure 19:

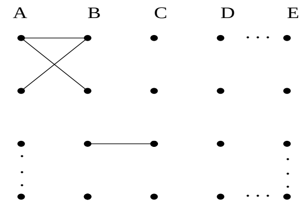


Figure 20:

Now consider the 4–element set T in Figure 20 and let B be the block containing T. As $B \neq T$, B contains an additional edge, e. Regardless of what the additional edge is, similar arguments force B to be complete. Therefore there do not exist any Steiner graphical 4BDs of type n^r for $n \geq 2$ and $r \geq 3$. ■

Lemma 3.6 *There do not exist any proper Steiner graphical* 5*BDs of type* n^r *for* $n \geq 2$ *and* $r \geq 3$.

Proof: We first show that for $r \geq 5$, B is forced to be complete by considering the 5–element

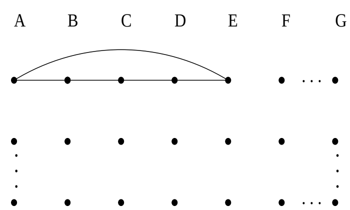


Figure 21:

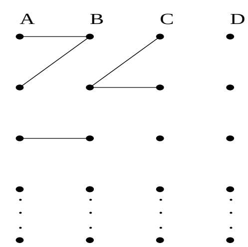


Figure 22:

set T in Figure 21. Let B be the block containing T. As $B \neq T$, B contains an additional edge, e. Regardless of what the additional edge is, arguments similar to previous lemmas force B to be complete. Thus, if there exists a Steiner graphical 5BD of type n^r, then $r = 3$ or $r = 4$.

Secondly, we show that $n \leq 3$. Assume that $n > 3$ and consider the 5–element set T in Figure 22. Let B be the block containing T. As $B \neq T$, B contains an additional edge, e. Regardless of what the additional edge is, arguments similar to previous lemmas force B to be complete. Therefore $n \leq 3$.

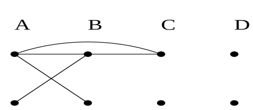


Figure 23:

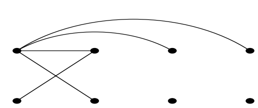


Figure 24:

Now suppose $n = 2$ and $r = 3$ and consider the 5–element set T in Figure 23. Note that we are examining only a portion of this figure. Let B be the block containing T. As $B \neq T$, B contains an additional edge, e. Up to symmetry, there are four possibilities for e: a_1c_2, a_2b_2, a_2c_1, and a_2c_2. In each of these cases, it is easy to show that B is forced to be complete. Therefore there does not exist a Steiner graphical 5BD of type 2^3.

Next consider the case $n = 2$ and $r = 4$. Again let T be the 5–element set in Figure 23. Attempts to cover T force the block $B = K(\text{A}, \text{B}, \text{C})$. Now consider the 5–element set in Figure 24. Attempts to cover this 5–element set forces either the block B_1 or B_2 in Figure 25. However, $|B \cap B_i| \geq 5 = t$ for $i = 1, 2$, which contradicts the fact that $\lambda = 1$. Therefore there does not exist a Steiner graphical 5BD of type 2^4.

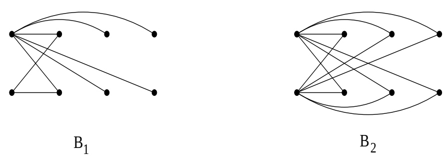


Figure 25:

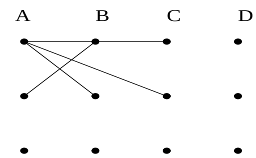


Figure 26:

Finally, consider the 5–element set T in Figure 26 where $n = 3$ and $r = 3$ or $r = 4$. A similar case analysis as in previous lemmas shows that B is forced to be complete regardless if $r = 3$ or 4. Therefore there does not exist a Steiner graphical 5BD of type 3^r for $r = 3$ or 4. ■

Lemma 3.7 *There do not exist any Steiner graphical* 6*BDs and* 7*BDs of type* 2^r *for* $r \geq 3$.

Proof: For $t = 6$, $n = 2$ and $r = 3$, Theorem 1.3 implies that $k = 7$. However, by divisibility conditions we know there cannot exist a $6-(12, 7, 1)$ design. Therefore there does not exist a Steiner graphical 6BD of type 2^3. Similarly, one can show that there does not exist a Steiner graphical 7BD of type 2^3.

For $t = 6$ and $r \geq 4$, consider the 6–element set T in Figure 27. Similar arguments as in previous lemmas show that the block containing T is forced to be complete. Therefore there does not exist a Steiner graphical 6BD of type 2^r for $r \geq 4$.

For $t = 7$ and $r \geq 4$, consider the 7–element set T_1 in Figure 28. To cover T_1 forces the block B_1 in Figure 29. To cover the 7–element set T_2 in Figure 28 forces the block B_2 in Figure 29. However $|B_1 \cap B_2| > 7$, which contradicts the fact that $\lambda = 1$. Therefore there does not exist a Steiner graphical 7BD of type 2^r for $r \geq 4$. ■

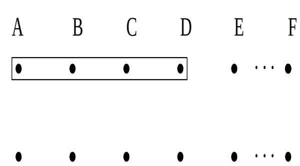


Figure 27:

Lemma 3.8 *There do not exist any proper Steiner graphical* t*BDs of type*

(a) 2^3, *for* $t \geq 8$.

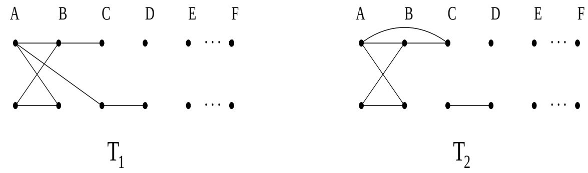


Figure 28:

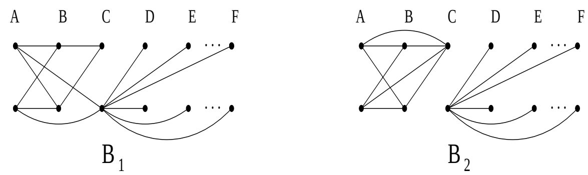


Figure 29:

(b) 2^4, *for* $t \geq 20$.

Proof: (a) Since $n = 2$ and $r = 3$, we have $v = 12$. For $8 \leq t \leq 11$, Theorem 1.3 implies that $k \leq \lfloor \frac{9+t}{2} \rfloor \leq t$, a contradiction. (b) Similarly, since $n = 2$ and $r = 4$, $v = 24$, and whenever $20 \leq t \leq 23$, we know, by Theorem 1.3, that $k \leq \lfloor \frac{21+t}{2} \rfloor \leq t$, a contradiction. Therefore there do not exist any proper Steiner graphical tBDs of type 2^3 for $t \geq 8$ and of type 2^4 for $t \geq 20$. ■

Lemma 3.9 *There do not exist any Steiner graphical* 8*BDs and* 9*BDs of type* 2^r.

Proof: For $t = 8$ and $t = 9$, note that $n = 2$ and so by Lemma 3.8, we know $r \geq 4$. Consider the 8–element set T in Figure 30 and the 9–element set T' in Figure 31. A similar argument as used in previous lemmas can be used to show that blocks containing T and T' are forced to be complete. Therefore there do not exist any Steiner graphical 8BDs and 9BDs of type 2^r. ■

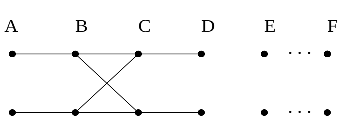


Figure 30:

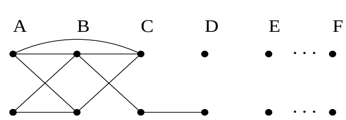


Figure 31:

We draw your attention to Table V. The table gives the lemma or theorem number used to show that there does not exist a Steiner graphical tBD of type 2^r for some small values of t ($3 \leq t \leq 99$) whenever $r \geq x$. (Recall that we write $t = \binom{x}{2} + y$ where $0 \leq y < x$ and $x \geq 3$.) The first column in the table gives the first digit for the value of t while the top row in the table gives the second digit. So, for example, Lemma 3.12 shows that there does not exist a Steiner graphical 12BD of type 2^r for $r \geq 5$.

Lemma 3.10 *If* $t = \binom{x}{2}$ *where* $r \geq x \geq 5$, *then there does not exist a Steiner graphical* t*BD of type* 2^r.

Table V: $n = 2$ **and** $r \geq x \geq 3$

t	0	1	2	3	4	5	6	7	8	9
0				3.4	3.5	3.6	3.7	3.7	3.9	3.9
1	3.10	3.11	3.12	3.22	3.22	3.10	3.11	3.12	3.22	3.22
2	3.22	3.10	3.11	3.12	3.13	3.22	3.22	3.22	3.10	3.11
3	3.12	3.13	3.14	3.15	3.22	3.22	3.10	3.11	3.12	3.13
4	3.14	3.15	3.16	3.22	3.22	3.10	3.11	3.12	3.13	3.14
5	3.15	3.16	3.17	3.18	3.22	3.10	3.11	3.12	3.13	3.14
6	3.15	3.16	3.17	3.18	3.19	3.20	3.10	3.11	3.12	3.13
7	3.14	3.15	3.16	3.17	3.18	3.19	3.20	3.21	3.10	3.11
8	3.12	3.13	3.14	3.15	3.16	3.17	3.18	3.19	3.20	3.21
9	3.19	3.10	3.11	3.12	3.13	3.14	3.15	3.16	3.17	3.18

Proof: We will illustrate the proof when $x = 6$ by considering the 15–element set T in Figure 32. Recall that the rectangle drawn across the first row of vertices indicates that the graph is complete in this region. Let B be the block containing T. As $B \neq T$, B contains an additional edge, e. If $e = a_2e_1$, then there exists a sequence of permutations which 15–generates Figure 7 from $T \cup \{a_2e_1\}$. By Lemma 1.2, B contains Figure 7 and hence by Lemma 3.1, B is complete.

So, if the additional edge e is not incident to a_1 and not incident to a_2, then the permutation (a_1a_2) displaces exactly one edge of $T \cup \{e\}$ and hence fixes t edges. Furthermore, the permutation forces $a_2e_1 \in B$; hence B is complete. Now assume that e is incident to a_1 or a_2. Up to symmetry, we have the following 14 cases for e to consider: a_1b_2, a_1c_2, a_1d_2, a_1e_2, a_1g_1, a_2b_2, a_2c_2, a_2d_2, a_2e_2, a_2g_1, a_1f_1, a_1f_2, a_2f_1, and a_2f_2. In all these cases it is easy to force $a_2e_1 \in B$. Thus B is complete in all cases.

If $x = 5$, then omit partite set B shown in Figure 32 and follow the argument above. For $x > 6$, add additional partite sets between the partite sets A and B such that the top vertex in each additional partite set is incident to a_2. In this case, some additional swapping is necessary, but can easily be done. Therefore there does not exist a Steiner graphical tBD of type 2^r when $t = \binom{x}{2}$ and $r \geq x \geq 5$. ■

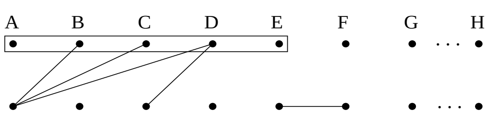


Figure 32:

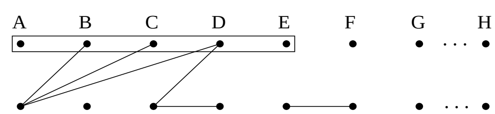


Figure 33:

Lemma 3.11 *If* $t = \binom{x}{2} + 1$ *where* $r \geq x \geq 5$, *then there does not exist a Steiner graphical* t*BD of type* 2^r.

Proof: Consider the t–element set T in Figure 33. Note that the figure is shown for $x = 6$. If $x = 5$, then omit the partite set B. For $x > 6$, add additional partite sets between the partite sets A and B such that the top vertex in each additional partite set is incident to a_2. The same strategy and techniques used in the proof of Lemma 3.10 can be used to show that the block containing T is forced to be complete. Therefore there does not exist a Steiner graphical tBD of type 2^r when

$t = \binom{x}{2} + 1$ and $r \geq x \geq 5$. ■

Lemma 3.12 *If $t = \binom{x}{2} + 2$ where $r \geq x \geq 5$, then there does not exist a Steiner graphical tBD of type 2^r.*

Proof: We illustrate the proof when $x = 5$ by considering the t–element set T in Figure 34. For $x > 5$, add additional partite sets between the partite sets C and D. Similar arguments as used in Lemma 3.10 can be used to show that the block containing T is forced to be complete. Complete details of the proof can be found in [7]. Therefore there does not exist a Steiner graphical tBD of type 2^r when $t = \binom{x}{2} + 2$, $r \geq x \geq 5$. ■

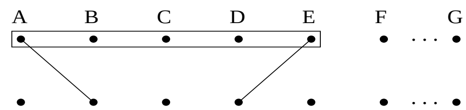


Figure 34:

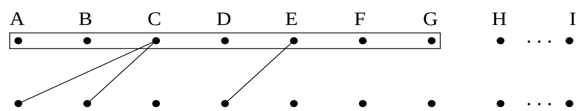


Figure 35:

Lemma 3.13 *If $t = \binom{x}{2} + 3$ where $r \geq x \geq 7$, then there does not exist a Steiner graphical tBD of type 2^r.*

Proof: We illustrate the proof when $x = 7$ by considering the t–element set T in Figure 35. For $x > 7$, add additional partite sets between the partite sets F and G. Similar arguments as used in Lemma 3.10 can be used to show that the block containing T is forced to be complete. Therefore there does not exist a Steiner graphical tBD of type 2^r when $t = \binom{x}{2} + 3$ and $r \geq x \geq 7$. ■

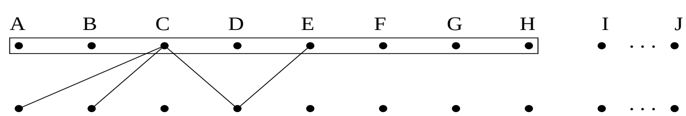


Figure 36:

Lemma 3.14 *If $t = \binom{x}{2} + 4$ where $r \geq x \geq 8$, then there does not exist a Steiner graphical tBD of type 2^r.*

Proof: We illustrate the proof when $x = 8$ by considering the t–element set T in Figure 36. If $x > 8$, then add additional partite sets between the partite sets G and H. Similar arguments as used in Lemma 3.10 can be used to show that the block containing T is forced to be complete. Therefore a Steiner graphical tBD of type 2^r does not exist when $t = \binom{x}{2} + 4$ and $r \geq x \geq 8$. ■

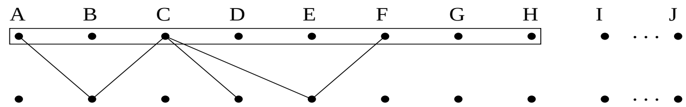


Figure 37:

Lemma 3.15 *If $t = \binom{x}{2} + 5$ where $r \geq x \geq 8$, then there does not exist a Steiner graphical tBD of type 2^r.*

Proof: We illustrate the proof when $x = 8$ by considering the t–element set T in Figure 37. If $x > 8$, then add additional partite sets between the partite sets G and H. Similar arguments to those used in Lemma 3.10 can be used to show that the block containing T is forced to be complete. Therefore a Steiner graphical tBD of type 2^r does not exist when $t = \binom{x}{2} + 5$ and $r \geq x \geq 8$. ∎

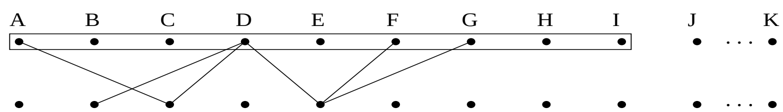


Figure 38:

Lemma 3.16 *If $t = \binom{x}{2} + 6$ where $r \geq x \geq 9$, then there does not exist a Steiner graphical tBD of type 2^r.*

Proof: We illustrate the proof when $x = 9$ by considering the t–element set T in Figure 38. If $x > 9$, then add additional partite sets between H and I. Similar arguments as used in Lemma 3.10 can be used to show that the block containing T is forced to be complete. Therefore there does not exist a Steiner graphical tBD of type 2^r when $t = \binom{x}{2} + 6$ and $r \geq x \geq 9$. ∎

Lemma 3.17 *If $t = \binom{x}{2} + 7$ where $r \geq x \geq 10$, then there does not exist a Steiner graphical tBD of type 2^r.*

Proof: We illustrate the proof when $x = 10$ by considering the t–element set T in Figure 39. If $x > 10$, then add additional partite sets between I and J. Similar arguments as used in Lemma 3.10 can be used to show that the block containing T is forced to be complete. Therefore there does not exist a Steiner graphical tBD of type 2^r when $t = \binom{x}{2} + 7$ and $r \geq x \geq 10$. ∎

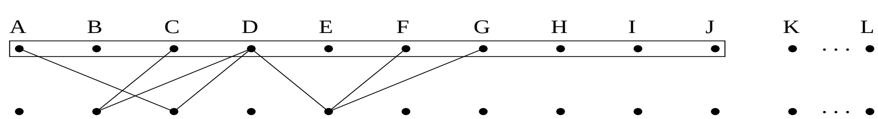


Figure 39:

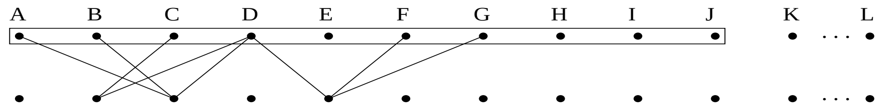


Figure 40:

Lemma 3.18 *If $t = \binom{x}{2} + 8$ where $r \geq x \geq 10$, then there does not exist a Steiner graphical tBD of type 2^r.*

Proof: Consider the t–element set T in Figure 40. Note that the figure is shown for $x = 10$. If $x > 10$, then add additional partite sets between the partite sets I and J. Similar arguments as used in Lemma 3.10 can be used to show that the block containing T is forced to be complete. Therefore a Steiner graphical tBD of type 2^r does not exist when $t = \binom{x}{2} + 8$ and $r \geq x \geq 10$. ■

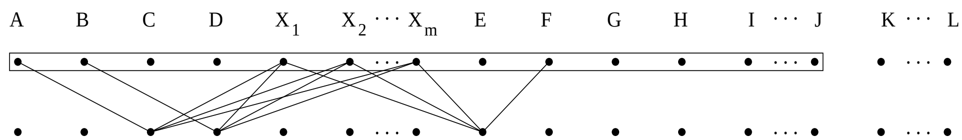


Figure 41:

Lemma 3.19 *If $t = \binom{x}{2} + y$ where $0 \leq y < x$, $r \geq x \geq 11$, $y \geq 9$, and $y \equiv 0\,(mod\ 3)$, then there does not exist a Steiner graphical tBD of type 2^r.*

Proof: Write $y = 3(m+1)$ where $m \geq 2$ and consider the t–element set T in Figure 41. Note that the top vertex in the partite set $X_i\,(1 \leq i \leq m)$ is incident to c_2, d_2, and e_2. The same strategy and techniques as used in the proof of Lemma 3.10 can be used to show that the block containing T is forced to be complete. ■

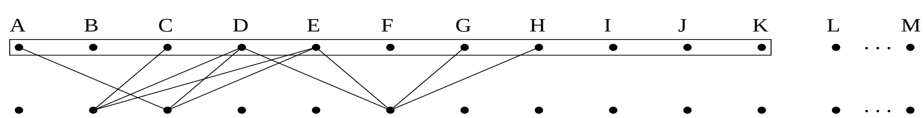


Figure 42:

Lemma 3.20 *If $t = \binom{x}{2} + y$ where $0 \leq y < x$, $r \geq x \geq 11$, $y \geq 10$, and $y \equiv 1\,(mod\ 3)$, then there does not exist a Steiner graphical tBD of type 2^r.*

Proof: We illustrate the proof when $x = 11$ and $y = 10$ by considering the t–element set T in Figure 42. For $x > 11$, add additional partite sets between the partite sets J and K. If $y > 10$, then add the necessary number of partite sets between D and E such that the top vertex in each additional partite set is incident to b_2, c_2, and f_2. The strategy and techniques used in the proof of Lemma 3.10 can be used to show that the block containing T is forced to be complete. ■

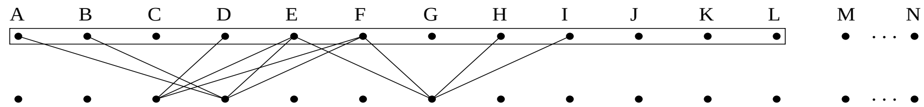


Figure 43:

Lemma 3.21 *If $t = \binom{x}{2} + y$ where $0 \leq y < x$, $r \geq x \geq 12$, $y \geq 11$, and $y \equiv 2 \,(mod\ 3)$, then there does not exist a Steiner graphical tBD of type 2^r.*

Proof: We illustrate the proof when $x = 12$ and $y = 11$ by considering the t–element set T in Figure 43. For $x > 12$, add additional partite sets between the partite sets K and L. If $y > 11$, then add the necessary number of partite sets between E and F such that the top vertex in each additional partite set is incident to c_2, d_2, and g_2. The strategy and techniques used in the proof of Lemma 3.10 can be used to show that the block containing T is forced to be complete. ■

Theorem 3.22 *If $t = \binom{x}{2} + y$, where $0 \leq y < x$ and $r \geq x \geq 3$, then there does not exist a Steiner graphical tBD of type 2^r.*

Proof: By Lemma 3.4 through Lemma 3.9, we know that there do not exist any Steiner graphical tBDs of type 2^r for $t = 3$, 4, 5, 6, 7, 8, or 9. By Lemma 3.10 through Lemma 3.21, we know that there do not exist any Steiner graphical tBDs of type 2^r except possibly for $t = 13$, 14, 18, 19, 20, 26, 27, 34, 35, 43, 44, and 54. In each case we consider a different t–element set and show the nonexistence of each Steiner graphical tBD of type 2^r. Complete details of these cases can be found in [7]. ■

Recall that $t = \binom{x}{2} + y$, where $0 \leq y < x$ and $x \geq 3$. Up to this point we have shown that there do not exist any Steiner graphical tBDs of type 2^r for all values of $t \geq 3$ when $r \geq x$. We now prove that there do not exist any Steiner graphical tBDs of type 2^r for all values of $t \geq 3$ when $r < x$. We first consider the case $x \leq 7$ and then close the section by considering the case $x \geq 8$. For $x \leq 7$, we need the following result.

Lemma 3.23 (a) *If $10 \leq t \leq 19$, then there do not exist any proper Steiner graphical tBDs of type 2^4.*

(b) *If $15 \leq t \leq 27$, then there do not exist any proper Steiner graphical tBDs of type 2^5.*

(c) *If $21 \leq t \leq 27$, then there do not exist any proper Steiner graphical tBDs of type 2^6.*

Proof: Case (a): If $13 \leq t \leq 19$, then taking the derived design with respect to $\bigcup_{r=2}^{4} K(\mathrm{A}_1, \mathrm{A}_i)$ we obtain a Steiner graphical sBD of type 2^3 where $1 \leq s \leq 7$. By Lemmas 3.4, 3.5, 3.6, 3.7, and 3.7, we know that there does not exist a Steiner graphical sBD of type 2^3 for $3 \leq s \leq 7$. Hence there does not exist a Steiner graphical tBD of type 2^4 for $15 \leq t \leq 19$. It remains to show that there do not exist any Steiner graphical tBDs of type 2^4 for $10 \leq t \leq 14$. In each case we consider a different t–element set and show the nonexistence of a Steiner graphical tBD of type 2^4. Complete details of these cases can be found in [7]. Therefore there do not exist any proper Steiner graphical tBDs of type 2^4 for $10 \leq t \leq 19$.

Case (b): We take the derived design with respect to $\bigcup_{r=2}^{5} K(\mathrm{A}_1, \mathrm{A}_i)$. Similar arguments and techniques as used above can be used to show the nonexistence of any proper Steiner graphical tBDs of type 2^5 for $15 \leq t \leq 27$.

Case (c): By taking the derived design with respect to $\bigcup_{r=2}^{6} K(\mathrm{A}_1, \mathrm{A}_i)$ and employing similar arguments and techniques as used in Case (a), one can show the nonexistence of any proper Steiner graphical tBDs of type 2^6 for $12 \leq t \leq 27$. ■

Theorem 3.24 *Write $t = \binom{x}{2} + y$, where $0 \leq y < x$. If $3 \leq r < x \leq 7$, then there do not exist any Steiner graphical tBDs of type 2^r.*

Proof: Note that $3 \leq r < x \leq 7$ implies that $6 \leq t \leq 27$. Lemma 3.7 proves that there does not exist a Steiner graphical 6BD or 7BD of type 2^3. To show the nonexistence of tBDs for the remaining values of t, we use Lemma 3.8, as well as, Lemma 3.23 ■

Given that $t = \binom{x}{2} + y$, where $0 \leq y < x$ and $x \geq 3$, we have proved that if $3 \leq r < x \leq 7$, then there do not exist any Steiner graphical tBDs of type 2^r. It remains to prove that there do not exist any Steiner graphical tBDs of type 2^r where $3 \leq r < x$ and $x \geq 8$. We first prove the following lemma.

Lemma 3.25 *If $t = 4(r-1) + 1$ and $r \geq 4$, then there does not exist a Steiner graphical tBD of type 2^r.*

Proof: If $r = 4, 5, 6$, then $t = 13, 17, 21$, respectively. Lemma 3.23 shows that there do not exist such designs. If $r = 7$, then $t = 25$ and Case 6 in the proof of Theorem 3.22 shows that there cannot exist such a design.

For $r \geq 8$, consider the $(4(r-1)+1)$–element set T in Figure 44 and let B be the block containing T. As $B \neq T$, B contains an additional edge, e. If $e = f_2h_1$, then we claim that B is forced to be complete. Swap partite set H with all of the $r-8$ partite sets to the right of H. Next swap A and B, swap A and D, swap B and D, and swap C and D. Now swap partite set F with each of the $r-6$ partite sets to the right of F. Up to this point $|B| \geq t+1+(r-8)+1+1+1+1+2(r-6) = t+3r-15$. Swapping partite sets E and F displaces the 4 edges incident to e_1, the 2 edges incident to f_1 and the 2 edges incident to f_2 and hence fixes $t + 3r - 15 - 8 = t + 3r - 23 \geq t$ as $r \geq 8$. Continue to swap partite sets until B contains all the edges which are between the top row of vertices and the bottom row of vertices. Now $|B| \geq 2\binom{r}{2} = r(r-1)$. Finally switch inside each partite set to force B to be complete. Note that each switch (e.g. (a_1a_2)) fixes all the edges between the remaining $(r-1)$ partite sets, which is $2\binom{r-1}{2} = (r-1)(r-2) \geq 4(r-1)+1 = t$ as $r \geq 8$. Thus if $f_2h_1 \in B$, then B is forced to be complete.

So, if e is not incident to g_1, g_2, h_1, h_2, then we can easily force $f_2h_1 \in B$ by swapping G and H. Hence B is forced to be complete. Now suppose e is incident to g_1, g_2, h_1, or h_2. Up to symmetry, we have several possibilities for e which we have classified into 3 different types:

1: $e \in K(\mathrm{A}, \mathrm{G}) \cup K(\mathrm{B}, \mathrm{G})$

2: $e \in K(\mathrm{A}, \mathrm{H}) \cup K(\mathrm{B}, \mathrm{H})$

and

3: $e \in K(\mathrm{C}, \mathrm{G}) \cup K(\mathrm{D}, \mathrm{G}) \cup K(\mathrm{E}, \mathrm{G}) \cup K(\mathrm{F}, \mathrm{G}) \cup K(\mathrm{C}, \mathrm{H}) \cup K(\mathrm{D}, \mathrm{H}) \cup K(\mathrm{E}, \mathrm{H}) \cup K(\mathrm{F}, \mathrm{H}) \cup K(\mathrm{G}, \mathrm{H}) \cup K(\mathrm{H}, \mathrm{I})$.

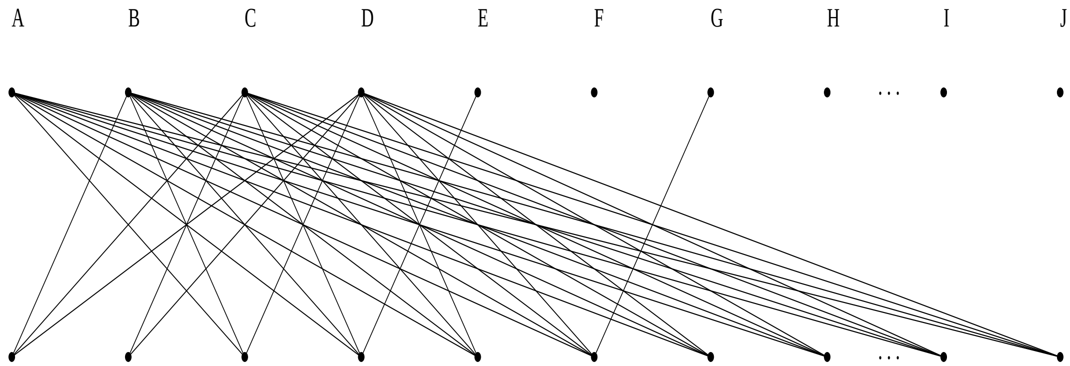


Figure 44:

Regardless of what the additional edge e is we can swap partite sets and always force $f_2h_1 \in B$; hence B is complete. Therefore there does not exist a Steiner graphical $(4(r-1)+1)$BD of type 2^r for $r \geq 8$. ■

Theorem 3.26 *If $t = \binom{x}{2} + y, 0 \leq y < x, 3 \leq r < x$ and $x \geq 8$, then there do not exist any Steiner graphical tBDs of type 2^r.*

Proof: We first note that $x \geq 8$ implies that $t \geq 28$. Secondly, note that $x \geq 8$ and $r < x$ implies that $t = \binom{x}{2} + y \geq \binom{x}{2} \geq 4(x-1) > 4(r-1)$. So by taking the derived design with respect to $\bigcup_{i=2}^{r} K(\mathrm{A}_1, \mathrm{A}_i)$ we obtain a Steiner graphical t'BD of type $2^{r'}$ where $t' = t - 4(r-1)$ and $r' = r-1$. Continue taking the derived design until either $r \geq x$ or $x < 8$.

If $3 < x < 8$, then we are done by Theorem 3.24. If $r \geq x \geq 3$, then we are done by Theorem 3.22. Otherwise it must be that $r \geq x$ and so $x = 2$. Therefore we must show that the Steiner bigraphical designs D_6 and D_7 do not extend and the designs D_{12}, D_{13}, D_{14}, D_{15}, and D_{16} do not extend. To do this we will make the following observations:

(a) there does not exist a Steiner graphical 9BD of type 2^3 by Lemma 3.8.

(b) there does not exist a Steiner graphical 17BD of type 2^5 by Lemma 3.23.

(c) there do not exist any Steiner graphical tBDs of type 2^r for $t = 4(r-1)+1$ and $r \geq 4$ by Lemma 3.25.

(d) there does not exist a Steiner graphical 14BD of type 2^4 by Lemma 3.23.

(e) there does not exist a Steiner graphical 18BD of type 2^5 by Lemma 3.23.

■

We end this section with a short summary. For $n = 2$, we write $t = \binom{x}{2} + y$, where $0 \leq y < x$ and $x \geq 3$. If $r \geq x$, then Theorem 3.22 proves that there do not exist any Steiner graphical tBDs of type 2^r. For $3 \leq r < x \leq 7$, Theorem 3.24 proves that there do not exist any Steiner graphical tBDs of type 2^r. Finally, Theorem 3.26 shows that there do not exist any Steiner graphical tBDs of type 2^r for $3 \leq r < x$ and $x \geq 8$. Therefore, if $t \geq 3$, then there do not exist any Steiner graphical tBDs of type 2^r for $r \geq 3$.

3.2 $t \geq 3$ and $n \geq 3$

We first show that there do not exist any Steiner graphical tBDs of type n^r for $t \geq n^2$ when $n \geq 3$ and $r \geq 3$. Secondly, we prove that there do not exist any Steiner graphical tBDs of type n^r for $t < n^2$ when $n \geq 3$ and $r \geq 3$ using Lemma 3.3. For $t \geq n^2$, we begin by considering $r = 3$. We refer the reader to [7] for the complete details.

Theorem 3.27 *There does not exist a Steiner graphical tBD of type n^3 for $t > n^2$ and $n \geq 3$.*

Proof: The proof given in [7] is organized into the following five cases: (i) $t = n^2+1$ for $n \geq 3$ (ii) $t = n^2+2$ for $n \geq 3$, (iii) $12 \leq t \leq 18$ where $n = 3$, (iv) $n^2+3 \leq t \leq 2n^2$ for $n \geq 4$, and (v) $t > 2n^2$ for $n \geq 3$. ∎

We now consider values of t such that $\binom{\ell}{2}n^2 < t \leq \binom{\ell+1}{2}n^2$ where $\ell \geq 2$ and $n \geq 3$ when $r - \ell \geq 2$ and then when $r - \ell = 1$. Again we refer the reader to [7] for the proofs.

Lemma 3.28 *If $t = \binom{\ell}{2}n^2 + 1$ where $\ell \geq 2$, $n \geq 3$, and $r - \ell \geq 2$, then there do not exist any Steiner graphical tBDs of type n^r.*

Lemma 3.29 *Given $\ell \geq 2$ and partite sets $\mathrm{A}_1, \mathrm{A}_2, \ldots, \mathrm{A}_\ell, \mathrm{B}, \mathrm{C}, \ldots,$ where $\mathrm{B} := \mathrm{A}_{\ell+1}, \mathrm{C} := \mathrm{A}_{\ell+2}$, etc. If*

$$\bigcup_{1 \leq i < j \leq \ell+1} K(\mathrm{A}_i, \mathrm{A}_j) \cup K(\mathrm{B}, \mathrm{C})$$

is contained in the block B containing T, then B is complete.

Theorem 3.30 *If $\binom{\ell}{2}n^2 < t \leq \binom{\ell+1}{2}n^2$ where $\ell \geq 2, n \geq 3$, and $r - \ell \geq 2$, then there do not exist any Steiner graphical tBDs of type n^r.*

Proof: Note that $r \geq 4$. If $t = \binom{\ell}{2}n^2 + 1$, then Lemma 3.28 shows that there does not exist a Steiner graphical tBD of type n^r.

For the remaining values for t, let $s = t - \binom{\ell}{2}n^2$ and label the partite sets as $\mathrm{A}_1, \mathrm{A}_2, \ldots, \mathrm{A}_\ell$, B, $\mathrm{C}, \ldots,$ where $\mathrm{B} := \mathrm{A}_{\ell+1}, \mathrm{C} := \mathrm{A}_{\ell+2}$, etc. We will construct a subgraph

$$T := \bigcup_{1 \leq i < j \leq \ell} K(\mathrm{A}_i, \mathrm{A}_j) \cup S$$

with t edges and where $|S| = s$. Note that $0 < s \leq ln^2 = \binom{l+1}{2}n^2 - \binom{l}{2}n^2$. We organize s into four cases: (i) $2 \leq s \leq \ell n - 1$, $\ell \geq 2$, (ii) $2n \leq s < 2n^2$, (iii) $\ell n \leq s < \ell n^2$, $\ell > 2$, and (iv) $s = \ell n^2$, $\ell \geq 2$. In each case, a set S is constructed and then Lemma 3.29 is used to force B to be complete. Therefore whenever $\binom{\ell}{2}n^2 < t \leq \binom{\ell+1}{2}n^2$ and $r - \ell \geq 2, \ell \geq 2, n \geq 3$, there do not exist any Steiner graphical tBDs of type n^r. ∎

Theorem 3.31 *If $\binom{\ell}{2}n^2 < t \leq \binom{\ell+1}{2}n^2$ where $\ell \geq 2, n \geq 3$, and $r - \ell = 1$, then there do not exist any Steiner graphical tBDs of type n^r.*

Proof: We first claim that if $r \geq 4$, then we can take the derived design through $(r-1)n^2$ edges. Note that $t > \binom{\ell}{2}n^2 = \binom{r-1}{2}n^2 \geq (r-1)n^2$ provided that $r \geq 4$. Now we take the derived design with respect to $\bigcup_{i=2}^{r} K(\mathrm{A}_1, \mathrm{A}_i)$ which has $(r-1)n^2$ edges. Thus the claim holds. After taking the derived design through $(r-1)n^2$ edges we obtain a Steiner graphical t'BD of type $n^{r'}$ where $t' = t-(r-1)n^2$ and $r' = r-1$. Continue to take the derived design until $t \le (r-1)n^2$ or $r=3$.

Suppose $t \le (r-1)n^2$. If $r-\ell \ge 2$, then Theorem 3.30 shows that there do not exist any Steiner graphical tBDs of type n^r. So suppose $r-\ell \le 1$. Then $\binom{\ell}{2}n^2 < t \le (r-1)n^2$ which implies $\binom{\ell}{2} < (r-1) \le \ell$. Simplifying we have that $\ell - 1 < 2$ or $\ell < 3$. Thus, $\ell = 2$ and $r = 3$ which implies that $n^2 < t \le 3n^2$

So, we need only show that there do not exist any Steiner graphical tBDs of type n^3 $(n \ge 3)$ for $n^2 < t \le 3n^2$. This has been proven by Theorem 3.27.

Therefore whenever $\binom{\ell}{2}n^2 < t \le \binom{\ell+1}{2}n^2$ and $\ell \ge 2, n \ge 3$, and $r-\ell = 1$ there do not exist any Steiner graphical tBDs of type n^r. ■

Lemma 3.32 *There do not exist any Steiner graphical tBDs of type n^r for $t = n^2$ when $n \ge 3$ and $r \ge 3$.*

Proof: First suppose $r = 3$ and consider $T = K(\text{A}, \text{B})$. One can easily show that the block containing T is $B = K(\text{A}, \text{B}, \text{C}) = K_{n,n,n}$, a contradiction. Therefore there does not exist a Steiner graphical n^2BD of type n^3, $n \ge 3$.

Now assume $r \ge 4$ and consider the t–element set

$$T := K(\text{A}, \text{B}) \cup \{a_1d_1, b_2c_2\} \setminus \{a_{n-1}b_n, a_nb_n\}.$$

Similar arguments and techniques as seen before force the block containing T to be complete. Therefore there does not exist a Steiner graphical n^2BD of type n^r, $n \ge 3$ and $r \ge 4$. ■

Theorem 3.33 *If $t \ge n^2$ where $n \ge 3$ and $r \ge 3$, then there do not exist any Steiner graphical tBDs of type n^r.*

Proof: Theorem 3.27 proves that there do not exist any Steiner graphical tBDs of type n^3 for $t > n^2$. Theorem 3.30 and Theorem 3.31 establish that there do not exist any Steiner graphical tBDs of type n^r for $t > n^2$ and $r \ge 4$. Finally, Lemma 3.32 proves there does not exist a Steiner graphical n^2BD of type n^r for $n \ge 3$ and $r \ge 3$. ■

We finish this section by proving that there do not exist any Steiner graphical tBDs of type n^r for $t < n^2$ when $n \ge 3$ and $r \ge 3$.

Lemma 3.34 *If $n = t \ge 3$, then there does not exist a Steiner graphical tBD of type n^r for $r \ge 3$.*

Proof: If $n = t = 3$, then Lemma 3.4 proves there does not exist a Steiner graphical 3BD of type 3^r for $r \ge 3$. So we may assume that $n = t \ge 4$. Let

$$T := \{a_ib_1 | 1 \le i \le n-2\} \cup \{b_nc_i | i = n-1, n\}$$

and let B be the block containing T. Similar arguments and techniques as seen before force B to be complete. Therefore there does not exist a Steiner graphical nBD of type n^r when $n \ge 4$ and $r \ge 3$. ■

Lemma 3.35 *If $t = n(n-1)$ and $n \ge 3$, there does not exist a Steiner graphical tBD of type n^r for $r \ge 3$.*

Proof: Consider the t–element set T in Figure 45. Similar arguments and techniques as seen before force the block containing T to be complete. Therefore there does not exist a Steiner graphical tBD of type n^r for $t = n(n-1)$ when $n \geq 4$ and $r \geq 3$.

It remains to show that when $n = 3$ and $t = n(n-1) = 6$, there does not exist a Steiner graphical 6BD of type n^r for $r \geq 3$. Consider the 6–element set in Figure 46 Similar arguments and techniques as seen before force the block containing T to be complete. Therefore there does not exist a Steiner graphical 6BD of type 3^r. ∎

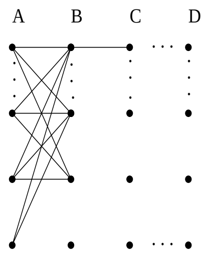


Figure 45:

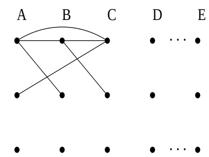


Figure 46:

Lemma 3.36 *If $3 \leq t < n^2$ and $n \geq 3$, then a Steiner graphical tBD of type n^r for $r \geq 3$, cannot have a block B containing $K(\mathrm{A},\mathrm{B})$.*

Proof: Note that throughout this proof we are assuming that the Steiner graphical tBD has a block B containing $K(\mathrm{A},\mathrm{B})$. If $n = 3$, then we need only consider $7 \leq t < 9$ as $t = 3, 4, 5, 6$ were dealt with in Lemmas 3.4, 3.5, 3.6, and 3.35, respectively. For $t = 7$, consider the 7–element set T in Figure 47 and let B' be the block containing T. Note that $B' \neq B$ and $|B' \cap K(\mathrm{A},\mathrm{B})| \leq t-1$ for otherwise $|B \cap B'| \geq t$. Regardless of what the additional edge $e \in B' \backslash T$ is incident to, we can always force an additional edge $e' \in K(\mathrm{A},\mathrm{B})$, which is a contradiction. For $t = 8$, take T to be the 7–element set in Figure 47 together with the edge $\{a_3b_3\}$. The same argument as in the $t = 7$ case forces a contradiction. Therefore we may assume that $n \geq 4$.

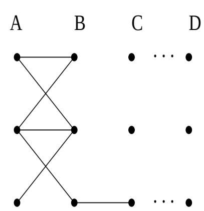


Figure 47:

For $n \geq 4$, we construct a t–element set T where $|T \cap K(\mathrm{A},\mathrm{B})| = t-1$. We organize t into 5 cases. Observe that these 5 cases cover $t \geq 9$. In each of these cases we will prove that there cannot exist a Steiner graphical tBD of type n^r for $n \geq 4$ and $r \geq 3$, if B is a block containing $K(\mathrm{A},\mathrm{B})$. Let B' be the block containing T and let $e \in B' \backslash T$. Observe that $B' \neq B$ and $|B' \cap K(\mathrm{A},\mathrm{B})| \leq t-1$ for otherwise $|B \cap B'| \geq t$.

Case (i): $t - 1 = \ell^2 + 2k$ where $1 \leq k < \ell < n$, $\ell \geq 3$. Consider the t–element set

$$T := \{a_ib_n | i = 2, \ldots, k+1\} \cup \{b_ia_n | i = 2, \ldots, k+1\} \cup \{a_ib_j | 1 \leq i, j \leq \ell\} \cup \{b_nc_n\}$$

Regardless of what the additional edge e is incident to, we can always force an additional edge $e' \in K(\text{A}, \text{B})$. However, this is a contradiction since $|B \cap B'| \geq t$.

Case (ii): $t-1 = \ell^2+2k+1$ where $1 \leq k < \ell < n$, $\ell \geq 3$. Add the edge $\{a_n b_n\}$ to the t–element set in Case (i) and apply the argument from Case (i).

Case (iii): $t-1 = \ell^2 - 1$ where $3 \leq \ell < n$. Consider the t–element set $T := \{a_i b_j | 1 \leq i, j \leq \ell < n\} \setminus \{a_{ell} b_{ell}\} \cup \{b_n c_n\}$. Regardless of what the additional edge e is incident to, we can always force an additional edge $e' \in K(\text{A}, \text{B})$. This is a contradiction since $|B \cap B'| \geq t$.

Case (iv): $t-1 = \ell^2$ where $3 \leq \ell < n$. Add the edge $\{a_n b_n\}$ to the t–element set in Case (iii) and apply the argument from Case (iii).

Case (v): $t-1 = \ell^2+1$ where $3 \leq \ell < n$. Add the edges $\{a_n b_n, a_n c_n\}$ to the t–element set in Case (iii) and apply the argument from Case (iii).

Therefore if $t \geq 9$, then there does not exist a Steiner graphical tBD of type n^r where $n \geq 4$, $r \geq 3$, and B is a block containing $K(\text{A}, \text{B})$.

Now suppose $3 \leq t < 9$ where $n \geq 4$ and $r \geq 3$. By Lemmas 3.4, 3.5, and 3.6, we know there cannot exist a Steiner graphical 3BD, 4BD, and 5BD when $n \geq 4$ and $r \geq 3$. If $t = 6$, then consider the 6–element set in Figure 48 and let B' be the block containing T. As $B' \neq T$, B' contains an additional edge, e. Regardless of what e is incident to, we can always force an additional edge $e' \in K(\text{A}, \text{B})$. This is a contradiction since $|B \cap B'| \geq t$. Suppose $t = 7$ and consider the 7–element set in Figure 49 and let B' be the block containing T. Regardless of what e is incident to, we can always force an additional edge $e' \in K(\text{A}, \text{B})$, a contradiction. Finally suppose $t = 8$ and consider the 8–element set in Figure 50. Let B' be the block containing T. Regardless of what e is incident to, we can always force an additional edge $e' \in K(\text{A}, \text{B})$, a contradiction. Therefore, if $3 \leq t < 9$, then there does not exist a Steiner graphical tBD of type n^r where $n \geq 4$, $r \geq 3$, and B is a block containing $K(\text{A}, \text{B})$. ∎

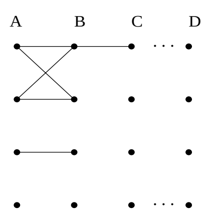


Figure 48:

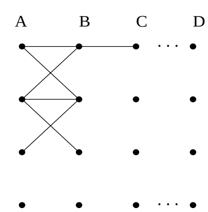


Figure 49:

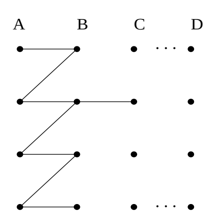


Figure 50:

Theorem 3.37 *If $3 \leq t < n^2$ and $n \geq 3$, then there do not exist any Steiner graphical tBDs of type n^r for $r \geq 3$.*

Proof: By Lemma 3.3, either there is a block B containing $K(\mathrm{A},\mathrm{B})$, or $t = 3, 4, 5$ (values for which there exists a bigraphical design in [5]), $t = n$, or $t = n(n-1)$.

Lemma 3.36 proves that there does not exist a Steiner graphical tBD of type n^r if there is a block B containing $K(\mathrm{A},\mathrm{B})$. In Lemma 3.4, 3.5, and 3.6, we proved that there does not exist a Steiner graphical 3BD, 4BD, or 5BD of type n^r for $n \geq 3$ and $r \geq 3$. When $t = n$ or $t = n(n-1)$, Lemma 3.34 and Lemma 3.35 established that there do not exist any Steiner graphical tBDs of type n^r for $n \geq 3$ and $r \geq 3$. ■

Main Theorem: The only proper Steiner graphical tBDs of type n^r are those given in Tables I, II, and III.

Proof: The designs D_1, D_2, D_3, D_4, and D_5 are the Steiner graphical tBDs of type 1^r found in [2]. Designs D_6, D_7, D_8, D_9, D_{10}, and D_{11} are the Steiner graphical tBDs of type n^2 found in [5].

Assume now that $n \geq 2$ and $r \geq 3$. Theorem 2.2 and Theorem 2.5 establish the proper Steiner graphical tBDs of type n^r for $t = 1$ and $t = 2$, respectively. These are the designs D_{12}, D_{13}, D_{14}, D_{15}, and D_{16} in Table III. For $t \geq 3$, we consider two cases for n: (i) $n = 2$ and (ii) $n \geq 3$. Theorem 3.22, Theorem 3.24, and Theorem 3.26 establish that there do not exist any proper Steiner graphical tBDs of type 2^r where $r \geq 3$ and $t \geq 3$.

Finally, Theorem 3.33 proves that there do not exist any proper Steiner graphical tBDs of type n^r where $n \geq 3, r \geq 3$, and $t \geq n^2$. Theorem 3.37 establishes that there do not exist any proper Steiner graphical tBDs of type n^r where $n \geq 3, r \geq 4$, and $3 \leq t < n^2$. ■

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