Small group divisible designs with block size four

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Abstract

In this paper we study the group divisible designs with block size four on at most 30 points. For all but three of the possible group types, we determine the existence or non-existence of the design.

1 Introduction and previous results

A *group-divisible design* (or GDD) is a triple $(X, \mathcal{G}, \mathcal{A})$, which satisfies the following properties:

1. $\mathcal{G}$ is a partition of X into subsets called *groups*
2. $\mathcal{A}$ is a set of subsets of X (called *blocks*) such that a group and a block contain at most one common point
3. every pair of points from distinct groups occurs in a unique block.

To avoid trivial cases, we will require that a GDD have more than one group.

The *group-type* or *type* of the GDD $(X, \mathcal{G}, \mathcal{A})$ is defined to be the multiset $\{|G| : G \in \mathcal{G}\}$. We sometimes use an exponential notation to denote types: a type ${t_1}^{u_1} \dots {t_k}^{u_k}$ denotes u_i occurrences of t_i, $1 \leq i \leq k$. A GDD $(X, \mathcal{G}, \mathcal{A})$ is said to be a $K-$GDD if $|A| \in K$ for every $A \in \mathcal{A}$. A GDD of type t^u is said to be *uniform.* In this paper we study $\{4\}-$GDDs, which we abbreviate to 4–GDDs.

The object of this paper is to investigate the existence of "small" 4–GDDs, having up to 30 points. Of the possible types, existence or non-existence was already known except for six cases. In three of these six cases, we construct a new design. This leaves three problematic cases unresolved.

We note that a similar investigation for 3–GDDs was recently done by Colbourn [11], who established necessary and sufficient conditions for 3–GDDs of all possible types on at most 60 points.

Let's first record some trivial necessary conditions for the existence of a 4–GDD of a specified type $t_1{}^{u_1} \dots t_k{}^{u_k}$.

Theorem 1.1 *Suppose there exists a* 4–GDD *of type* $T = \{s_1, \dots, s_n\}$*, and denote* $v = \sum_{i=1}^{n} s_i$*. Then the following conditions hold:*

1. $n \geq 4$.
2. $v \equiv s_i \pmod 3$ *for* $1 \leq i \leq n$.
3. $3s_i + s_j \leq v$ *for all* $i \neq j$.
4. $v(v-1) \equiv \sum_{i=1}^{n} s_i(s_i - 1) \bmod 12$.

Proof. First, since the block size is four, there must be at least four groups.

Second, a point in a group of size s_i occurs in $(v - s_i)/3$ blocks in the design, which must be an integer.

Third, consider a point x in a group G of size s_j, and the s_i blocks that contain both x and a point from a fixed group $H \neq G$ of size s_i. Each of these blocks contains two points not in $G \cup H$, and these $2s_i$ points must be distinct. Thus $v \geq s_j + s_i + 2s_i$.

Finally, each block covers six pairs, and a group of size s_i covers $\binom{s_i}{2}$ pairs, so

$$\left(\binom{v}{2} - \sum_{i=1}^{n} \binom{s_i}{2}\right) \equiv 0 \pmod 6.$$

□

Several classes of 4–GDDs have been previously constructed. We will summarize some of the known existence results in the remainder of this section. The following result is due to Brouwer, Schrijver and Hanani [9].

Theorem 1.2 [9] *There exists a* 4–GDD *of type* t^u *if and only if the necessary conditions are satisfied, with the two exceptions of types* 2^4 *and* 6^4*, for which* 4–GDD*s do not exist.*

Remarks.

1. The two non-existent GDDs are equivalent to (non-existent) transversal designs TD$(4, 2)$ and TD$(4, 6)$ respectively. Non-existence of the GDD of type 2^4 is trivial; non-existence of the GDD of type 6^4 was first shown by Tarry in 1900 [20] (who considered the equivalent formulation of a pair of orthogonal Latin squares of order 6).

2. D. Mesner has pointed out to us that most of the uniform 4–GDDs with $v \leq 30$ were already known as early as 1954. In fact, the table by Bose, Clatworthy and Shrikhande [4] shows the existence of all the uniform 4–GDDs with $v \leq 30$ except the designs of type 2^{10} and 6^5. The designs in [4] were constructed for use in statistical design of experiments. (An updated, more extensive table was published by Clatworthy in 1973 [10].)

Table 1: Existence of 4–GDDs of type t^u, $u \geq 4$

t	u	necessary conditions
$1,5 \pmod 6$	$1,4 \pmod{12}$	$u \geq 4$
$2,4 \pmod 6$	$1 \pmod 3$	$u \geq 4$, $(t,u) \neq (2,4)$
$3 \pmod 6$	$0,1 \pmod 4$	$u \geq 4$
$0 \pmod 6$	anything	$u \geq 4$, $(t,u) \neq (6,4)$

Thus, 4–GDDs of type t^u exist as indicated in Table 1.

Considerable work has been done on incomplete pairwise balanced designs with block size four. Such a design, on v points with a hole of size w, is denoted by $(v,w;4)$–IPBD. When a point of the hole is deleted from a $(v,w;4)$–IPBD, we obtain a 4–GDD of type $(w-1)^1 3^{(v-w)/3}$. If we just take the hole as a group, then we produce a 4–GDD of type $w^1 1^{v-w}$.

In fact, several papers have culminated in the determination of necessary and sufficient conditions for existence of $(v,w;4)$–IPBDs; see Brouwer [6], Brouwer and Lenz [7, 8], Bermond and Bond [3], Wei and Zhu [21, 22] and Rees and Stinson [19, 18]. When translated into the language of 4–GDDs, the following theorem results.

Theorem 1.3 *There exists a* 4–GDD *of type* $t^1 1^u$ *or* $t^1 3^u$ *if and only if the necessary conditions are satisfied.*

Hence, 4–GDDs of these types exist as indicated in Table 2.

Table 2: Existence of 4–GDDs of types $t^1 1^u$ and $t^1 3^u$

t	u	type	necessary conditions
$1,7 \pmod{12}$	$0,3 \pmod{12}$	$t^1 1^u$	$u \geq 2t+1$
$4,10 \pmod{12}$	$0,9 \pmod{12}$	$t^1 1^u$	$u \geq 2t+1$
$0,6 \pmod{12}$	$0,1 \pmod 4$	$t^1 3^u$	$u \geq (2t+3)/3$
$3,9 \pmod{12}$	$0,3 \pmod 4$	$t^1 3^u$	$u \geq (2t+3)/3$

Resolvable 3–GDDs also give rise to 4–GDDs. Suppose we have a resolvable 3–GDD of type t^u. There are $t(u-1)$ parallel classes in the design. If we adjoin a new point ∞_i to all the blocks of the ith parallel class, and then form a new group consisting of the infinite points, then we have a 4–GDD of type $t^u(t(u-1)/2)^1$. Conversely, from a 4–GDD of this type, one can obtain a resolvable 3–GDD.

The problem of resolvable 3–GDDs was first solved in the cases $t = 1,3$ (as Kirkman triple systems) by Ray-Chaudhuri and Wilson [14]; and next in the case $t = 2$ (as nearly Kirkman triple systems) by Kotzig and Rosa [12], Baker and Wilson [2], Brouwer [5] and Rees and Stinson [16]. The problem for general t was studied by Rees and Stinson [16], Assaf and Hartman [1] and then completed by Rees [15].

The necessary conditions for the existence of a resolvable 3–GDD of type t^u are that $t(u-1)$ is even, $tu \equiv 0 \pmod 3$, and $u \geq 3$. As a result of the above papers, the necessary conditions were shown to be sufficient, with the three exceptions $(t,u) = (2,3), (6,3)$ and

(2, 6) which do not exist. (The first two exceptions were already noted in Theorem 1.2; the third exception would be a nearly Kirkman triple system on 12 points, if it were to exist.)

We restate these results in terms of 4–GDDs as follows.

Theorem 1.4 *There exists a* 4–GDD *of type* $t^u(t(u-1)/2)^1$ *if and only if the necessary conditions are satisfied, with the three exceptions of types* 2^4, 6^4 *and* $2^6 5^1$, *for which* 4–GDD*s do not exist.*

We summarize the 4–GDDs of these types in Table 3.

Table 3: Existence of 4–GDDs of type $t^u(t(u-1)/2)^1$

t	u	necessary conditions
$1, 5 \pmod 6$	$3 \pmod 6$	$u \geq 3$
$2, 4 \pmod 6$	$0 \pmod 3$	$u \geq 3$, $(t, u) \neq (2, 3), (2, 6)$
$3 \pmod 6$	odd	$u \geq 3$
$0 \pmod 6$	anything	$u \geq 3$, $(t, u) \neq (6, 3)$

Finally, Brouwer has studied the following class of 4–GDDs.

Theorem 1.5 **[6]** *There exists a* 4–GDD *of type* $2^u 5^1$ *if and only if* $u \equiv 0 \pmod 3$, $u \geq 9$.

Remark. The necessary conditions from Theorem 1.1 are $u \equiv 0 \pmod 3$, $u \geq 6$. However, a 4–GDD of type $2^6 5^1$ does not exist, as noted in Theorem 1.4.

2 New designs

We were able to find three new designs. These 4–GDDs have types $3^4 6^3$, $2^8 5^1 8^1$, and $1^{14} 7^2$. In each case, we used the familiar method of first identifying a potential automorphism, then constructing a tactical decomposition (by hand) and, finally, lifting the tactical decomposition to a design by means of a backtrack algorithm.

Here are the three designs we found in this way.

A 4–GDD of type $3^4 6^3$	
automorphism	$(0\ 1\ 2)(3\ 4 \ldots 11)(12\ 13 \ldots 20)(21\ 22 \ldots 29)$
base blocks	$\{3, 13, 21, 23\}$ $\{11, 14, 19, 21\}$ $\{6, 8, 21, 22\}$ $\{5, 10, 12, 14\}$ $\{0, 16, 17, 21\}$ $\{2, 4, 21, 25\}$ $\{0, 6, 7, 12\}$
groups	$\{0, 1, 2\}$ $\{3, 6, 9\}$ $\{4, 7, 10\}$ $\{5, 8, 11\}$ $\{12, 15, 18, 21, 24, 27\}$ $\{13, 16, 19, 22, 25, 28\}$ $\{14, 17, 20, 23, 26, 29\}$

A 4−GDD of type $2^8 5^1 8^1$	
automorphism	$(0\,1\ldots7)(8\,9\ldots15)(16)(17\,18\,19\,20)(21\,22\ldots28)$
base blocks	$\{12,13,18,21\}$ $\{7,14,17,21\}$ $\{2,11,19,21\}$ $\{5,9,16,21\}$ $\{4,10,15,21\}$ $\{0,1,20,21\}$ $\{3,6,8,21\}$ $\{0,2,4,6\}$ $\{8,10,12,14\}$ (two orbits of length 2)
groups	$\{0,8\}$ $\{1,9\}$ $\{2,10\}$ $\{3,11\}$ $\{4,12\}$ $\{5,13\}$ $\{6,14\}$ $\{7,15\}$ $\{16,17,18,19,20\}$ $\{21,22,23,24,25,26,27,28\}$

A 4−GDD of type $1^{14} 7^2$	
automorphism	$(0)(1)(2\,3\ldots7)(8\,9\ldots13)(14)(15\,16\ldots20)(21)(22\,23\ldots27)$
base blocks	$\{8,13,16,22\}$ $\{2,12,18,22\}$ $\{5,10,17,22\}$ $\{2,10,15,21\}$ $\{4,11,14,22\}$ $\{6,7,15,22\}$ $\{1,3,20,22\}$ $\{0,9,19,22\}$ $\{0,2,4,6\}$ $\{1,8,10,12\}$ (two orbits of length 2) $\{2,5,8,11\}$ (orbit of length 3) $\{0,1,14,21\}$ (orbit of length 1)
groups	$\{0\}$ $\{1\}$ $\{2\}$ $\{3\}$ $\{4\}$ $\{5\}$ $\{6\}$ $\{7\}$ $\{8\}$ $\{9\}$ $\{10\}$ $\{11\}$ $\{12\}$ $\{13\}$ $\{14,15,16,17,18,19,20\}$ $\{21,22,23,24,25,26,27\}$

Observe that the (block) orbit of length three in the 4–GDD of type $1^{14}7^2$ contains three disjoint blocks that miss the groups of size seven. By taking one, two or three of these blocks to be groups, we also get 4–GDDs of types $1^{10}4^1 7^2$, $1^6 4^2 7^2$ and $1^2 4^3 7^2$.

There remain three types for which we were unable to construct designs. In each case, there were some potential automorphisms and tactical decompositions that looked promising, but exhaustive searches showed that none of the decompositions we tried could be lifted to a design.

3 Summary

The following tables summarize existence results for 4–GDDs on at most 30 points.

v	group type	existence	remarks
4	1^4	YES	Theorem 1.2
8	2^4	NO	Theorem 1.2
12	3^4	YES	Theorem 1.2
13	1^{13}	YES	Theorem 1.2
	$1^9 4^1$	YES	From 4–GDD of type 1^{13}
14	2^7	YES	Theorem 1.2
15	3^5	YES	Theorem 1.2
16	1^{16}	YES	Theorem 1.2
	$1^{12} 4^1$	YES	From 4–GDD of type 4^4
	$1^8 4^2$	YES	From 4–GDD of type 4^4
	$1^4 4^3$	YES	From 4–GDD of type 4^4
	4^4	YES	Theorem 1.2
17	$2^6 5^1$	NO	Theorem 1.5
20	2^{10}	YES	Theorem 1.2
	5^4	YES	Theorem 1.2
21	$3^5 6^1$	YES	Theorem 1.3
22	$1^{15} 7^1$	YES	Theorem 1.3
23	$2^9 5^1$	YES	Theorem 1.5
24	3^8	YES	Theorem 1.2
	$3^4 6^2$	YES	Rees and Stinson [19]
	6^4	NO	Theorem 1.2
25	1^{25}	YES	Theorem 1.2
	$1^{21} 4^1$	YES	From 4–GDD of type $1^1 4^6$
	$1^{17} 4^2$	YES	From 4–GDD of type $1^1 4^6$
	$1^{13} 4^3$	YES	From 4–GDD of type $1^1 4^6$
	$1^9 4^4$	YES	From 4–GDD of type $1^1 4^6$
	$1^5 4^5$	YES	From 4–GDD of type $1^1 4^6$
	$1^1 4^6$	YES	Kramer, Magliveras and Mathon [13]
26	2^{13}	YES	Theorem 1.2
	$2^3 5^4$	???	
	$2^9 8^1$	YES	Theorem 1.4
27	3^9	YES	Theorem 1.2
	$3^5 6^2$	???	
	$3^1 6^4$	YES	Rees and Stinson [17]

v	group type	existence	remarks
28	1^{28}	YES	Theorem 1.2
	$1^{24}4^1$	YES	From 4–GDD of type 4^7
	$1^{20}4^2$	YES	From 4–GDD of type 4^7
	$1^{16}4^3$	YES	From 4–GDD of type 4^7
	$1^{12}4^4$	YES	From 4–GDD of type 4^7
	$1^8 4^5$	YES	From 4–GDD of type 4^7
	$1^4 4^6$	YES	From 4–GDD of type 4^7
	4^7	YES	Theorem 1.2
	$1^{14}7^2$	YES	From 4–GDD of type $1^2 4^3 7^2$
	$1^{10}4^1 7^2$	YES	From 4–GDD of type $1^2 4^3 7^2$
	$1^6 4^2 7^2$	YES	From 4–GDD of type $1^2 4^3 7^2$
	$1^2 4^3 7^2$	YES	this paper
	7^4	YES	Theorem 1.2
29	$2^{12}5^1$	YES	Theorem 1.5
	$2^2 5^5$	???	
	$2^8 5^1 8^1$	YES	this paper
30	$3^8 6^1$	YES	Theorem 1.3
	$3^4 6^3$	YES	this paper
	6^5	YES	Theorem 1.2
	$3^7 9^1$	YES	Theorem 1.3

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