SMALL ORTHOGONAL MAIN EFFECT PLANS WITH FOUR FACTORS

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ABSTRACT

In this paper we study orthogonal main effect plans with four factors. A table of such designs, where each factor has at most 10 levels, and there are at most 40 runs, is generated. We determine the spectrum of the degrees of freedom of pure error for these designs.

INTRODUCTION

A *main-effect plan* (MEP) has f *rows* (or *factors*), n *columns* (or *runs*) and s_i *symbols* (or *levels*) in row $i, 1 \leq i \leq f$. We will represent an MEP with these parameters by $s_1 \times s_2 \times \ldots \times s_f//n$. Let $N_i[x]$ be the number of times that symbol x appears in row i. The *replication array* for a factor contains the number of times each level of the factor appears in the design. Thus the replication array for factor I consists of the $N_i[x]$ entries where $1 \leq x \leq s_i$. Let N_{ij} be the $s_i \times s_j$ *incidence matrix* of rows i and j. The (x, y) position of N_{ij} is the number of runs in the MEP with x in row i and y in row j. For the main effects to be estimated orthogonally, we require, in addition, that $N_{ij}[x,y] = N_i[x]N_j[y]/n$ for all pairs of rows i and j

(see Plackett (1946), Addelman (1962), Dey (1985)). This is often called the *condition of proportional frequencies.* An *orthogonal array* of strength 2, degree k, order s and index λ, denoted by $\mathrm{OA}_\lambda(k,s)$, is a $\underbrace{s \times s \times \cdots \times s}_{k}//s^2\lambda$ OMEP.

When runs in an experiment are expensive and/or time-consuming, the design with the minimum number of runs is often preferred. An OMEP is said to be *minimal* if, given $s_1, s_2, \ldots, s_k$, n is the smallest number of runs for which an $s_1 \times s_2 \times \cdots \times s_k//n$ OMEP exists. Algorithms for determining the minimal number of runs, n_{min}, given $(s_1, s_2, \ldots, s_k)$, are presented in Street (1994) and Gallant and Colbourn (1998).

A run is said to be *repeated* if it occurs more than once as a column of the array. Pigeon and McAllister (1989) discussed the advantages of having some repeated runs in an OMEP to provide an estimate of pure error. When an estimate of pure error is required, either a design with some repeated runs is selected, or additional runs are included in the experiment. Examples of the latter case are given in Joo, Hool and Curtis (1998), Maio, Vonholst, Wenclawiak and Darskus (1997) and Glaser and Shulman (1996), in which runs were added to the original experiment in order to obtain an estimate of pure error. However, adding these runs prevented the main effects from being orthogonally estimated. If runs are expensive and an estimate of pure error is required, then an orthogonal minimal design with some repeated runs is preferred.

Of a sample of 61 applications papers taken from the Current Contents Database 1993-1998 using the search string 'fractional factorial design', 14 papers used designs with four factors. In this paper we consider minimal OMEPs with four factors, assuming without loss that $s_1 \le s_2 \le s_3 \le s_4$, $S_i = \{1, 2, \ldots, s_i\}$ and that $N_i[1] \le N_i[2] \le \cdots \le N_i[s_i]$, $i = 1, 2, 3, 4$. Using the method of Street (1994), it is easy to show that $N_{3,4}[1,1] = 1$ and so $n_{min} = N_3[1]N_4[1]$. From the orthogonality condition for $N_{3,4}$, we see that $N_3[1]$ divides $N_3[x]$, $1 \le x \le s_3$ and $N_4[1]$ divides $N_4[y]$, $1 \le y \le s_4$. Considering the entries in $N_{2,3}$ we see that $N_3[1]N_4[1]$ divides $N_2[i]N_3[1]$ so $N_4[1]$ divides $N_2[x]$ and considering $N_{2,4}$ we see that $N_3[1]$ divides $N_2[x]$. Similarly we get that $N_4[1]$ divides $N_1[x]$ and $N_3[1]$ divides $N_1[x]$. Let $g = \gcd(N_3[1], N_4[1])$ and let $N_3[1] = gg_3$ and $N_4[1] = gg_4$. Then $N_2[x]$ and $N_1[x]$ are each multiples of gg_3g_4. We say that an OMEP has *level replication as equal as possible* if

$$\begin{aligned} |N_i[x]/N_i[1] - N_i[m]/N_i[1]| &\le 1, \text{ for } i = 3, 4 \text{ and} \\ |N_i[x]/gg_3g_4 - N_i[m]/gg_3g_4| &\le 1, \text{ for } i = 1, 2, \text{ where } 1 \le m \le s_i. \end{aligned}$$

If the array has m_i runs that are each repeated i times, then the *repeated run sequence* (RRS) of the array is $\text{RRS} = 1^{m_1}2^{m_2}\cdots n^{m_n}$. Note that $\sum_i im_i =$

n. The *degrees of freedom of pure error* (DFPE) of the experimental design represented by the array is

$$\text{DFPE} = \sum_{i=1}^{n} m_i(i-1).$$

Example 1. Some $2 \times 2 \times 3 \times 4//16$ OMEPs.

$$\begin{array}{cccccccccccccccc}
1&1&2&2&1&1&2&2&2&2&2&2&1&1&1&1\\
1&1&2&2&2&2&1&1&1&2&1&2&1&2&1&2\\
1&1&1&1&2&2&2&2&3&3&3&3&3&3&3&3\\
1&2&3&4&1&2&3&4&1&1&2&2&3&3&4&4
\end{array}$$

$\text{RRS} = 1^{16}$, $\text{DFPE} = 0$.

(a)

$$\begin{array}{cccccccccccccccc}
1&1&2&2&1&2&2&1&\mathbf{2}&\mathbf{2}&1&2&1&1&1&2\\
1&2&1&2&1&1&2&2&\mathbf{2}&\mathbf{2}&2&1&1&2&1&1\\
1&1&1&1&2&2&2&2&\mathbf{3}&\mathbf{3}&3&3&3&3&3&3\\
1&2&3&4&1&2&3&4&\mathbf{1}&\mathbf{1}&2&2&3&3&4&4
\end{array}$$

$\text{RRS} = 1^{14}2^{1}$, $\text{DFPE} = 1$

(b)

$$\begin{array}{cccccccccccccccc}
1&1&2&2&1&1&2&2&\mathbf{2}&\mathbf{2}&\mathbf{2}&\mathbf{2}&1&1&1&1\\
1&2&1&2&1&2&2&1&\mathbf{2}&\mathbf{2}&\mathbf{1}&\mathbf{1}&1&2&1&2\\
1&1&1&1&2&2&2&2&\mathbf{3}&\mathbf{3}&\mathbf{3}&\mathbf{3}&3&3&3&3\\
1&2&3&4&1&2&3&4&\mathbf{1}&\mathbf{1}&\mathbf{2}&\mathbf{2}&3&3&4&4
\end{array}$$

$\text{RRS} = 1^{12}2^{2}$, $\text{DFPE} = 2$

(c)

$$\begin{array}{cccccccccccccccc}
1&1&2&2&1&1&2&2&\mathbf{2}&\mathbf{2}&\mathbf{2}&\mathbf{2}&\mathbf{1}&\mathbf{1}&\mathbf{1}&\mathbf{1}\\
1&2&1&2&1&2&1&2&\mathbf{2}&\mathbf{2}&\mathbf{1}&\mathbf{1}&\mathbf{2}&\mathbf{2}&\mathbf{1}&\mathbf{1}\\
1&1&1&1&2&2&2&2&\mathbf{3}&\mathbf{3}&\mathbf{3}&\mathbf{3}&\mathbf{3}&\mathbf{3}&\mathbf{3}&\mathbf{3}\\
1&2&3&4&1&2&3&4&\mathbf{1}&\mathbf{1}&\mathbf{2}&\mathbf{2}&\mathbf{3}&\mathbf{3}&\mathbf{4}&\mathbf{4}
\end{array}$$

$\text{RRS} = 1^{8}2^{4}$, $\text{DFPE} = 4$

(d)

Ideally we would like to use OMEPs with equal replication since Cheng (1980) has shown that such OMEPs are universally optimal. Lewis and John (1976) have shown that in an OMEP with unequal replication the orthogonal contrasts are those that correspond to the weighted hypotheses and these depend on the particular fraction that is used. In an equally replicated experiment the orthogonal contrasts are the usual contrasts, corresponding to the unweighted hypotheses which are of most interest to experimenters. The weights depend on the replication and so the more equal the replication the less weighted are the comparisons. For most factor level combinations the designs with equal replication have too many runs and so we compromise and use designs with the minimal number of runs and with level replication as equal as possible. For those factors which are equi-replicate, the design is universally optimal, using the same proof as in Cheng (1980). An indication of the savings to be made can be obtained from the graph in Figure 1, in which the two values for the number of runs are plotted for each of the 98

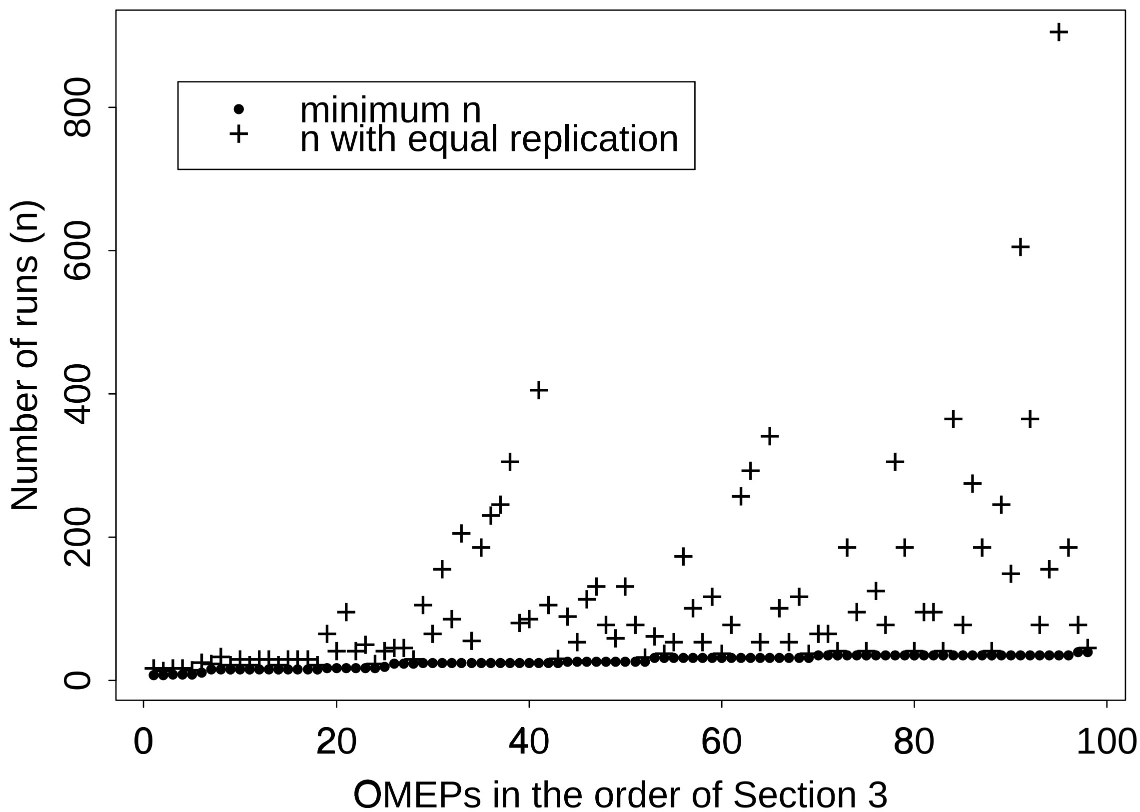

Figure 1: Comparison of minimum n with equal replication n.

combinations of $s_1 \times s_2 \times s_3 \times s_4$ in the list of OMEPs in the final section of the paper.

It can be seen from the graph that an insistence on equal replication of levels for each of the factors can significantly increase the number of runs required. Of the 98 cases, 19 have equal level replication with the minimum value of n, while 20 cases require the full factorial, i.e. $n = s_1 s_2 s_3 s_4$. The remaining 59 cases require a value of n which is greater than the minimum, but less than $s_1 s_2 s_3 s_4$. The difference between the two values can be small, for example the $2 \times 2 \times 3 \times 3$ OMEP has a minimum n value of 9, while the value of n with equal level replication is 12. However, as the size of the parameters s_1, s_2, s_3 and s_4 increases, the value of n with equal level replication can be very large. For example, consider the $2 \times 3 \times 4 \times 5$ OMEP, for which the minimum number of runs is 25. If, however, we wish to have equal replication of levels for each of the factors then the number of runs required is 60. The $4 \times 5 \times 6 \times 6$ OMEP has a much larger difference between the two values of n. In this case the minimum n is 36, while the value of n with equal level replication is 360.

The next section gives a recursive construction for OMEPs and a construction based on incomplete mutually orthogonal Latin squares. Using these results, and the techniques of Street (1994) and Burgess and Street (1994), we give, in the final section, a tabulation of four factor minimal OMEPs, in which each factor has at most 10 levels, and with at most 40 runs, for all possible values of DFPE. Some of the OMEPs listed in the table are already available in the literature. However, this table is different because it provides information about the number of repeated runs, which is important if an estimation of pure error is required. The table is also self-contained and any of the OMEPs listed can be easily and quickly constructed from the information given. Other tables, such as the one given in Dey (1985) do not provide a choice of DFPE values, and often other information such as Latin squares, orthogonal arrays or Hadamard matrices are required in order to construct an OMEP. In some cases the number of runs required is less than that given in Dey's table. See for example $2 \times 2 \times 2 \times 5//12$ (16 in Dey), $2 \times 2 \times 4 \times 9//40$ (81 in Dey), $2 \times 2 \times 4 \times 10//40$ (50 in Dey) and the 27 OMEPs which can be done in 36 runs, but Dey lists these as requiring 49 or 50 runs.

CONSTRUCTIONS

One well-known method of obtaining one OMEP from another is by 'collapsing levels', introduced in Addelman (1962). In its simplest form this is a many-to-one correspondence which maps the s levels of a factor onto $t(< s)$ levels.

Theorem 1 *(Addelman (1962)) Let* $S_i = \{1, 2, ..., s_i\}$, $1 \leq i \leq k$, *be the level sets corresponding to the factors of a* $s_1 \times s_2 \times \cdots \times s_k//n$ OMEP A. *If, for each* i, *a map* $f_i : S_i \mapsto T_i$ *is chosen such that* $t_i = |T_i| \leq s_i = |S_i|$, *then applying* f_i *to the levels in factor* i *of* A, $1 \leq i \leq k$, *gives a* $t_1 \times t_2 \times \cdots \times t_k//n$ OMEP B.

To obtain the OMEP B constructed in Theorem 1 we would say:

Collapse A via $f_1(1), \cdots, f_1(s_1)/f_2(1), \cdots, f_2(s_2)/\cdots/f_k(1), \cdots, f_k(s_k)$.

Example 2. The $2 \times 2 \times 2 \times 3//16$ OMEP below has RRS $= 1^8 2^4$ and DFPE $= 4$. It was obtained by performing a collapse of example 1 (a) via 12/12/122/1123.

1	1	2	2	1	1	2	2	2	2	2	2	1	1	1	1
1	1	2	2	2	2	1	1	1	2	1	2	1	2	1	2
1	1	1	1	2	2	2	2	2	2	2	2	2	2	2	2
1	1	2	3	1	1	2	3	1	1	1	1	2	2	3	3

The next result gives conditions under which two OMEPs can be juxtaposed to get an OMEP with more runs and with more levels for one of the factors. For a proof see Gallant and Colbourn (1998).

Theorem 2 *If there is a* $s_1 \times s_2 \times \cdots \times s_k//n_1$ OMEP A *and a* $t_1 \times t_2 \times \cdots \times t_k//n_2$ OMEP B *such that*

1. $s_i = t_i$ *for all* i *except possibly for some* $i = i_0$,
2. $n_2 = \mu n_1$,
3. $N_{Bi}[x] = \mu N_{Ai}[x]$ *for all* x *and all* $i \neq i_0$
4. *for all* x, $N_{Ai_0}[x]$ *and* $N_{Bi_0}[x]$ *are multiples of some* r_{i_0}.

Then there is a $s_1 \times s_2 \times \cdots \times (s_{i_0} + t_{i_0}) \times \cdots \times s_k//(n_1 + n_2)$ OMEP.

Example 3. Consider a $2 \times 2 \times 2 \times 3//8$ OMEP and a $2 \times 2 \times 2 \times 4//8$ OMEP. Apply Theorem 2 with $i_0 = 4$. Then $\mu = 1$ and in the first three rows $N_{Bi}[x] = N_{Ai}[x]$. In the fourth row all the $N_{A4}[x]$'s and $N_{B4}[x]$'s are multiples of 2. Hence we get a $2 \times 2 \times 2 \times 7//16$ OMEP. The DFPE is the sum of the DFPE in each of the original OMEPs.

1	1	2	2	2	2	1	1	1	1	2	2	2	2	1	1
1	2	1	2	2	1	2	1	1	2	1	2	2	1	2	1
1	1	1	1	2	2	2	2	1	1	1	1	2	2	2	2
1	2	3	3	1	2	3	3	4	5	6	7	4	5	6	7

Recall that a pair of mutually orthogonal Latin squares of order s_1 corresponds to an orthogonal array of strength 2 (Raghavarao (1971), for instance), and so to an $s_1 \times s_1 \times s_1 \times s_1 // s_1{}^2$ OMEP. Applying Theorem 2 to two such OMEPs will give $s_1 \times s_1 \times s_1 \times s_4 // 2s_1{}^2$ minimal OMEPs where $s_1 < s_4 \leq 2s_1$ and with a range of values for DFPE.

Theorem 3 *Suppose there exists a pair of MOLS of order s_1. Then there exists a $s_1 \times s_1 \times s_1 \times (2s_1 - p)//2s_1{}^2$ OMEP $1 \leq p \leq s_1 - 1$ with* DFPE$=$ qf, $1 \leq q \leq p$ *and* $0 \leq f \leq s_1, f \neq s_1 - 1$.

Proof. A pair of MOLS of order s_1 is a $s_1 \times s_1 \times s_1 \times s_1 // s_1{}^2$ OMEP. Using Theorem 2 with two of these gives a $s_1 \times s_1 \times s_1 \times 2s_1 // 2s_1{}^2$. It is possible to collapse levels so that there are repeated runs. Let σ be a permutation of s_1 symbols and suppose that σ has f fixed points. Apply σ to the symbols in the third row of the second $s_1 \times s_1 \times s_1 \times s_1 // s_1{}^2$ OMEP. Collapse levels in factor 4 of the $s_1 \times s_1 \times s_1 \times s_1 // 2s_1{}^2$ OMEP so that the final number of levels is $2s_1 - p$ and DFPE$=$ qf, by collapsing $x + s_1$ to x for $1 \leq x \leq q$. A further $p - q$ levels need to be collapsed. For $x = q + 1$ to p, collapse $x + s_1$ to y for any $y \neq x$. The fixed points of σ appear with all the symbols in the other three rows equally often and so for each level which is collapsed by mapping $x + s_1$ to x there are f repeated pairs of runs. Since σ is a permutation of s_1 things there can be any number of fixed points between 0 and s_1 except $s_1 - 1$.

Example 4. Let $s_1 = 5$ and consider the two MOLS:

1 2 3 4 5		1 3 5 2 4
2 3 4 5 1		2 4 1 3 5
3 4 5 1 2	and	3 5 2 4 1
4 5 1 2 3		4 1 3 5 2
5 1 2 3 4		5 2 4 1 3

They can be used to construct two $5 \times 5 \times 5 \times 5//25$ OMEPs and using Theorem 2 we obtain the $5 \times 5 \times 5 \times 10//50$ OMEP given below, where $0 = 10$.

```
1111122222333334444455555 1111122222333334444455555
1234512345123451234512345 1234512345123451234512345
1234523451345124512351234 1234523451345124512351234
1352424135352414135252413 6807979680807969680707968
```

Now apply the permutation (12) to the third row of the second $5 \times 5 \times 5 \times 5//25$ OMEP. Thus there are 3 fixed points. Now collapse by equating 6 and 1 in row 4. The runs with 6 in row 4 before the collapsing and symbols 3, 4 or 5 in row 3 are all now repeats of the runs in the first $5 \times 5 \times 5 \times 5//25$ OMEP. Thus we have constructed the $5 \times 5 \times 5 \times 9//50$ OMEP with DFPE$= 3$ given below.

```
1111122222333334444455555 1111122222333334444455555
1234512345123451234512345 1234512345123451234512345
1234523451345124512351234 2134513452345214521352134
1352424135352414135252413 1807979180807919180707918
```

We can get a $5 \times 5 \times 5 \times 8//50$ OMEP with 4 repeated runs by using the same initial OMEP and the permutation (123) and collapsing 6 to 1 and 7 to 2. To get 2 repeated runs we could collapse 6 to 1 and 7 to 3.

It is possible to extend the previous result to adjoining more than 2 copies of the OMEP.

Theorem 4 *Suppose there exist m MOLS of order s_1. Then there exists a $s_1 \times s_1 \times s_1 \times (ms_1 - p)//ms_1{}^2$ OMEP, $1 \leq p \leq s_1 - 1$, with* DFPE$= qf$, *$1 \leq q \leq p$ and $0 \leq f \leq s_1$, $f \neq s_1 - 1$.*

The OMEPs constructed by Theorems 3 and 4 are not always minimal. For example, if $(ms_1 - p)^2 < ms_1^2$, then an $s_1 \times s_1 \times s_1 \times (ms_1 - p)//(ms_1 - p)^2$ OMEP can be constructed by collapsing levels in an $(ms_1 - p) \times (ms_1 - p) \times (ms_1 - p) \times (ms_1 - p)//(ms_1 - p)^2$ OMEP. These exist when $ms_1 - p \notin \{2, 6\}$.

An easy upper bound on DFPE comes from the observation that each pair (x, y), $x \in S_i$, $y \in S_j$ must occur at least once in rows i and j. Thus

$$\text{DFPE} \leq n - s_i s_j. \tag{1}$$

This bound is best when s_i and s_j are the largest of the values and can sometimes be improved by considering a third row, see Street (1994). The OMEPs in Theorems 3 and 4 could never have DFPE equal to one less than the upper bound. The next result shows that this observation is independent of the method of construction of the designs.

Theorem 5 *Consider a minimal* $s_1 \times s_2 \times s_3 \times s_4//n$ OMEP *where* $n = s_3(s_4 + z)$ *say. Then the upper bound for* DFPE *is* $s_3 z$ *and there is no design with* DFPE $= s_3 z - 1$.

Proof. The replication vector for the third factor has all entries equal to $s_4 + z$ and the replication vector for the fourth factor has all entries multiples of s_3 (because of the form of n and the proportionality requirement) so there are $s_4 - z$ entries equal to s_3 and z entries equal to $2s_3$ (if we want the replication to be as equal as possible).

Repeated runs must come from those levels of the fourth factor with replication $2s_3$. Each of these levels have exactly two occurrences of each level of the third factor and so all repeated runs occur in pairs.

Consider a design with DFPE$= s_3 z - 1$. In $z-1$ of the levels of the fourth factor all levels of the third factor are involved in repeated runs. In the *zth* level there are only $s_3 - 1$ levels of the third factor involved in repeated runs. The final level of the third factor can only not be involved in repeated runs if there are different levels of the second and first factors available to fill the places. But the proportionality requirement means that each of the z levels of the fourth factor has the same number of the occurrences of each level of each of the other factors. So if there are $s_3 - 1$ repeats in a level of factor 4 then the only levels left must correspond to a repeated pair.

The same argument can be used to establish the following result.

Theorem 6 *Consider a minimal* $s_1 \times s_2 \times s_3 \times s_4//n$ OMEP *with* $n = (s_3 + z)s_4$ *say. Then the upper bound for the* DFPE *is* $s_4 z$ *and there is no design with* DFPE $= s_4 z - 1$.

There are relationships between OMEPs and incomplete arrays or Latin squares with holes.

Using the notation of Horton (1974) an IA(n, k, s) is an $s \times (n^2 - k^2)$ array on n objects with a distinguished subset of k objects such that the ordered pairs obtained by superimposing any two rows gives the n^2 ordered pairs except for the k^2 ordered pairs from the distinguished subset. So when $k = 2$ and $s = 4$ these are a pair of MOLS of order n with a 2×2 subsquare missing, and they exist if and only if $n \geq 6$, see Heinrich (1991).

Theorem 7 *An IA*$(n, 2, 4)$ *is equivalent to an* $(n-1) \times (n-1) \times (n-1) \times n//n^2$ OMEP *with* DFPE=*2.*

Proof. In one square we put 4 $n-1$'s in the 2×2 hole and change all n's to $n-1$'s. In the other square we put a 2×2 Latin square on the symbols $n-1$ and n. Then we label the rows and columns so that we can write a $(n-1) \times (n-1) \times (n-1) \times n//n^2$ OMEP. The OMEP will always have DFPE $= 2$ which is maximum possible for these parameters.

Remark 8 *This also establishes that there are no* OMEP*s of type* $3 \times 3 \times 3 \times 4//16$ *or* $4 \times 4 \times 4 \times 5//25$ *with* DFPE $= 2$.

In general an $\text{IA}(n, k, 4)$ gives a $(n-k+1) \times (n-k+1) \times (n-k+1) \times n//n^2$ OMEP with DFPE $= k(k-1)$ (and $k < n$ of course).

Example 5. For example, $n = 6$, $k = 2$, gives the $\text{IA}(6, 2, 4)$, written as two incomplete MOLS,

5	6	3	4	1	2
2	1	6	5	3	4
6	5	1	2	4	3
4	3	5	6	2	1
1	4	2	3		
3	2	4	1		

and

1	2	5	6	3	4
6	5	1	2	4	3
4	3	6	5	1	2
5	6	4	3	2	1
2	4	3	1		
3	1	2	4		

In the first square put $\boxed{\begin{matrix} 5 & 6 \\ 6 & 5 \end{matrix}}$ in the hole. In the second square, fill the hole with 5's and change all the 6's to 5's. This gives the following $5 \times 5 \times 5 \times 6//36$ OMEP with DFPE $= 2$:

1 1 1 1 1 1 2 2 2 2 2 2 3 3 3 3 3 3 4 4 4 4 4 4 5 5 5 5 5 5 5 5 5 5 5 5

1 2 3 4 5 5 1 2 3 4 5 5 1 2 3 4 5 5 1 2 3 4 5 5 1 2 3 4 1 2 3 4 5 5 5 5

1 2 5 5 3 4 5 5 1 2 4 3 4 3 5 5 1 2 5 5 4 3 2 1 2 4 3 1 3 1 2 4 5 5 5 5

5 6 3 4 1 2 2 1 6 5 3 4 6 5 1 2 4 3 4 3 5 6 2 1 1 4 2 3 3 2 4 1 5 5 6 6

TABLE OF OMEPs

In this section minimal orthogonal main effect plans with four factors, each with at most 10 levels, and at most 40 runs, are listed. The listed OMEPs have level replication as equal as possible and unlisted OMEPs with DFPE less than the upper bound (1) were shown not to exist by a simple backtracking algorithm. Repeated runs are often required in an experiment to provide an estimate for pure error, and for many OMEPs listed in the table there is a choice of DFPE values. However, if an $s_1 \times s_2 \times s_3 \times s_4$ OMEP is not listed with any repeated runs, then an OMEP with larger parameter values can be constructed and appropriate factor levels collapsed to obtain the required OMEP with some repeated runs. For example, the $2 \times 2 \times 2 \times 2//8$ has no repeated runs. An OMEP with DFPE=1 can be obtained by collapsing the levels of the fourth factor in the $2 \times 2 \times 2 \times 5//12$

resulting in the $2 \times 2 \times 2 \times 2//12$ OMEP, or alternatively an OMEP with DFPE=4 can be obtained by collapsing the levels of the last two factors in the $2 \times 2 \times 3 \times 3//9$ resulting in the $2 \times 2 \times 2 \times 2//9$ OMEP.

1. $2 \times 2 \times 2 \times 2//8, \quad \text{DFPE} = 0:$
$$\boxed{\begin{array}{l} 12122121 \\ 12121212 \\ 11221122 \\ 11112222 \end{array}}$$
2. $2 \times 2 \times 2 \times 3//8, \quad \text{DFPE} = 0:$
$$\boxed{\begin{array}{l} 11222211 \\ 12122112 \\ 11112222 \\ 12331233 \end{array}}$$
3. $2 \times 2 \times 2 \times 3//8, \quad \text{DFPE} = 1:$ Does not exist by Theorem 5.
4. $2 \times 2 \times 2 \times 4//8, \quad \text{DFPE} = 0:$
$$\boxed{\begin{array}{l} 11222211 \\ 12122121 \\ 11112222 \\ 12341234 \end{array}}$$
5. $2 \times 2 \times 3 \times 3//9, \quad \text{DFPE} = 0:$ Collapse 7 via 123/123/122/122
6. $2 \times 3 \times 3 \times 3//9, \quad \text{DFPE} = 0:$ Collapse 7 via 123/123/123/122
7. $3 \times 3 \times 3 \times 3//9, \quad \text{DFPE} = 0:$
$$\boxed{\begin{array}{l} 123312231 \\ 123231312 \\ 123123123 \\ 111222333 \end{array}}$$
8. $2 \times 2 \times 2 \times 5//12, \quad \text{DFPE} = 0:$
$$\boxed{\begin{array}{l} 121122122211 \\ 121212212121 \\ 111111222222 \\ 551234551234 \end{array}}$$
9. $2 \times 2 \times 2 \times 6//16, \quad \text{DFPE} = 0:$
$$\boxed{\begin{array}{l} 1111222222221111 \\ 1212112212122211 \\ 1111111122222222 \\ 5566123455661234 \end{array}}$$
10. $2 \times 2 \times 2 \times 6//16, \quad \text{DFPE} = 2:$
$$\boxed{\begin{array}{l} 1112122222122111 \\ 1121212222211211 \\ 1111111122222222 \\ 5566123455661234 \end{array}}$$

11. $2 \times 2 \times 2 \times 6//16$, DFPE $= 4$:

$$\boxed{\begin{array}{l} 1111222222221111 \\ 1122112222112211 \\ 1111111122222222 \\ 5566123455661234 \end{array}}$$

12. $2 \times 2 \times 2 \times 7//16$, DFPE $= 0$: Collapse 15 via 12/12/12/12345676.
13. $2 \times 2 \times 2 \times 7//16$, DFPE $= 1$: Does not exist by Theorem 5.
14. $2 \times 2 \times 2 \times 7//16$, DFPE $= 2$: Collapse 15 via 12/12/12/12345677.
15. $2 \times 2 \times 2 \times 8//16$, DFPE $= 0$:

$$\boxed{\begin{array}{l} 1111222222221111 \\ 1122112222112211 \\ 1111111122222222 \\ 1234567812345678 \end{array}}$$

16. $2 \times 2 \times 3 \times 4//16$, DFPE $= 0$: Collapse 34 via 1234/1233/1221/1221
17. $2 \times 2 \times 3 \times 4//16$, DFPE $= 1$:

$$\boxed{\begin{array}{l} 1122122122121112 \\ 1212112222211211 \\ 1111222233333333 \\ 1234123411223344 \end{array}}$$

18. $2 \times 2 \times 3 \times 4//16$, DFPE $= 2$: Collapse 23 via 12/123/112/1234
19. $2 \times 2 \times 3 \times 4//16$, DFPE $= 3$: Does not exist by Theorem 6.
20. $2 \times 2 \times 3 \times 4//16$, DFPE $= 4$: Collapse 34 via 1234/1233/1221/1212.
21. $2 \times 2 \times 4 \times 4//16$, DFPE $= 0$: Collapse 34 via 1234/1234/1221/1221
22. $2 \times 3 \times 3 \times 4//16$, DFPE $= 0$: Collapse 34 via 1234/1233/1233/1221
23. $2 \times 3 \times 3 \times 4//16$, DFPE $= 1$:

$$\boxed{\begin{array}{l} 1122112222221111 \\ 1323233133121323 \\ 1111222233333333 \\ 1234123411223344 \end{array}}$$

24. $2 \times 3 \times 3 \times 4//16$, DFPE $= 2$: Collapse 34 via 1234/1233/1231/1212.
25. $2 \times 3 \times 3 \times 4//16$, DFPE $= 3$: Does not exist by Theorem 6.
26. $2 \times 3 \times 4 \times 4//16$, DFPE $= 0$: Collapse 34 via 1234/1234/1233/1221
27. $2 \times 4 \times 4 \times 4//16$, DFPE $= 0$: Collapse 34 via 1234/1234/1234/1221
28. $3 \times 3 \times 3 \times 4//16$, DFPE $= 0$: Collapse 34 via 1234/1233/1233/1233.
29. $3 \times 3 \times 3 \times 4//16$, DFPE $= 1$: Collapse 34 via 1234/1233/1231/1232.
30. $3 \times 3 \times 3 \times 4//16$, DFPE $= 2$: Does not exist by Theorem 7
31. $3 \times 3 \times 3 \times 4//16$, DFPE $= 3$: Does not exist by Theorem 6.
32. $3 \times 3 \times 4 \times 4//16$, DFPE $= 0$: Collapse 34 via 1234/1234/1233/1233.
33. $3 \times 4 \times 4 \times 4//16$, DFPE $= 0$: Collapse 34 via 1234/1234/1234/1233.
34. $4 \times 4 \times 4 \times 4//16$, DFPE $= 0$:

$$\boxed{\begin{array}{l} 1342421324313124 \\ 1234214334124321 \\ 1234123412341234 \\ 1111222233334444 \end{array}}$$

35. $2 \times 2 \times 3 \times 5//18$, DFPE $= 0$: Collapse 49 via 123/122/122/123454.
36. $2 \times 2 \times 3 \times 5//18$, DFPE $= 1$: Collapse 49 via 123/122/122/123455.
37. $2 \times 2 \times 3 \times 5//18$, DFPE $= 2$: Does not exist by Theorem 5.
38. $2 \times 2 \times 3 \times 5//18$, DFPE $= 3$: Collapse 49 via 123/122/122/123453.
39. $2 \times 2 \times 3 \times 6//18$, DFPE $= 0$: Collapse 49 via 123/122/122/123456.
40. $2 \times 3 \times 3 \times 5//18$, DFPE $= 0$: Collapse 49 via 123/123/122/123455.
41. $2 \times 3 \times 3 \times 5//18$, DFPE $= 1$: Collapse 46 via 123/123/122/12345.
42. $2 \times 3 \times 3 \times 5//18$, DFPE $= 2$: Does not exist by Theorem 5.
43. $2 \times 3 \times 3 \times 5//18$, DFPE $= 3$: Collapse 49 via 123/123/122/123453.
44. $2 \times 3 \times 3 \times 6//18$, DFPE $= 0$: Collapse 49 via 123/123/122/123456.
45. $3 \times 3 \times 3 \times 5//18$, DFPE $= 0$: Collapse 49 via 123/123/123/123455.
46. $3 \times 3 \times 3 \times 5//18$, DFPE $= 1$:

$$\boxed{\begin{array}{l} 123123123123112233 \\ 231321312132121233 \\ 321132231312211233 \\ 111222333444555555 \end{array}}$$

47. $3 \times 3 \times 3 \times 5//18$, DFPE $= 2$: Does not exist by Theorem 5.
48. $3 \times 3 \times 3 \times 5//18$, DFPE $= 3$: Collapse 49 via 123/123/123/123453.
49. $3 \times 3 \times 3 \times 6//18$, DFPE $= 0$: Juxtapose 7 and 7 with distinct symbols on row 1.
50. $2 \times 2 \times 2 \times 9//20$, DFPE $= 0$:

$$\boxed{\begin{array}{l} 12121212121212121122 \\ 12121212212121212121 \\ 12211221122112211221 \\ 11223344556677889999 \end{array}}$$

51. $2 \times 2 \times 2 \times 9//20$, DFPE $= 1$: Does not exist by Theorem 5.
52. $2 \times 2 \times 2 \times 10//24$, DFPE $= 0$:

$$\boxed{\begin{array}{l} 222222111121111111222222 \\ 111222121212222111211212 \\ 111111111111222222222222 \\ 123456789900123456789900 \end{array}}$$

53. $2 \times 2 \times 2 \times 10//24$, DFPE $= 2$:

$$\boxed{\begin{array}{l} 111111222222222222111111 \\ 111222111222222111221211 \\ 111111111111222222222222 \\ 123456789900123456789900 \end{array}}$$

54. $2 \times 2 \times 2 \times 10//24$, DFPE $= 4$:

```
112212111222221121222111
111111222222222222111111
111111111111222222222222
990012345678990012345678
```

55. $2 \times 2 \times 4 \times 5//24$, DFPE $= 0$: Juxtapose 4 and 16 with distinct symbols on row 3.
56. $2 \times 2 \times 4 \times 5//24$, DFPE $= 1$: Juxtapose 4 and 17 with distinct symbols on row 3.
57. $2 \times 2 \times 4 \times 5//24$, DFPE $= 2$: Juxtapose 4 and 18 with distinct symbols on row 3.
58. $2 \times 2 \times 4 \times 5//24$, DFPE $= 3$: Does not exist by Theorem 5.
59. $2 \times 2 \times 4 \times 5//24$, DFPE $= 4$: Juxtapose 4 and 20 with distinct symbols on row 3.
60. $2 \times 2 \times 4 \times 6//24$, DFPE $= 0$: Juxtapose 4 and 21 with distinct symbols on row 3.
61. $2 \times 2 \times 5 \times 5//25$, DFPE $= 0$: Collapse 90 via 12345/12345/12221/12221
62. $2 \times 3 \times 4 \times 5//25$, DFPE $= 0$: Collapse 90 via 12345/12344/12332/12221
63. $2 \times 3 \times 4 \times 5//25$, DFPE $= 1$: Collapse 90 via 12345/12344/12332/11222
64. $2 \times 3 \times 4 \times 5//25$, DFPE $= 2$: Collapse 90 via 12345/12344/12331/11222
65. $2 \times 3 \times 4 \times 5//25$, DFPE $= 3$:

```
1122212212222112212112212
1223323132232133321332221
1111122222333334444444444
1234512345123451122334455
```

66. $2 \times 3 \times 4 \times 5//25$, DFPE $= 4$: Does not exist by Theorem 6.
67. $2 \times 3 \times 5 \times 5//25$, DFPE $= 0$: Collapse 90 via 12345/12345/12332/12221
68. $2 \times 4 \times 4 \times 5//25$, DFPE $= 0$: Collapse 90 via 12345/12344/12344/12122.
69. $2 \times 4 \times 4 \times 5//25$, DFPE $= 1$: Collapse 90 via 12345/12344/12344/12221
70. $2 \times 4 \times 4 \times 5//25$, DFPE $= 2$:

```
1122211222222112222111212
1234434214241434413442312
1111122222333334444444444
1234512345123451122334455
```

71. $2 \times 4 \times 4 \times 5//25$, DFPE $= 4$: Does not exist by Theorem 6.
72. $2 \times 4 \times 5 \times 5//25$, DFPE $= 0$: Collapse 90 via 12345/12345/12344/12221
73. $2 \times 5 \times 5 \times 5//25$, DFPE $= 0$: Collapse 90 via 12345/12345/12345/12221
74. $3 \times 3 \times 4 \times 5//25$, DFPE $= 0$: Collapse 90 via 12345/12344/12332/12332
75. $3 \times 3 \times 4 \times 5//25$, DFPE $= 1$: Collapse 90 via 12345/12344/12332/11233
76. $3 \times 3 \times 4 \times 5//25$, DFPE $= 2$: Collapse 90 via 12345/12344/12331/11223
77. $3 \times 3 \times 4 \times 5//25$, DFPE $= 4$: Does not exist by Theorem 6.
78. $3 \times 3 \times 5 \times 5//25$, DFPE $= 0$: Collapse 90 via 12345/12345/12332/12332
79. $3 \times 4 \times 4 \times 5//25$, DFPE $= 0$: Collapse 90 via 12345/12344/12344/12332
80. $3 \times 4 \times 4 \times 5//25$, DFPE $= 1$: Collapse 90 via 12345/12344/12344/12233.

81. $3 \times 4 \times 4 \times 5//25$, DFPE $= 4$: Does not exist by Theorem 6.
82. $3 \times 4 \times 5 \times 5//25$, DFPE $= 0$: Collapse 90 via 12345/12345/12344/12332
83. $3 \times 5 \times 5 \times 5//25$, DFPE $= 0$: Collapse 90 via 12345/12345/12345/12332
84. $4 \times 4 \times 4 \times 5//25$, DFPE $= 0$: Collapse 90 via 12345/12344/12344/12344.
85. $4 \times 4 \times 4 \times 5//25$, DFPE $= 1$: Collapse 90 via 12345/12344/12344/12234.
86. $4 \times 4 \times 4 \times 5//25$, DFPE $= 2$: Does not exist by Theorem 7
87. $4 \times 4 \times 4 \times 5//25$, DFPE $= 4$: Does not exist by Theorem 6.
88. $4 \times 4 \times 5 \times 5//25$, DFPE $= 0$: Collapse 90 via 12345/12345/12344/12344.
89. $4 \times 5 \times 5 \times 5//25$, DFPE $= 0$: Collapse 90 via 12345/12345/12345/12344.
90. $5 \times 5 \times 5 \times 5//25$, DFPE $= 0$:

$$\boxed{\begin{array}{l} 1111122222333334444455555 \\ 1234512345123451234512345 \\ 1234523451345124512351234 \\ 1543221543321544321554321 \end{array}}$$

91. $2 \times 2 \times 3 \times 7//27$, DFPE $= 0$: Collapse 126 via 112/122/123/123456775.
92. $2 \times 2 \times 3 \times 7//27$, DFPE $= 1$: Collapse 126 via 122/122/123/123456777.
93. $2 \times 2 \times 3 \times 7//27$, DFPE $= 2$: Collapse 126 via 121/122/123/123456773.
94. $2 \times 2 \times 3 \times 7//27$, DFPE $= 3$: Collapse 126 via 122/122/123/123456776.
95. $2 \times 2 \times 3 \times 7//27$, DFPE $= 4$: Collapse 126 via 121/122/123/123456776.
96. $2 \times 2 \times 3 \times 7//27$, DFPE $= 5$: Does not exist by Theorem 5.
97. $2 \times 2 \times 3 \times 7//27$, DFPE $= 6$: Collapse 126 via 122/122/123/123456756.
98. $2 \times 2 \times 3 \times 8//27$, DFPE $= 0$: Collapse 126 via 121/122/123/123456788.
99. $2 \times 2 \times 3 \times 8//27$, DFPE $= 1$: Collapse 126 via 122/122/123/123456788.
100. $2 \times 2 \times 3 \times 8//27$, DFPE $= 2$: Does not exist by Theorem 5.
101. $2 \times 2 \times 3 \times 8//27$, DFPE $= 3$: Collapse 126 via 122/122/123/123456786.
102. $2 \times 2 \times 3 \times 9//27$, DFPE $= 0$: Collapse 126 via 122/122/123/123456789.
103. $2 \times 3 \times 3 \times 7//27$, DFPE $= 0$: Collapse 126 via 122/123/123/123456777.
104. $2 \times 3 \times 3 \times 7//27$, DFPE $= 1$: Collapse 126 via 121/123/123/123456773.
105. $2 \times 3 \times 3 \times 7//27$, DFPE $= 2$: Collapse 126 via 123/122/123/123456733.
106. $2 \times 3 \times 3 \times 7//27$, DFPE $= 3$: Collapse 126 via 122/123/123/123456776.
107. $2 \times 3 \times 3 \times 7//27$, DFPE $= 4$: Collapse 126 via 121/123/123/123456753.
108. $2 \times 3 \times 3 \times 7//27$, DFPE $= 5$: Does not exist by Theorem 5.
109. $2 \times 3 \times 3 \times 7//27$, DFPE $= 6$: Collapse 126 via 122/123/123/123456756.
110. $2 \times 3 \times 3 \times 8//27$, DFPE $= 0$: Juxtapose 6 and 40 with distinct symbols on row 4.
111. $2 \times 3 \times 3 \times 8//27$, DFPE $= 1$: Juxtapose 6 and 41 with distinct symbols on row 4.
112. $2 \times 3 \times 3 \times 8//27$, DFPE $= 2$: Does not exist by Theorem 5.
113. $2 \times 3 \times 3 \times 8//27$, DFPE $= 3$: Juxtapose 6 and 43 with distinct symbols on row 4.
114. $2 \times 3 \times 3 \times 9//27$, DFPE $= 0$: Juxtapose 6 and 44 with distinct symbols on row 4.
115. $3 \times 3 \times 3 \times 7//27$, DFPE $= 0$: Collapse 126 via 123/123/123/123456777.
116. $3 \times 3 \times 3 \times 7//27$, DFPE $= 1$: Collapse 126 via 123/123/123/123456773.

117. $3 \times 3 \times 3 \times 7//27$, DFPE $= 2$: Collapse 126 via 123/123/123/123456723.
118. $3 \times 3 \times 3 \times 7//27$, DFPE $= 3$: Collapse 126 via 123/123/123/123456776.
119. $3 \times 3 \times 3 \times 7//27$, DFPE $= 4$: Collapse 126 via 123/123/123/123456753.
120. $3 \times 3 \times 3 \times 7//27$, DFPE $= 5$: Does not exist by Theorem 5.
121. $3 \times 3 \times 3 \times 7//27$, DFPE $= 6$: Collapse 126 via 123/123/123/123456756.
122. $3 \times 3 \times 3 \times 8//27$, DFPE $= 0$: Juxtapose 7 and 45 with distinct symbols on row 4.
123. $3 \times 3 \times 3 \times 8//27$, DFPE $= 1$: Juxtapose 7 and 46 with distinct symbols on row 4.
124. $3 \times 3 \times 3 \times 8//27$, DFPE $= 2$: Does not exist by Theorem 5.
125. $3 \times 3 \times 3 \times 8//27$, DFPE $= 3$: Juxtapose 7 and 48 with distinct symbols on row 4.
126. $3 \times 3 \times 3 \times 9//27$, DFPE $= 0$: Juxtapose 7 and 49 with distinct symbols on row 4.
127. $2 \times 2 \times 4 \times 7//32$, DFPE $= 0$: Juxtapose 12 and 12 with distinct symbols on row 1.
128. $2 \times 2 \times 4 \times 7//32$, DFPE $= 1$: Collapse 158 via 12/1234/1221/1234567
129. $2 \times 2 \times 4 \times 7//32$, DFPE $= 2$: Juxtapose 12 and 14 with distinct symbols on row 1.
130. $2 \times 2 \times 4 \times 7//32$, DFPE $= 3$: Does not exist by Theorem 5.
131. $2 \times 2 \times 4 \times 7//32$, DFPE $= 4$: Juxtapose 14 and 14 with distinct symbols on row 1.
132. $2 \times 2 \times 4 \times 8//32$, DFPE $= 0$: Juxtapose 15 and 15 with distinct symbols on row 1.
133. $2 \times 3 \times 4 \times 6//32$, DFPE $= 0$: Collapse 199 via 1234/1233/1221/1234562
134. $2 \times 3 \times 4 \times 6//32$, DFPE $= 1$: Collapse 199 via 1234/1233/1221/1234566
135. $2 \times 3 \times 4 \times 6//32$, DFPE $= 2$: Juxtapose 22 and 24 with distinct symbols on row 2.
136. $2 \times 3 \times 4 \times 6//32$, DFPE $= 3$: Juxtapose 23 and 24 with distinct symbols on row 2.
137. $2 \times 3 \times 4 \times 6//32$, DFPE $= 4$: Juxtapose 24 and 24 with distinct symbols on row 2.
138. $2 \times 3 \times 4 \times 6//32$, DFPE $= 5$: Collapse 158 via 12/1234/1232/1123456
139. $2 \times 3 \times 4 \times 6//32$, DFPE $= 6$: Collapse 168 via 112/123/1234/123456.
140. $2 \times 3 \times 4 \times 6//32$, DFPE $= 7$: Does not exist by Theorem 5.
141. $2 \times 3 \times 4 \times 6//32$, DFPE $= 8$: Collapse 171 via 123/112/1234/123456
142. $2 \times 3 \times 4 \times 7//32$, DFPE $= 0$: Collapse 199 via 1234/1233/1221/1234567
143. $2 \times 3 \times 4 \times 7//32$, DFPE $= 1$: Collapse 158 via 12/1234/1232/1234567
144. $2 \times 3 \times 4 \times 7//32$, DFPE $= 2$: Juxtapose 24 and 26 with distinct symbols on row 3.
145. $2 \times 3 \times 4 \times 7//32$, DFPE $= 3$: Does not exist by Theorem 5.
146. $2 \times 3 \times 4 \times 7//32$, DFPE $= 4$:

11112222111122222222111122221111
11223333333311222211333333332211
11111111222222223333333344444444
12345677123456771234567712345677

147. $2 \times 3 \times 4 \times 8//32$, DFPE $= 0$: Juxtapose 26 and 26 with distinct symbols on row 3.
148. $2 \times 4 \times 4 \times 6//32$, DFPE $= 0$: Collapse 199 via 1234/1234/1221/1234562
149. $2 \times 4 \times 4 \times 6//32$, DFPE $= 1$: Collapse 199 via 1234/1234/1221/1234566
150. $2 \times 4 \times 4 \times 6//32$, DFPE $= 2$: Collapse 201 via 1234/1234/1221/1234566
151. $2 \times 4 \times 4 \times 6//32$, DFPE $= 3$: Collapse 158 via 12/1234/1234/1234565
152. $2 \times 4 \times 4 \times 6//32$, DFPE $= 4$: Collapse 196 via 1234/1234/1221/123456
153. $2 \times 4 \times 4 \times 6//32$, DFPE $= 5$:

11112222111212222212211122221111
11223344334141222243141344132231
11111111222222223333333344444444
55661234556612345566123455661234

154. $2 \times 4 \times 4 \times 6//32$, DFPE $= 6$:

11112222111122222222111122221111
11223344334411222213441344132231
11111111222222223333333344444444
55661234556612345566123455661234

155. $2 \times 4 \times 4 \times 6//32$, DFPE $= 7$: Does not exist by Theorem 5.
156. $2 \times 4 \times 4 \times 6//32$, DFPE $= 8$: Collapse 201 via 1234/1234/1122/1234556
157. $2 \times 4 \times 4 \times 7//32$, DFPE $= 0$: Collapse 199 via 1234/1234/1221/1234567
158. $2 \times 4 \times 4 \times 7//32$, DFPE $= 1$:

11112222111122222222111122221111
11223344334412122213441344312123
11111111222222223333333344444444
12345677123456771234567712345677

159. $2 \times 4 \times 4 \times 7//32$, DFPE $= 2$: Collapse 201 via 1234/1234/1221/1234567
160. $2 \times 4 \times 4 \times 7//32$, DFPE $= 3$: Does not exist by Theorem 5.
161. $2 \times 4 \times 4 \times 7//32$, DFPE $= 4$: Collapse 201 via 1234/1234/1122/1234567
162. $2 \times 4 \times 4 \times 8//32$, DFPE $= 0$: Juxtapose 27 and 27 with distinct symbols on row 2.
163. $3 \times 3 \times 4 \times 6//32$, DFPE $= 0$: Juxtapose 28 and 28 with distinct symbols on row 1.
164. $3 \times 3 \times 4 \times 6//32$, DFPE $= 1$: Juxtapose 28 and 29 with distinct symbols on row 1.
165. $3 \times 3 \times 4 \times 6//32$, DFPE $= 2$: Juxtapose 29 and 29 with distinct symbols on row 1.
166. $3 \times 3 \times 4 \times 6//32$, DFPE $= 3$: Collapse 188 via 123/1233/1234/1234566.
167. $3 \times 3 \times 4 \times 6//32$, DFPE $= 4$: Collapse 177 via 123/123/1234/12345664.
168. $3 \times 3 \times 4 \times 6//32$, DFPE $= 5$: Collapse 177 via 123/123/1234/12345644.

169. $3 \times 3 \times 4 \times 6//32$, DFPE $= 6$:

$$\boxed{\begin{array}{l} 11223333221133333333112233332211 \\ 11332233232333113211332333321123 \\ 11111111222222223333333344444444 \\ 12345566123455661234556612345566 \end{array}}$$

170. $3 \times 3 \times 4 \times 6//32$, DFPE $= 7$: Does not exist by Theorem 5.
171. $3 \times 3 \times 4 \times 6//32$, DFPE $= 8$: Collapse 177 via 123/123/1234/12345634.
172. $3 \times 3 \times 4 \times 7//32$, DFPE $= 0$: Juxtapose 28 and 32 with distinct symbols on row 3.
173. $3 \times 3 \times 4 \times 7//32$, DFPE $= 1$: Juxtapose 29 and 32 with distinct symbols on row 3.
174. $3 \times 3 \times 4 \times 7//32$, DFPE $= 2$: Collapse 188 via 123/1233/1234/1234567.
175. $3 \times 3 \times 4 \times 7//32$, DFPE $= 3$: Does not exist by Theorem 5.
176. $3 \times 3 \times 4 \times 7//32$, DFPE $= 4$: Collapse 177 via 123/123/1234/12345674.
177. $3 \times 3 \times 4 \times 8//32$, DFPE $= 0$: Juxtapose 32 and 32 with distinct symbols on row 3.
178. $3 \times 4 \times 4 \times 6//32$, DFPE $= 0$: Juxtapose 32 and 32 with distinct symbols on row 1.
179. $3 \times 4 \times 4 \times 6//32$, DFPE $= 1$: Collapse 187 via 123/1234/1234/1234526.
180. $3 \times 4 \times 4 \times 6//32$, DFPE $= 2$: Collapse 187 via 123/1234/1234/1234565.
181. $3 \times 4 \times 4 \times 6//32$, DFPE $= 3$: Collapse 201 via 1234/1231/1234/1234566
182. $3 \times 4 \times 4 \times 6//32$, DFPE $= 4$: Collapse 191 via 123/1234/1234/1234566.
183. $3 \times 4 \times 4 \times 6//32$, DFPE $= 5$:

$$\boxed{\begin{array}{l} 11232333223333113312123333311322 \\ 12123344414122332323441134341122 \\ 11111111222222223333333344444444 \\ 12345566123455661234556612345566 \end{array}}$$

184. $3 \times 4 \times 4 \times 6//32$, DFPE $= 6$:

$$\boxed{\begin{array}{l} 11332233223311333312331233213312 \\ 12123344212144333434112243432211 \\ 11111111222222223333333344444444 \\ 12345566123455661234556612345566 \end{array}}$$

185. $3 \times 4 \times 4 \times 6//32$, DFPE $= 7$: Does not exist by Theorem 5.
186. $3 \times 4 \times 4 \times 6//32$, DFPE $= 8$: Collapse 191 via 123/1234/1234/1234536.
187. $3 \times 4 \times 4 \times 7//32$, DFPE $= 0$: Juxtapose 32 and 33 with distinct symbols on row 2.
188. $3 \times 4 \times 4 \times 7//32$, DFPE $= 1$: Collapse 200 via 1233/1234/1234/1234567.
189. $3 \times 4 \times 4 \times 7//32$, DFPE $= 2$: Collapse 201 via 1234/1234/1232/1234567
190. $3 \times 4 \times 4 \times 7//32$, DFPE $= 3$: Does not exist by Theorem 5.
191. $3 \times 4 \times 4 \times 7//32$, DFPE $= 4$: Collapse 192 via 123/1234/1234/12345674.
192. $3 \times 4 \times 4 \times 8//32$, DFPE $= 0$: Juxtapose 33 and 33 with distinct symbols on row 2.
193. $4 \times 4 \times 4 \times 6//32$, DFPE $= 0$: Juxtapose 27 and 34 with distinct symbols on

row 1.

194. $4 \times 4 \times 4 \times 6//32$, DFPE $= 1$: Collapse 199 via 1234/1234/1234/1234566.

195. $4 \times 4 \times 4 \times 6//32$, DFPE $= 2$: Collapse 200 via 1234/1234/1234/1234564.

196. $4 \times 4 \times 4 \times 6//32$, DFPE $= 4$: Collapse 204 via 1234/1234/1234/12345664.

197. $4 \times 4 \times 4 \times 6//32$, DFPE $= 7$: Does not exist by Theorem 5.

198. $4 \times 4 \times 4 \times 6//32$, DFPE $= 8$: Collapse 204 via 1234/1234/1234/12345634.

199. $4 \times 4 \times 4 \times 7//32$, DFPE $= 0$: Juxtapose 33 and 34 with distinct symbols on row 1.

200. $4 \times 4 \times 4 \times 7//32$, DFPE $= 1$: First an 4x4x4x4//32 OMEP was constructed by juxtaposing 2 pairs of MOLS(16) and then a 4x4x4x8//32 OMEP was obtained by changing the symbols of the the last square to be disjoint from the other three. Then the last row was collapsed to 7 symbols.

201. $4 \times 4 \times 4 \times 7//32$, DFPE $= 2$:

```
11223344221144333434121243432112
11223344334411224213243124314231
11111111222222223333333344444444
12345677123456771234567712345677
```

202. $4 \times 4 \times 4 \times 7//32$, DFPE $= 3$: Does not exist by Theorem 5.

203. $4 \times 4 \times 4 \times 7//32$, DFPE $= 4$: Collapse 204 via 1234/1234/1234/12345674.

204. $4 \times 4 \times 4 \times 8//32$, DFPE $= 0$: Juxtapose 34 and 34 with distinct symbols on row 1.

205. $2 \times 2 \times 3 \times 10//36$, DFPE $= 0$: Juxtapose 35 and 35 with distinct symbols on row 4.

206. $2 \times 2 \times 3 \times 10//36$, DFPE $= 1$: Juxtapose 35 and 36 with distinct symbols on row 4.

207. $2 \times 2 \times 3 \times 10//36$, DFPE $= 2$: Juxtapose 36 and 36 with distinct symbols on row 4.

208. $2 \times 2 \times 3 \times 10//36$, DFPE $= 3$: Juxtapose 35 and 38 with distinct symbols on row 4.

209. $2 \times 2 \times 3 \times 10//36$, DFPE $= 4$: Juxtapose 36 and 38 with distinct symbols on row 4.

210. $2 \times 2 \times 3 \times 10//36$, DFPE $= 5$: Does not exist by Theorem 5.

211. $2 \times 2 \times 3 \times 10//36$, DFPE $= 6$: Juxtapose 38 and 38 with distinct symbols on row 4.

212. $2 \times 2 \times 5 \times 6//36$, DFPE $= 0$: Collapse 247 via 12/123456/123455/122211

213. $2 \times 2 \times 5 \times 6//36$, DFPE $= 1$: Collapse 247 via 12/123456/123455/122112

214. $2 \times 2 \times 5 \times 6//36$, DFPE $= 2$: Collapse 247 via 12/123456/123455/112221

215. $2 \times 2 \times 5 \times 6//36$, DFPE $= 3$: Collapse 247 via 12/123456/123451/122121

216. $2 \times 2 \times 5 \times 6//36$, DFPE $= 4$:

```
111222112122111222221211222222111111
111222112212222111112122222211121211
111111222222333333444444555555555555
123456123456123456123456112233445566
```

217. $2 \times 2 \times 5 \times 6//36$, DFPE $= 5$: Does not exist by Theorem 6.

218. $2 \times 2 \times 5 \times 6//36$, DFPE $= 6$:

```
111222111222112221221112222222111111
111222122112122112222111221111222211
111111222222333333444444555555555555
123456123456123456123456112233445566
```

219. $2 \times 2 \times 6 \times 6//36$, DFPE $= 0$: Collapse 247 via 12/123456/123456/122211
220. $2 \times 3 \times 3 \times 10//36$, DFPE $= 0$: Juxtapose 40 and 40 with distinct symbols on row 4.
221. $2 \times 3 \times 3 \times 10//36$, DFPE $= 1$: Juxtapose 40 and 41 with distinct symbols on row 4.
222. $2 \times 3 \times 3 \times 10//36$, DFPE $= 2$: Juxtapose 41 and 41 with distinct symbols on row 4.
223. $2 \times 3 \times 3 \times 10//36$, DFPE $= 3$: Juxtapose 40 and 43 with distinct symbols on row 4.
224. $2 \times 3 \times 3 \times 10//36$, DFPE $= 4$: Juxtapose 41 and 43 with distinct symbols on row 4.
225. $2 \times 3 \times 3 \times 10//36$, DFPE $= 5$: Does not exist by Theorem 5.
226. $2 \times 3 \times 3 \times 10//36$, DFPE $= 6$: Juxtapose 43 and 43 with distinct symbols on row 4.
227. $2 \times 3 \times 5 \times 6//36$, DFPE $= 0$: Collapse 294 via 12345/123456/123321/122121
228. $2 \times 3 \times 5 \times 6//36$, DFPE $= 1$: Collapse 247 via 12/123456/123455/122331
229. $2 \times 3 \times 5 \times 6//36$, DFPE $= 2$: Collapse 294 via 12345/123456/123321/122211
230. $2 \times 3 \times 5 \times 6//36$, DFPE $= 3$: Collapse 266 via 123/1221/12345/123456
231. $2 \times 3 \times 5 \times 6//36$, DFPE $= 4$: Collapse 275 via 123/123456/123455/122211
232. $2 \times 3 \times 5 \times 6//36$, DFPE $= 5$: Does not exist by Theorem 6.
233. $2 \times 3 \times 5 \times 6//36$, DFPE $= 6$:

```
111222111222111222222111222222111111
112233123123323121312231223311331122
111111222222333333444444555555555555
123456123456123456123456112233445566
```

234. $2 \times 3 \times 6 \times 6//36$, DFPE $= 0$: Collapse 275 via 123/123456/123456/122211
235. $2 \times 4 \times 5 \times 6//36$, DFPE $= 0$: Collapse 294 via 12345/123456/123443/122211
236. $2 \times 4 \times 5 \times 6//36$, DFPE $= 1$: Collapse 288 via 1234/123456/123454/112221
237. $2 \times 4 \times 5 \times 6//36$, DFPE $= 2$: Collapse 294 via 12345/123456/123344/122211
238. $2 \times 4 \times 5 \times 6//36$, DFPE $= 3$: Collapse 289 via 11212/12324/12345/123456
239. $2 \times 4 \times 5 \times 6//36$, DFPE $= 4$: Collapse 294 via 12345/123456/123344/121221
240. $2 \times 4 \times 5 \times 6//36$, DFPE $= 5$: Does not exist by Theorem 6.
241. $2 \times 4 \times 6 \times 6//36$, DFPE $= 0$: Collapse 288 via 1234/123456/123456/122211
242. $2 \times 5 \times 5 \times 6//36$, DFPE $= 0$: Collapse 294 via 12345/123456/123455/122211
243. $2 \times 5 \times 5 \times 6//36$, DFPE $= 1$: Collapse 247 via 12/123456/123455/123445
244. $2 \times 5 \times 5 \times 6//36$, DFPE $= 2$: Collapse 294 via 12345/123456/123455/121221
245. $2 \times 5 \times 5 \times 6//36$, DFPE $= 5$: Does not exist by Theorem 6.
246. $2 \times 5 \times 6 \times 6//36$, DFPE $= 0$: Collapse 294 via 12345/123456/123456/122211
247. $2 \times 6 \times 6 \times 6//36$, DFPE $= 0$:

$$\boxed{\begin{array}{c}111222112122121122212211221211222111\\123456214563435612362145546231651324\\111111222222333333444444555555666666\\123456123456123456123456123456123456\end{array}}$$

248. $3 \times 3 \times 3 \times 10//36$, DFPE $= 0$: Juxtapose 45 and 45 with distinct symbols on row 4.
249. $3 \times 3 \times 3 \times 10//36$, DFPE $= 1$: Juxtapose 45 and 46 with distinct symbols on row 4.
250. $3 \times 3 \times 3 \times 10//36$, DFPE $= 2$: Juxtapose 46 and 46 with distinct symbols on row 4.
251. $3 \times 3 \times 3 \times 10//36$, DFPE $= 3$: Juxtapose 45 and 48 with distinct symbols on row 4.
252. $3 \times 3 \times 3 \times 10//36$, DFPE $= 4$: Juxtapose 46 and 48 with distinct symbols on row 4.
253. $3 \times 3 \times 3 \times 10//36$, DFPE $= 5$: Does not exist by Theorem 5.
254. $3 \times 3 \times 3 \times 10//36$, DFPE $= 6$: Juxtapose 48 and 48 with distinct symbols on row 4.
255. $3 \times 3 \times 5 \times 6//36$, DFPE $= 0$: Juxtapose 44 and 49 with distinct symbols on row 1.
256. $3 \times 3 \times 5 \times 6//36$, DFPE $= 1$: Juxtapose 45 and 46 with distinct symbols on row 1.
257. $3 \times 3 \times 5 \times 6//36$, DFPE $= 2$: Juxtapose 46 and 46 with distinct symbols on row 1.
258. $3 \times 3 \times 5 \times 6//36$, DFPE $= 3$: Juxtapose 45 and 48 with distinct symbols on row 1.
259. $3 \times 3 \times 5 \times 6//36$, DFPE $= 4$: Juxtapose 46 and 48 with distinct symbols on row 1.
260. $3 \times 3 \times 5 \times 6//36$, DFPE $= 5$: Does not exist by Theorem 6.
261. $3 \times 3 \times 5 \times 6//36$, DFPE $= 6$: Juxtapose 48 and 48 with distinct symbols on row 1.
262. $3 \times 3 \times 6 \times 6//36$, DFPE $= 0$: Juxtapose 49 and 49 with distinct symbols on row 1.
263. $3 \times 4 \times 5 \times 6//36$, DFPE $= 0$: Collapse 294 via 12345/123456/123443/123321
264. $3 \times 4 \times 5 \times 6//36$, DFPE $= 1$: Collapse 288 via 1234/123456/123454/112332
265. $3 \times 4 \times 5 \times 6//36$, DFPE $= 2$: Collapse 294 via 12345/123456/123344/123231
266. $3 \times 4 \times 5 \times 6//36$, DFPE $= 3$:

$$\boxed{\begin{array}{c}112233112233231312231321332233111212\\123344213344434132434213334412443231\\111111222222333333444444555555555555\\123456123456123456123456112233445566\end{array}}$$

267. $3 \times 4 \times 5 \times 6//36$, DFPE $= 4$: Collapse 294 via 12345/123456/123344/121233
268. $3 \times 4 \times 5 \times 6//36$, DFPE $= 5$: Does not exist by Theorem 6.
269. $3 \times 4 \times 6 \times 6//36$, DFPE $= 0$: Collapse 288 via 1234/123456/123456/123321
270. $3 \times 5 \times 5 \times 6//36$, DFPE $= 0$: Collapse 294 via 12345/123456/123455/123321

271. $3 \times 5 \times 5 \times 6//36$, DFPE $= 1$:

112233112233223311231312332313112212
123455214355551234535142345452551231
111111222222333333444444555555555555
123456123456123456123456112233445566

272. $3 \times 5 \times 5 \times 6//36$, DFPE $= 2$: Collapse 294 via 12345/123456/123455/121332

273. $3 \times 5 \times 5 \times 6//36$, DFPE $= 5$: Does not exist by Theorem 6.

274. $3 \times 5 \times 6 \times 6//36$, DFPE $= 0$: Collapse 294 via 12345/123456/123456/123321

275. $3 \times 6 \times 6 \times 6//36$, DFPE $= 0$:

112233112233223311223311331122331122
123456214365561234652143345612436521
111111222222333333444444555555666666
123456123456123456123456123456123456

276. $4 \times 4 \times 5 \times 6//36$, DFPE $= 0$: Collapse 294 via 12345/123456/123443/123443

277. $4 \times 4 \times 5 \times 6//36$, DFPE $= 1$: Collapse 288 via 1234/123456/123454/112344

278. $4 \times 4 \times 5 \times 6//36$, DFPE $= 2$: Collapse 294 via 12345/123456/123344/123244

279. $4 \times 4 \times 5 \times 6//36$, DFPE $= 3$: Collapse 289 via 11234/12324/12345/123456

280. $4 \times 4 \times 5 \times 6//36$, DFPE $= 4$: Collapse 291 via 12314/12334/12345/123456

281. $4 \times 4 \times 5 \times 6//36$, DFPE $= 5$: Does not exist by Theorem 6.

282. $4 \times 4 \times 6 \times 6//36$, DFPE $= 0$: Collapse 288 via 1234/123456/123456/123443

283. $4 \times 5 \times 5 \times 6//36$, DFPE $= 0$: Collapse 294 via 12345/123456/123455/123443

284. $4 \times 5 \times 5 \times 6//36$, DFPE $= 1$: Collapse 290 via 12345/12345/12334/123456

285. $4 \times 5 \times 5 \times 6//36$, DFPE $= 2$: Collapse 294 via 12345/123456/123455/121344

286. $4 \times 5 \times 5 \times 6//36$, DFPE $= 5$: Does not exist by Theorem 6.

287. $4 \times 5 \times 6 \times 6//36$, DFPE $= 0$: Collapse 294 via 12345/123456/123456/123443

288. $4 \times 6 \times 6 \times 6//36$, DFPE $= 0$:

123344213344341423342413434231434132
123456341265516342635124254613462531
111111222222333333444444555555666666
123456123456123456123456123456123456

289. $5 \times 5 \times 5 \times 6//36$, DFPE $= 0$:

123455214355345512435521555512123434
123455341255552134554312241355554213
111111222222333333444444555555555555
123456123456123456123456112233445566

290. $5 \times 5 \times 5 \times 6//36$, DFPE $= 1$:

123455215534341525452153553545231412
123455342155455312235541555114522334
111111222222333333444444555555555555
123456123456123456123456112233445566

291. $5 \times 5 \times 5 \times 6//36$, DFPE $= 2$:

```
153524452531545213515342322314415555
545231535142452513351524212143435555
111111222222333333444444555555555555
123456123456123456123456112233445566
```

292. $5 \times 5 \times 5 \times 6//36$, DFPE $= 5$: Does not exist by Theorem 6.
293. $5 \times 5 \times 6 \times 6//36$, DFPE $= 0$: Collapse 294 via 12345/123455/123456/123456
294. $5 \times 6 \times 6 \times 6//36$, DFPE $= 0$:

```
123455214355351524452513535142545231
123456341265516342635124264531452613
111111222222333333444444555555666666
123456123456123456123456123456123456
```

295. $6 \times 6 \times 6 \times 6//36$, DFPE $= 0$:This would corresponds to a pair of orthogonal Latin squares of order 6, which Does not exist.
296. $2 \times 2 \times 4 \times 9//40$, DFPE $= 0$: Juxtapose 50 and 50 with distinct symbols on row 1.
297. $2 \times 2 \times 4 \times 9//40$, DFPE $= 1$:

```
1111122222111122221222222111112222111112
1111122222222211112111112222122222211111
1111111111222222222233333333334444444444
1234567899123456789912345678991234567899
```

298. $2 \times 2 \times 4 \times 9//40$, DFPE $= 2$:

```
1111122222111112222222222111112222211111
1111122222222221111111112222122222111112
1111111111222222222233333333334444444444
1234567899123456789912345678991234567899
```

299. $2 \times 2 \times 4 \times 9//40$, DFPE $= 3$: Does not exist by Theorem 5.
300. $2 \times 2 \times 4 \times 9//40$, DFPE $= 4$: Collapse 301 via 12/12/1234/1234567898.
301. $2 \times 2 \times 4 \times 10//40$, DFPE $= 0$:

```
1122221111222211112222111122221111222211
1212212112122121121221211212212112122121
1234123412341234123412341234123412341234
1111222233334444555566667777888899990000
```

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