A FAMILY OF RESOLVABLE REGULAR GRAPH DESIGNS

Donald L. Kreher
Department of Mathematical Sciences,
Michigan Technological University,
Houghton, MI 49931, USA

Gordon F. Royle
Department of Computer Science,
University of Western Australia,
Perth, WA 6009, Australia

W.D. Wallis
Department of Mathematics,
Southern Illinois University,
Carbondale, IL 62901, USA

ABSTRACT

A *regular graph design* $RGD(v,k;r)$ is a design on v points with blocks of size k and constant replication number r, such that any two points belong to either λ_1 or λ_1+1 common blocks, for some constant λ_1. We investigate *resolvable* regular graph designs with block size 4. In particular we determine the spectrum of such designs for $v \le 16$.

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1. Introduction

A regular graph design $RGD(v, k; r)$ is a collection of blocks of size k on a v-set such that every element occurs in r blocks and any pair of objects occur together in either λ_1 or λ_2 blocks, where λ_1 is some constant and $\lambda_2 = \lambda_1 + 1$. It is easy to see that there are b blocks, where

$$bk = vr,$$

and that $\lambda_1 = \lfloor \lambda \rfloor$, where

$$\lambda = r(k-1)/(v-1)$$

If λ is an integer, then an RGD is a balanced incomplete block design; the interesting cases are usually those where λ is not an integer, so that no $BIBD$ exists.

Given elements x and y, let λ_{xy} denote the number of blocks containing $\{x, y\}$; for fixed x, write $n_i(x)$ for the number of elements y such that $\lambda_{xy} = \lambda_i$. We observe that

$$n_1(x)\lambda_1 + n_2(x)\lambda_2 = r(k-1),$$

by counting (with multiplicity preserved) all elements other than x in the blocks containing x. As $n_1(x) + n_2(x) = v - 1$ and $\lambda_2 = \lambda_1 + 1$,

$$(v-1)\lambda_1 + n_2(x) = r(k-1),$$

and $n_2(x) = r(k-1) - (v-1)\lfloor r(k-1)/(v-1) \rfloor = r(k-1) - (v-1)\lambda_1$, so $n_1(x)$ and $n_2(x)$ are independent of the choice of x. We write $d = n_2(x)$. The graph formed by taking v elements as the vertices and joining x to y if and only if $\lambda_{xy} = \lambda_2$ is a regular graph of degree d, which is called the *graph of the design*; the edges are called the *special pairs* of the design. Balanced incomplete block designs are regular graph designs whose graph is empty.

Regular graph designs were introduced by John and Mitchell [5], and have been studied because it is conjectured [5]that if a regular graph design exists for given parameters then it is optimal. This has been further discussed by various authors — see the survey by Cheng [2].

It is interesting to consider regular graph designs with various special properties. In this paper we discuss *resolvable regular graph designs* (*RRGDs*). A design is resolvable if its blocks can be partitioned into r *parallel classes* or *resolutions* such that each element occurs in exactly one block in each class. We discuss the existence of resolvable regular graph designs with block size 4, on 8, 12 and 16 points.

2. Another view of the problem.

Resolvable regular graph designs were also introduced in a different way in [3]. The authors discuss a bridge club consisting of 12 couples. At each meeting the club is divided into 3 groups of 4 couples, and each couple competes against each other couple in its group. The club meets eight times, and it is required that each couple plays against each other couple 2 or 3 times. So what is required is an $RRGD(12, 4; 8)$.

The authors of [3]did not succeed in producing such a design. Let us define the *imbalance* of a design as

$$\sum_{x<y} \Big(\lambda_{xy} - \lfloor\lambda\rfloor\Big)2,$$

as was done by Brown [1]and essentially also done in [3]. Then an $RRGD$ will have imbalance 12 (the theoretical minimum). A design with imbalance 14 was found in [3]. A design with imbalance 12 — an $RRGD(12, 4; 8)$ — is constructed in section 5 below.

3. The case v = 8.

We first discuss the case $v = 8$. A design with $r = 7$ can be obtained by taking the parallel class

$$\infty\ 0\ 1\ 3, \quad 2\ 4\ 5\ 6$$

and forming six more classes by successively adding 1 to each entry (where the numbers are treated as integers modulo 7, and ∞ is the “infinity” element satisfying $\infty + 1 = \infty$).

This $RRGD(8,4;7)$ is of course a resolvable $(8,14,7,4,3) - BIBD$. An $RRGD(8,4;6)$ can be obtained by deleting one parallel class from the $RRGD(8,4;7)$. An $RRGD(8,4;1)$ exists — simply take one parallel class.

If any $RRGD(8,4;r)$ exists, one can construct an $RRGD(8,4;7+r)$ by appending a copy of an $RRGD(8,4;7)$ to it. So we can construct an $RRGD(8,4;r)$ whenever $r \equiv 0, 1$ or $6 \pmod 7$.

In general, suppose an $RRGD(8,4;r)$ exists. Then $\lambda = 3r/7$ and $d = 3r - 7\lambda_1$. Consider the occurrence of three treatments, 1, 2 and 3 say. We can classify the parallel classes according to which other elements from this set occur in the same block as 1. Say there are

a	classes of form	1 2 3 -,	- - - -
b	classes of form	1 2 - -,	3 - - -
c	classes of form	1 3 - -,	2 - - -
e	classes of form	1 - - -,	2 3 - -.

Then

$$\begin{array}{lcl} a + b + c + e &=& r \\ a + b &=& \lambda_{12} \\ a + c &=& \lambda_{13} \\ a + e &=& \lambda_{23}. \end{array}$$

Adding, we see that $r + \lambda_{12} + \lambda_{13} + \lambda_{23}$ is even for every possible choice of the three treatments 1, 2 and 3.

(i) Say $r \equiv 2 \pmod 7$: $r = 7q + 2$. Then $\lambda = 3q + \frac{6}{7}$ so $\lambda_1 = 3q$, $\lambda_2 = 3q + 1$ and $d = 6$. The underlying graph consists of K_8 minus a one-factor and it is possible to find three treatments 1, 2 and 3 such that $\lambda_{12} = \lambda_{13} = \lambda_{23} = 3q + 1$. For this choice $r + \lambda_{12} + \lambda_{13} + \lambda_{23} = 7q + 2 + 3(3q + 1) = 16q + 5$, which is odd — a contradiction.

(ii) Say $r \equiv 3 \pmod 7$: $r = 7q+3$. Then $\lambda = 3q+1+\frac{2}{7}$, so $\lambda_1 = 3q+1$, $\lambda_2 = 3q+2$, $d = 2$. The underlying graph is regular of degree 2, so it is a union of cycles which covers all the vertices. If there is more than one cycle in the union, we can assume that vertex 1 is in one cycle and vertices 2 and 3 are adjacent vertices in another one. If the graph is a single cycle of length 8, we can take it to be $(1, 4, 2, 3, \dots)$. In either case

$\lambda_{12} = \lambda_{13} = 3q+1$, $\lambda_{23} = 3q+2$, and $r+\lambda_{12}+\lambda_{13}+\lambda_{23} = 7q+3+2(3q+1)+3q+2 = 16q+7$ which is odd — a contradiction.

From the above we see that there can never be an $RRGD(8,4;r)$ with $r \equiv 2$ or $3 \pmod 7$. But suppose there were an $RRGD(8,4;r)$ where $r \equiv 5 \pmod 7$. Such a design might contain repeated classes; say no class occurs more than n times. One could then construct an $RRGD(8,4;35n-r)$ as follows. There are 35 possible parallel classes on 8 objects. Take n copies of each of those classes. Then delete all the parallel classes in the $RRGD(8,4;r)$ (if a class was repeated, delete the appropriate number of copies). But this design has $35n-r$ parallel classes, and $35n-r \equiv 2 \pmod 7$, which is impossible. Similarly we can rule out $4 \pmod 7$. So we have

Theorem 1. *There is an $RRGD(8,4;r)$ is and only if $r \equiv 0, 1$ or $6 \pmod 7$.* □

4. The case v = 12.

Lemma 2.1. *There is no $RRGD(12,4;r)$ when $2 \le r \le 7$.*

Proof. (i) Say $r = 2$ or 3, so that $\lambda < 1$. Then $\lambda_1 = 0$ and $\lambda_2 = 1$. There is no pair of elements occurring together twice. On the other hand, if 1234 is a block in the first parallel class, one of the three blocks in the second parallel class must contain at least two of $\{1,2,3,4\}$. So $\lambda_{xy} \ge 2$ for some pair $\{x,y\}$ — a contradiction.

(ii) Say $4 \le r \le 7$. Now $\lambda_1 = 1$, $\lambda_2 = 2$, $d = 3r-11$. The graph contains $6(3r-11)$ edges.

Consider a block 1234 in the first parallel class. In each subsequent parallel class, there is at least one block in which at least one pair of elements of $\{1,2,3,4\}$ appear together. Since no pair can occur three times, each of these subsequent occurrences must be a different pair. So $\{1,2,3,4\}$ contains at least $r-1$ special pairs; adding over all blocks, we get at least $3r(r-1)$. This process counts every special pair twice. So

$$3r(r-1) \le 2 \cdot 6 \cdot (3r-11),$$

or $3r2 - 39r + 132 \leq 0$. But $3r2 - 39r + 132$ is positive definite. $\square$

Now an $RRBD(12, 4; 11)$ is a resolvable $(12, 33, 11, 4, 3)$– $BIBD$. These designs are known [4], and one solution is obtained by circulating

$$\infty 014, \quad 2368, \quad 579T \qquad (\mathrm{mod}\ 11).$$

We can get an $RRBD(12, 4; 10)$ by deleting one class from this. An $RRBD(12, 4; 1)$ is of course trivial.

An $RRBD(12, 4; r)$ was constructed by computer for $r = 8, 9, 13, 14, 15, 16, 17$ and 18. (The $RRBD(12, 4; 8)$ was found using a hill-climbing technique; the others were all found by specifying a set of automorphisms for the design – see the next section.) These designs are shown in Table 1. So

Theorem 2. *There is an $RRBD(12, 4; r)$ with $r > 1$ if and only if $r \geq 8$.* $\square$

5. Generation of Table 1.

The $RRBD(12, 4; r)$ for $r = 9$ and $13 \leq r \leq 18$ were generated by the following technique. For the given parameters we constructed all regular graph designs which had a certain automorphism. The designs were tested for isomorphism. Then exemplars of the isomorphism classes were tested for resolvability. In most cases we found a solution using the automorphism $(0123456789TE)$, although other automorphisms had to be used for $r = 9$ and $r = 18$. Larger order automorphisms give shorter computations.

This technique failed for $r = 8$. We attempted to find a design with an automorphism group of order 4, but were unsuccessful. (We did not try all possibilities – there are a number of different presentations of 4-element groups – but we did enough to become discouraged.) The computation method would take too long with a group of order 3.

Instead we used the following hill-climbing technique. We represented a resolvable design on 12 points with $r = 8$ by an 8×12 array with each row partitioned into three parts; the three parts contain the elements of the three blocks in a given parallel class. The *weight* of an array was defined to be the number of pairs of treatments which occur together in less than two, or more than three, blocks in the corresponding design. So we were seeking an array of weight 0.

The program takes an input array, selects a row at random, selects two blocks in that row at random, selects an element at random from each of those blocks, and interchanges the elements. The weight of the new array is calculated; if it is no greater than before then the new array becomes the input, otherwise the old input is retained. Then the process is repeated.

We started with the "worst-possible" array

$$\begin{array}{ccc}
0\,1\,2\,3 & 4\,5\,6\,7 & 8\,9\,T\,E \\
0\,1\,2\,3 & 4\,5\,6\,7 & 8\,9\,T\,E \\
0\,1\,2\,3 & 4\,5\,6\,7 & 8\,9\,T\,E \\
0\,1\,2\,3 & 4\,5\,6\,7 & 8\,9\,T\,E \\
0\,1\,2\,3 & 4\,5\,6\,7 & 8\,9\,T\,E \\
0\,1\,2\,3 & 4\,5\,6\,7 & 8\,9\,T\,E \\
0\,1\,2\,3 & 4\,5\,6\,7 & 8\,9\,T\,E \\
0\,1\,2\,3 & 4\,5\,6\,7 & 8\,9\,T\,E,
\end{array}$$

of weight 66. If a run had not succeeded after $20,000$ iterations it was terminated; such a run took about 7 seconds of cpu time on a SPARC ELC.

A solution was found after 50 runs.

The designs constructed are displayed in Table 1 as follows. An automorphism ϕ of the design is given. Then some parallel classes are displayed. The design consists of all classes formed by applying ϕ to the given classes. (For convenience, the *period* of the class – the number of different classes which result from applying ϕ to the class – is stated.)

$r = 8.$ Develop using $(0\ 4\ 8)\ (1\ 5\ 9)\ (2\ 6\ \mathrm{T})\ (3\ 7\ \mathrm{E})$:

0359 148T 267E (period 3)
0678 134E 259T (period 3)
0123 4567 89TE (period 1)
017T 245E 3689 (period 1)

$r = 9.$ Develop using $(0\ 1\ 2\ 3\ 4\ 5)\ (6\ 7\ 8\ 9\ \mathrm{T}\ \mathrm{E})$:

07TE 2456 1389 (period 6),
0369 127E 458T (period 3)

$r = 13$ Develop modulo 12:

0135 2679 48TE (period 12),
0369 147T 258E (period 1).

$r = 14$ Develop modulo 12:

013T 2457 689E (period 4),
0159 248T 367E (period 6),
016E 2789 345T (period 4).

$r = 15$ To the solution for r = 14, append

0369 147T 258E.

$r = 16$ Develop modulo 12:

015E 2489 367T (period 12),
079T 124E 3568 (period 4).

$r = 17.$ To the solution for $r = 16$, append

0369 147T 258E

$r = 18$ Develop using $(0\ 2\ 4)\ (1\ 3\ 5)\ (6\ 8\ \mathrm{T})\ (7\ 9\ \mathrm{E})$:

0125 348E 679T (period 3)
068T 179E 2345 (period 3)
029E 158T 3467 (period 3)
0267 1489 35TE (period 3)
046E 1378 259T (period 3)
018E 237T 4569 (period 1)
0369 147T 258E (period 1)
019T 236E 4578 (period 1)

TABLE 1

6. A complete search for RGD(12,4;8)′n.

There are 5775 possible parallel classes on twelve points. So, after relabeling one class as 0123 4567 89TE, there are (5775)7/7! different ways to choose eight classes. At first sight constructing and testing all these would take an infeasible amount of time. (The estimate in [3] is more than 1013 years.) So, in order to construct all possible $RGD(12,4;8)$'s, some restrictions on the structure must be found.

Suppose an $RGD(12,4;8)$ exists. Then $\lambda_2 = 3$, so the special pairs occur together in three blocks. There are 12 special pairs and each object belongs to two of them. So, if we count each occurrence of each special pair in the design, we get 36 occurrences.

If $abcd$ is any block in parallel class 1 of the design, it has intersection size at least 2 with at least one block in each later parallel class. So there are at least 13 occurrences of two elements from $\{a,b,c,d\}$ in the same block – 6 in $abcd$ and at least 7 others. So $\{a,b,c,d\}$ contains at least one special pair. So every block contains at least one special pair. We shall refer to a block containing exactly one special pair as *good*. On the other hand, no block can contain 5 or 6 special pairs – in particular there are no repeated blocks – because no object belongs to 3 special pairs. Writing a_i for the number of blocks containing precisely i special pairs,

$$a_1 + a_2 + a_3 + a_4 = 24,$$

$$a_1 + 2a_2 + 3a_3 + 4a_4 = 36.$$

So $a_2 + 2a_3 + 3a_4 = 12$, $a_2 + a_3 + a_4 \leq 12$, and $a_1 \geq 12$. So there must be at least two parallel classes, each of which contains at least two good blocks.

Without loss of generality suppose 0123 and 4567 are good blocks and the first parallel class is 0123 4567 89TE. In every subsequent parallel class there must be precisely one block with one pair from 0123 and one block with one pair from 4567, or else there will be more than 13 total appearances of pairs from those blocks. So there are only two possible patterns:

$$xxyy \quad xyzz \quad xyzz \qquad \text{or} \qquad xxyz \quad xyyz \quad xyzz$$

(where x means "one of 0123", y means "one of 4567" and z means "one of $89TE$".) We'll say that classes of these kinds meet the first class *in type I* and *in type II* respectively.

Without loss of generality, suppose the second parallel class also contains two good blocks. There are two cases: if classes 1 and 2 meet in type I, class 2 may be taken without loss of generality as

$$0145 \quad 2689 \quad 37TE$$

and if they meet in type II, it is

$$0148 \quad 2569 \quad 37TE.$$

For each choice of the second parallel class we carried out the following computation. First we found S_1 and S_2, the sets of all possible parallel classes which meet the first and second class respectively either in type I or type II, and set $S = S_1 \cap S_2$. Then S was partitioned into orbits under the action of the set of permutations which stabilized the first two classes; the orbits were ordered arbitrarily as T_0, T_1, ..., and it was assumed:

> if i is the smallest integer such that the $RRGD$ contains a member of T_i, then the orbit leader of T_i is included.

So for each i we found all possible $RRGD$'s containing the first two classes, the orbit leader of T_i, and five other classes chosen from $T_i \cup T_{i+1} \cup \ldots$.

No further isomorph reduction was carried out, although clearly the computation could have been further reduced by more theoretical work. But the exhaustive computation was now feasible, and 66 designs were found. After testing for isomorphism we found there were two non-isomorphic designs, which are listed in Table 2. We have:

Theorem. *There are precisely two isomorphism classes of $RRGD(12, 4; 8)$.* □

In each case develop using (048) (159) $(26T)$ $(37E)$:

I.	0123	4567	89TE	(period 1)
	017T	245E	3689	(period 1)
	0359	148T	267E	(period 3)
	0678	134E	259T	(period 3)
II.	0123	4567	89TE	(period 1)
	025E	178T	3469	(period 1)
	0149	268E	357T	(period 3)
	017E	2348	569T	(period 3)

THE $RRGD(12,4;8)$'s

TABLE 2

7. The case v = 16.

There is an affine plane on 16 points. This consists of five parallel classes of blocks of size 4. Any selection of r of these classes forms an $RRGD(16,4;r)$, for $1 \leq r \leq 5$. Adding a copy of the affine plane to an $RRGD(16,4;r)$ forms an $RRGD(16,4;r+5)$. So

Theorem 3. *There is an $RRGD(16,4;r)$ for all r.*

REFERENCES

[1] R.B. Brown, Nonexistence of a regular graph design with $v = 17$ and $k = 6$, Discrete Math. **68** (1988), 315–318.

[2] C.S. Cheng, Optimal properties of balanced incomplete block and other designs, *Coding Theory and Design Theory Part II*, (Springer, 1990), 54–64.

[3] B.S. Elenbogen and B.R. Maxim, Scheduling a bridge club (a case study in discrete optimization), Mag. **65** (1992), 18–26.

[4] M. Hall Jr., *Combinatorial Theory (2nd Ed)*, (Wiley, 1986).

[5] J.A. John and T.J. Mitchell, Optimal incomplete block designs, J. Royal. Statist. Soc. **39B** (1977), 39–43.