The bigraphical t–wise balanced designs of index 1

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ABSTRACT

The t–wise balanced designs of index 1 whose point set are the edges of $K_{m,n}$, the complete bipartite graph, and that have the subgroup of all automorphisms that fix the two independent sets of $K_{m,n}$ as an automorphism group are studied. All such designs are found.

1. INTRODUCTION

A *t–wise balanced design* (tBD) of type t–$(v, \mathcal{K}, \lambda)$ is a pair $(X, \mathcal{B})$ where X is a v–element set of points and $\mathcal{B}$ is a family of distinct subsets of X called blocks such that: (1) $\mathcal{K}$ is a set of positive integers strictly between t and v; (2) the size of every block is in $\mathcal{K}$; and (3) every t–element subset of X is contained in exactly λ blocks. Note that X is not a block. Also if we give the parameters of a specific tBD, then we will choose a minimal $\mathcal{K}$. The index of a tBD is λ. All of the designs in this paper have index 1.

For example a 3–$(16, \{4, 6\}, 1)$ can be constructed as follows. The 16 points will be the 16 cells of a 4 by 4 array. There are sixteen 6–element blocks and sixty 4–element blocks. The sixteen 6-element blocks are obtained by choosing the 6 cells different from and in the same row or column as a given cell. The 2 by 2 sub-arrays account for thirty-six of the 4–element blocks and the twenty-four ways of choosing 4 cells with one from each row and column account for the remaining 4–element blocks. Alternatively this design can be described as taking all subgraphs of $K_{4,4}$ that are isomorphic to $K_{1,3}\dot{\cup}K_{3,1}$, to $K_{2,2}$ and to a one–factor. The rows and columns of the 4 by 4 array are the vertices and and the cells are the edges. This motivates the following definition.

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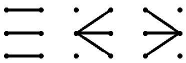
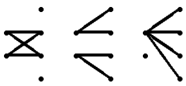
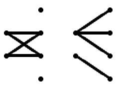
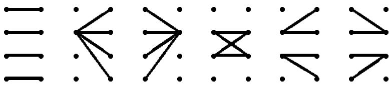
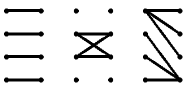
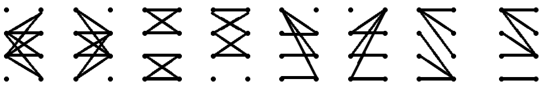
A t–wise balanced design $(X, \mathcal{B})$ of type t–$(m \cdot n, \mathcal{K}, \lambda)$ is *bigraphical* if X is the set of edges of the complete bipartite graph $K_{m,n}$ and whenever B is a block and α is an automorphism of $K_{m,n}$ (that fixes the independent sets), then $\alpha(B)$ is also a block. That is $\mathcal{B}$ is a union of orbits under the action of the subgroup $\mathcal{G}$ of the automorphism group of $K_{m,n}$ that preserves the two independent sets. ($\mathcal{G} = \textbf{aut}\ K_{m,n}$ if $m \neq n$.) Without loss we will always assume that $m \leq n$. A t–wise balanced design on the complete graph is called a *graphical* tBD and they were studied in [?].

In the subsequent discussion we prove that the only bigraphical t–wise balanced designs of index 1 are those listed in table I.

TABLE I. The bigraphical t–wise balanced designs of index 1.

	Parameters	Bigraphical representation
D_1:	$2 \leq m \leq n$ 1–$(mn, n, 1)$	$K_{1,n}$
D_2:	$2 \leq m \leq n$ 1–$(mn, m, 1)$	$K_{m,1}$
D_3:	$m = n = 2$ 1–$(4, 2, 1)$	
D_4:	$m = n = 3$ 2–$(9, 3, 1)$	
D_5:	$m = 2,\ n = 4$ 3–$(8, 4, 1)$	
D_6:	$m = 2,\ n = 4$ 3–$(8, 4, 1)$	
D_7:	$m = n = 4$ 3–$(16, 4, 1)$	
D_8:	$m = n = 4$ 3–$(16, \{4, 6\}, 1)$	
D_9:	$m = n = 4$ 5–$(16, \{6, 8\}, 1)$	

Before proceeding with the proof we mention a few more definitions and notational conventions. Throughout we fix a certain set of m vertices called *the left* and a certain set of n vertices called *the right*. All of our subgraphs will only contains edges $\{x, y\}$ that have one vertex x from the left and one vertex y from the right. There are no multiple edges. The edge $\{x, y\}$ is denoted by xy. A subgraph obtained by choosing a collection $\mathcal{L}$ of i left vertices and a collection $\mathcal{R}$ of j right vertices and all edges xy with $x \in \mathcal{L}$ and $y \in \mathcal{R}$ is denoted by $K_{i,j}$. Note that although $K_{i,j} \cong K_{j,i}$ as graphs we consider them as being from different classes of subgraphs of $K_{m,n}$. This is because we restrict our attention only to those automorphisms of $K_{m,n}$ that map left vertices to left vertices and right vertices to right vertices. The *orbit of a subgraph* of $K_{m,n}$ will mean the orbit under this restricted set of automorphisms. An automorphism that interchanges exactly two vertices is called a *switch* and its application to the graph is called *switching*. The *reverse* of a graph is the graph obtained by interchanging its left and right vertices. So, for example the reverse of $K_{i,j}$ is $K_{j,i}$. Of course "reverse" is not one of the restricted automorphisms. The notation $G_1 \dot{\cup} G_2$ will denote the disjoint union of the subgraphs G_1 and G_2. The symbols A, B will always be blocks and the symbol T will be a t-element subset of edges. The letters x, x', x'', a_i, u, u' will invariably be left vertices and y, y', y'', b_i, v, v' will be right vertices.

2. THE PROOF

It is straight forward to check that the designs D_1, D_2, D_3, D_4, D_5, D_6, D_7, D_8 and D_9, listed in Table I are indeed bigraphical t-wise balanced designs. So, we have:

Lemma 1. *Table I contains bigraphical t-wise balanced designs of index 1.*

In the sequence of lemmas that follows we prove that there are no others. So, suppose that $(X, \mathcal{B})$ is a bigraphical t–$(m \cdot n, \mathcal{K}, 1)$.

Lemma 2. *If A and B are distinct blocks, then $|A \cap B| \leq |A| - 2$, and if equality holds then $|A| = t + 1$.*

Proof. $|A \cap B| \leq t - 1 \leq |A| - 2$. □

We will frequently make use of the fact that if T is a t–element subset of the block B that is fixed by the automorphism α, then α also fixes B.

Lemma 3. $2 \leq m, n$.

Proof. If $m = 1$, then a block is a $K_{1,k}$ for some $t < k < n$. Switching an endpoint with an isolated point is an automorphism that fixes a t-element subset of the block. Thus, by the above, the block is the complete bipartite graph, i.e. it has size $v = m \cdot n$ which is not permitted. □

Lemma 4. *If $t = 1$, then the design is: D_1 , D_2 or D_3.*

Proof. Let xy be in block B. If deg $x > 1$, then switching on the right forces B to be a $K_{1,n}$. The orbit of B then yields design D_1. Similarly, deg $y > 1$ yields D_2. Otherwise, deg $x = 1 =$ deg y for any xy in any block B. By switching, one easily forces $m = n = 2$ and design D_3. $\square$

Lemma 5. *If $2 \leq t \leq n$, then either*

(i) $2 \leq m \leq n = t+1$ *and* $K_{1,t}\dot{\cup}K_{m-1,1}$ *is a block; or*

(ii) $2 \leq m$, $t+1 \leq n$ *and* $K_{1,n}$ *is a block;*

(and not both, obviously).

Proof. Let B be the block containing the t-element set $T = K_{1,t}$ and let x be its vertex of degree t. If $t = n$, then $B = K_{m,n}$, so $n \geq t+1$. If B contains any other edge incident to x, then B contains $K_{1,n}$, and can contain no more, lest $B = K_{m,n}$; so we are in case (ii). Hence we may assume B contains no other edges incident with x. If B contains an edge incident to an end in T but not to x, then switching on the left shows that B contains a $K_{m,t}$. Now switching on the right forces $B = K_{m,n}$. Consequently, B contains an edge disjoint from T.Using switches that fix T we see that $B = K_{1,t}\dot{\cup}K_{m-1,n-t}$. A switch on the right will force a contradiction unless $n = t+1$. $\square$

Lemma 6. *If $2 \leq t \leq m, n$, then $m = n = t+1$, and all 1–factors are blocks.*

Proof. By Lemma 5, $m, n > t$. Let T be a matching of size t and B be the block containing it. Well $B \neq T$ so B contains an additional edge e. If e is incident to two ends of T, then switches that permute the edges of T show that B contains a $K_{t,t}$. Further switching gives $B = K_{m,n}$ and this is a contradiction. If $e = xy$ is incident to exactly one end x of T, then switching y with isolated vertices and permuting the edges of T shows that B contains a $K_{t,n-t}$ additional to and incident with the matching T. Now switching y with vertex of T will force an edge incident to two ends of T. This we have already seen lead to a contradiction. The only possibility is for e to be incident to none of ends of T. Then switches that fix T point–wise show that B contains $T\dot{\cup}K_{m-t,n-t}$. Now additional switches produce a contradiction unless $m = n = t+1$. $\square$

Lemma 7. *If $2 \leq t \leq m, n$, then the design is D_4, D_7 or D_8.*

Proof. By Lemma 6, $m = n = t+1$. Let B be the block containing the t set T in Figure 1. By Lemma 6, since all 1-factors are blocks, B cannot contain t independent edges. We use this fact to obtain contradictions when $t \geq 4$. So suppose $t \geq 4$. Then no vertex can have degree greater than or equal to t in B by Lemma 5, so every edge of B meeting x' or x'' avoids meeting y, y' and y'', and every edge meeting x meets y, y' or y''. If B contains the edge a_ib_j, $i \neq j$, then permuting the $t-2$ edges $\{a_1b_1, \ldots a_{t-2}b_{t-2}\}$ shows that B contains $K_{t-2,t-2}$. Now switching x and x' gives a contradiction, since $(t-2)^2 \geq t$. If B contains an edge a_iy (or a_iy'), then permuting the $t-2$ edges $\{a_1b_1, \ldots a_{t-2}b_{t-2}\}$ and switching y and y' shows that B contains $K_{t-2,2}$ on $\{a_1, \ldots, a_{t-2}, y, y'\}$. Switching b_{t-2} and y gives a contradiction. If B

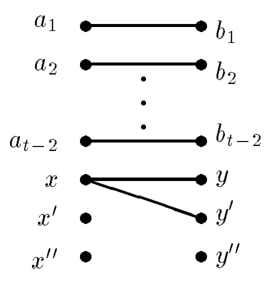


Figure 1.

contains an edge $b_i x'$ (or $b_i x''$) then permuting the $t-2$ edges $\{a_1b_1, \ldots a_{t-2}b_{t-2}\}$ and switching x' and x'' shows that B contains $K_{2,t-2}$ on $\{x', x'', b_1, \ldots, b_{t-2}\}$. Switching a_{t-2} and x gives a contradiction, since $2(t-2) \geq t$. If B contains the edge $a_i y''$, then permuting the $t-2$ edges $\{a_1b_1, \ldots a_{t-2}b_{t-2}\}$ shows that B contains $K_{t-2,1}$ on $\{a_1, \ldots, a_{t-2}, y''\}$. Now again switching x and x' gives a contradiction, since $(t-2)+(t-2) \geq t$. Finally, if B contains the edge xy'', then switching a_{t-2} and x'' gives a contradiction. Therefore $t \leq 3$.

Now suppose $t = 3$. If case (ii) of Lemma 5 holds, then each $K_{1,4}$ and each $K_{4,1}$ are blocks. In particular, no vertex of B can have degree greater than 2, so $B = T \cup \{a_1, y''\}$. Similarly the reverse of B is also block, and the design is D_7 . If case (i) of Lemma 5 holds, then B is $K_{1,3}\dot{\cup}K_{3,1}$ and the design is D_8 .

If $t = 2$, the design is D_4 . □

Lemma 8. *If $m = 2 \leq t$, then the design is D_5 or D_6.*

Proof. If B is any block, then by Lemma 2 there are two types of blocks: *either*

(I) all vertices on the right have degree one in B, *or*

(II) all vertices on the right have degree zero or two in B.

Since $t \geq 2$ there must be one block B of type (II). Let n_i be the number of vertices on the right that have degree i in B, $i \in \{0, 2\}$. Thus $n_0 + n_2 = n$ and since $B \neq K_{2,n}$ we have $1 \leq n_2 \leq n-1$. Switching a degree 0 vertex with a degree 2 vertex on the right and applying Lemma 2 shows that $t = 2n_2 - 1$ so $n_2 \geq 2$ and there is a unique orbit of blocks of type (II). But if n_2 exceeds 2 the block that covers a t-element set consisting a K_{2,n_2-2} and any three additional edges that have degree one on the right can not be in the orbit of B. Therefore $n_2 = 2$ and so $t = 3$. Efforts to cover $K_{1,2}\dot{\cup}K_{1,1}$ force $n = 4$ and the design to be D_5 or D_6. □

Definition: The *derived design* of a t–$(v, \mathcal{K}, \lambda)$ design $(X, \mathcal{B})$ with respect to $S \subseteq X$ is $(X', \mathcal{B}')$ where $X = X \setminus S$ and $\mathcal{B}' = \{B \setminus S : S \subseteq B \in \mathcal{B}\}$. If $|S| = j \leq t$, it is a $(t-j)$–$(v-j, \mathcal{K}', \lambda)$ design; where $\mathcal{K}' = \{k - j : k \in \mathcal{K}\}$.

Main Theorem: *The only bigraphical t–wise balanced designs of index 1 are those listed in Table I.*

Proof. If not, choose an unlisted bigraphical t–$(m \cdot n, \mathcal{K}, 1)$ design $(X, \mathcal{B})$ with $m + n$ as small as possible. We may assume

$$3 \leq m \leq n, \qquad m+1 \leq t \leq mn-2$$

Thus the derived design with respect to a $K_{m,1}$ is a bigraphical $(t-m)$–$(m \cdot (n-1), \mathcal{K}', 1)$ design and must by the choice of m and n appear in Table I. (It is impossible for the derived design to be the reverse of a design in Table I since $m \leq n$.) There are then several cases:

<u>Case 1.</u> *The derived design with respect to $K_{m,1}$ is D_1.*

Then $m = t-1$, $4 \leq t \leq n+1$ and the graph in Figure 2 is a block. If B is any block not in the same orbit it cannot cover any t–element set already covered, so if u and v are vertices on the left and right respectively their degrees in B must satisfy

$$\begin{aligned} \deg(u) + \deg(v) &\leq t-1 \text{ if } uv \notin B; \\ \deg(u) + \deg(v) &\leq t \text{ if } uv \in B. \end{aligned}$$

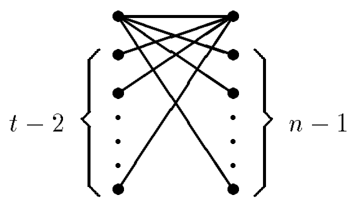


Figure 2.

So if B is the block containing the t–element set T in Figure 3 then no further edge of B can meet x, x', y or y'. Then the appropriate switches show that B contains a $K_{t-3,n-1}$ and is the block given in

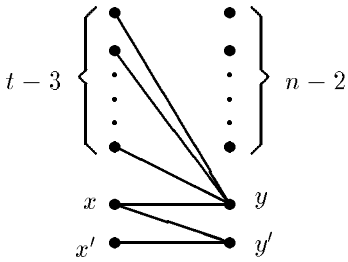


Figure 3.

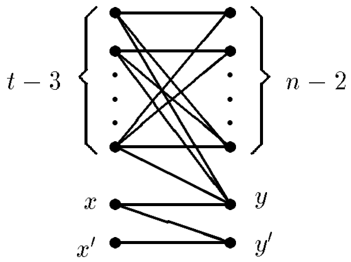


Figure 4.

Figure 4. Therefore using the above degree conditions we see that $n = 3$ and $t = 4$. But now B reduces to a path of length 5 and non adjacent vertices u, v of degree 2 in such a path violate the condition that $\deg(u) = \deg(v) \leq t-1$.

<u>Case 2.</u> *The derived design with respect to* $K_{m,1}$ *is* D_2.

Then $m = t-1$, $4 \leq t \leq n+1$ and $K_{m,2} \in \mathcal{B}$. Now the degree conditions for any other block are

$$\deg(v) + \deg(v') \leq t-1$$

for any pair of vertices v,v', both on the right. So if B is the block containing the t-element set in Figure 5 no further edge of B can meet y, y' or y''. Thus $n \geq 4$, and every vertex on the right other than y has degree 1 in B, so switching y and y' shows that $n \leq t-1$, whence $m = n = t-1$. Switching x' with any other vertex on the left except x shows that B is the block in Figure 6. Now switching y' with any other vertex on the right other than y, y' or y'' shows by Lemma 2 that $|B| = t+1$,

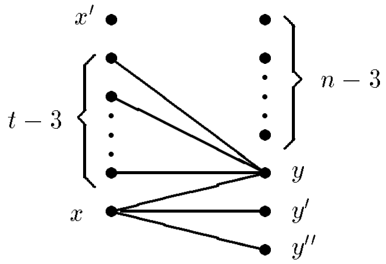
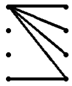
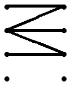
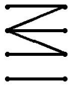


Figure 5.

so $t = 5$, $m = n = 4$. Now  must be covered by $K_{2,4}$ so any other block must satisfy the same degree conditions for vertices on the left. Thus $K_{1,3} \dot{\cup} K_{2,1}$ must be covered by $K_{1,3} \dot{\cup} K_{3,1}$. Now to cover , we can't use , for

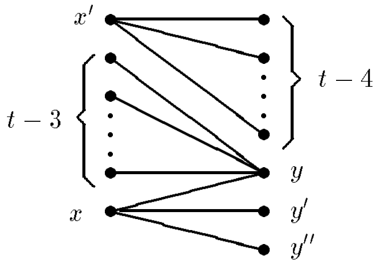
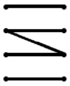
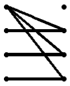
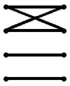
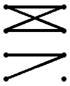
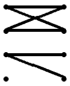
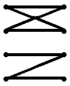


Figure 6.

then  would be covered twice. Thus  is a block and by symmetry, so is its reverse. Now the 6-cycle must be a block, as it is the only way to cover the path of length 5 without re-covering some other 5–element set. This leaves the 5-sets

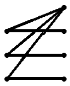
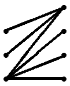
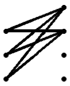
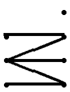
T_1 T_2 T_3 T_4

to be covered. They can be all covered by $K_{2,2} \dot{\cup} K_{2,2}$, giving the known design D_9. If  $\in \mathcal{B}$, there is no way to cover T_2. If  $\in \mathcal{B}$, there is no way to cover T_3. Similarly,  $\notin \mathcal{B}$. The only alternative is  $\in \mathcal{B}$, but that covers T_1 three times.

<u>Case 3.</u> *The derived design with respect to* $K_{m,1}$ *is* D_3.

Then $t = 3$, $m = 2$ and $n = 3$. This contradicts $m \geq 3$.

<u>Case 4.</u> *The derived design with respect to* $K_{m,1}$ *is* D_4.

Then $t = 5$, $m = 3$, $n = 4$ and ,  and  must be blocks, but now  cannot be covered.

<u>Case 5.</u> *The derived design with respect to* $K_{m,1}$ *is* D_5.

Then $t = 5$, $m = 2$ and $n = 5$. This contradicts $m \geq 3$.

Case 6. *The derived design with respect to $K_{m,1}$ is D_6.*

Then $t = 5$, $m = 2$ and $n = 5$. This contradicts $m \geq 3$.

Case 7. *The derived design with respect to $K_{m,1}$ is D_7.*

Then $t = 7$, $m = 4$, $n = 5$ and the derived design with respect to $K_{1,5}$ would be bigraphical 2–$(3 \cdot 5, \mathcal{K}'', 1)$ design. There is no such design in Table I.

Case 8. *The derived design with respect to $K_{m,1}$ is D_8.*

Then $t = 7$, $m = 4$, $n = 5$ and the derived design with respect to $K_{1,5}$ would be bigraphical 2–$(3 \cdot 5, \mathcal{K}'', 1)$ design. There is no such design in Table I.

Case 9. *The derived design with respect to $K_{m,1}$ is D_9.*

Then $t = 9$, $m = 4$, $n = 5$ and the derived design with respect to $K_{1,5}$ would be bigraphical 4–$(3 \cdot 5, \mathcal{K}'', 1)$ design. There is no such design in Table I.

There are no other cases to consider. □

3. REVELATION

When $d \geq 4$ a construction in [?] due to R. Wilson is given for a 5–$(2^d, \{6, 8\}, 1)$ design $(X, \mathcal{B})$. He takes X to be a d–dimensional vector space over GF(2). The 8–element blocks in $\mathcal{B}$ are the 3–dimensional affine subspaces and a 6-element subset $\{\vec{x_1}, \ldots \vec{x_6}\}$ is a block if and only if $\vec{x_1} + \cdots + \vec{x_6} = \vec{0}$. Our bigraphical design D_9 is precisely this design when $d = 4$. Indeed assign in some order the pairs $(0, 0)$, $(0, 1)$, $(1, 0)$ and $(1, 1)$ to each of the 4 vertices of the left of D_9 and to each of the 4 vertices of the right of D_9. An edge xy has now been assigned a 4–dimensional vector over GF(2) and the bigraphical blocks are easily seen to satisfy the construction given by R. Wilson.

REFERENCES

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