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The bigraphical t–wise balanced designs of index two

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ABSTRACT

In this paper we study the group $S_m \times S_n$ acting on the mn ordered pairs and classify all t–wise balanced designs of index 2 that have such an automorphism group.

1. INTRODUCTION

A *t–wise balanced design* (tBD) of type t–$(v, \mathcal{K}, \lambda)$ is a pair $(X, \mathcal{B})$ where X is a v–element set of points and $\mathcal{B}$ is a family of subsets of X called blocks such that: (1) $\mathcal{K}$ is a set of positive integers strictly between t and v; (2) the size of every block is in $\mathcal{K}$; and (3) every t–element subset of X is contained in exactly λ blocks. Note that X is not a block. If $\mathcal{B}$ contains all k-element subsets of X for some k, then $(X, \mathcal{B})$ is said to be a *trivial design.* If we give the parameters of a specific tBD, then we will choose a minimal $\mathcal{K}$. If $|\mathcal{K}| = 1$, then the tBD is called a t–(v, k, λ) design, where $\mathcal{K} = \{k\}$. The index of a tBD is λ. All of the

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designs in this paper have index 2. For some interesting results and conjectures on t–wise balanced designs the reader is directed to [5].

A fundamental question in the theory of combinatorial designs is to ask *what are the designs that can be obtained with a particular type of automorphism group.* In this paper we study the group $S_m \times S_n$ acting on the set of $m \cdot n$ ordered pairs and classify all tBDs of index 2 that have such an automorphism group. It is natural to think of the $m \cdot n$ ordered pairs as the edges of $K_{m,n}$ and this motivates the following definition:

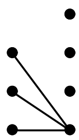
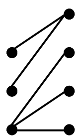
Definition: *Bigraphical t-wise balanced designs.*

A t–wise balanced design $(X, \mathcal{B})$ of type t–$(m \cdot n, \mathcal{K}, \lambda)$ is *bigraphical* if X is the set of edges of the complete bipartite graph $K_{m,n}$ and whenever B is a block and α is an automorphism of $K_{m,n}$ (that fixes the independent sets), then $\alpha(B)$ is also a block.

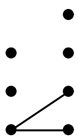
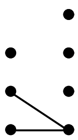
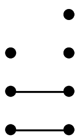
Thus, in a bigraphical tBD, the set of blocks $\mathcal{B}$ is a union of orbits under the action of the subgroup $\mathcal{G}$ of the automorphism group of $K_{m,n}$ that preserves the two independent sets. ($\mathcal{G} =$ **aut** $K_{m,n}$ if $m \neq n$.) Without loss we will always assume that $m \leq n$. In Section 2 we prove that the only bigraphical t–wise balanced designs of index 2 are those listed in the Appendix.

The following is an example of a bigraphical 2-wise balanced design of index 2.

Example: A bigraphical 2–$(12, \{3, 5\}, 2)$ on $K_{3,4}$ is given by taking as blocks all of the subgraphs isomorphic to

$B_1 =$  and $B_2 =$  .

This accounts for 4 and 12 blocks respectively. There are three types of pairs of edges:

$T_1 =$  , $T_2 =$  and $T_3 =$  .

Observe that T_1 is covered twice by the orbit of B_2, T_2 is covered once by the orbit of B_1 and once by the orbit of B_2, and T_3 is covered twice by the orbit of B_2.

The bigraphical designs of index 1 were studied in [4]. Repeating all of the blocks in a design of index 1 yields a design of index 2. The bigraphical designs of index 2 obtained this way from the bigraphical designs of index 1 are given in the Appendix as designs $D_2, D_3, D_{10}, D_{18}, D_{32}, D_{33}, D_{40}, D_{41}$ and D_{53}. A bigraphical 3-(77,20,3135283200) design on $K_{7,11}$ is given in [1]. When $n = m$ the full automorphism group of $K_{m,n}$ is the wreath product $S_n \sim S_2$ which acts as a rank 3 permutation group on the edges. Using this observation Tonchev constructs the 3-(16,6,2) design D_{27} with $S_4 \sim S_2$ as an automorphism group and shows that this is the only symmetric 2-(n^2, k, λ) design that can be constructed in this way, [10]. A t–wise balanced design on the complete graph is called a *graphical* tBD. These were studied in [2]. An example of graphical 2–(21,5,3) appears in [6] and other examples appear in [9]. de Caen and Kreher found all 3-hypergraphical Steiner quadruple systems of order twenty in [3]. These are 3–$(20, 4, 1)$ designs on the complete hypergraph $K_6^{(3)}$, i.e. the set of all twenty 3-element subsets of a 6-element set.

2. THE PROOF

Before proceeding with the proof we mention a few more definitions and notational conventions. Throughout we fix a certain set of m vertices called *the left* and a certain set of n vertices called *the right*. All of our subgraphs will only contain edges $\{a_i, b_j\}$ that have one vertex a_i from the left and one vertex b_j from the right. There are no multiple edges. The edge $\{a_i, b_j\}$ is denoted by a_ib_j. A subgraph obtained by choosing a collection $\mathcal{L}$ of i left vertices and a collection $\mathcal{R}$ of j right vertices and all edges xy with $x \in \mathcal{L}$ and $y \in \mathcal{R}$ is denoted by $K_{i,j}$. Note that although $K_{i,j} \cong K_{j,i}$ as graphs we consider them as being from different classes of subgraphs of $K_{m,n}$. This is because we restrict our attention only to those automorphisms of $K_{m,n}$ that map left vertices to left vertices and right vertices to right vertices. The *orbit of a subgraph* of $K_{m,n}$ will mean the orbit under this restricted set of automorphisms. The *reverse* of a graph is the graph obtained by interchanging its left and right vertices. So, for example the reverse of $K_{i,j}$ is $K_{j,i}$. Of course "reverse" is not one of the restricted automorphisms. The notation $G_1 \dot{\cup} G_2$ will denote the disjoint union of the subgraphs G_1 and G_2. The symbols B or B_i will always be blocks and the symbols T or T_i will be a t-element subset of edges. The symbols x, x' and a_i will invariably be left vertices and y, y' and b_i will be right vertices. A *left degree sequence* of a subgraph G is an unordered list of the number of edges to which each vertex on the left of G is incident. A *right degree sequence* of G is similarly defined. We will adopt the phrase "B covers T" to mean there is a subgraph B' isomorphic to B that contains T.

It is straightforward to check that the designs D_i, $1 \leq i \leq 53$ listed in the Appendix are indeed bigraphical t-wise balanced designs. Furthermore, using ideas found in [7] and computational tools available in the *design theory toolchest*, [8], a simple backtracking algorithm was constructed to find all t–$(m \cdot n, \mathcal{K}, 2)$ designs with automorphism group $S_m \times S_n$. The search was successfully completed for all m, n and t with $2 \leq m \leq n \leq 5$ and $t \leq 6$. The running time was less than one day on a Sun S10/40. So, we have:

Lemma 1 *The Appendix contains bigraphical t-wise balanced designs of index 2. Furthermore, if $2 \leq m \leq n \leq 5$ and $t \leq 6$, then every t–$(m \cdot n, \mathcal{K}, 2)$ is listed in the Appendix.*

In the sequence of lemmas that follows we prove that there are no others. So, suppose that $(X, \mathcal{B})$ is a nontrivial bigraphical t–$(m \cdot n, \mathcal{K}, 2)$. Suppose $m \leq n$.

Lemma 2 *Let $(X, \mathcal{B})$ be a bigraphical t–wise balanced design of type t–$(m \cdot n, \mathcal{K}, 2)$. Suppose σ is an automorphism, B is a block and $|T| = t$. If $T \subseteq B$, $T \subseteq \sigma B$, and $T \subseteq \sigma^2 B$, then $B = \sigma^2 B$.*

Proof. T can be contained in only two blocks, so either (i) $B = \sigma B$, (ii) $B = \sigma^2 B$, or (iii) $\sigma B = \sigma^2 B$. These three cases are easily dealt with as follows:

(i) $B = \sigma B \Rightarrow \sigma B = \sigma(\sigma B) = \sigma^2 B \Rightarrow B = \sigma^2 B$;

(ii) $B = \sigma^2 B$; and

(iii) $\sigma B = \sigma^2 B \Rightarrow \sigma B = \sigma(\sigma B) \Rightarrow B = \sigma B \Rightarrow B = \sigma^2 B$.

$\square$

Corollary 2.1 *Let $(X, \mathcal{B})$ be a bigraphical t–wise balanced design of type $t - (m \cdot n, \mathcal{K}, 2)$. Suppose σ is an automorphism, B is a block and $|T| = t$. If $\sigma T = T$ and $T \subseteq B$, then $B = \sigma^2 B$.*

Proof. $\sigma T = T$ and $\sigma T \subseteq \sigma B \Rightarrow T \subseteq \sigma B$. $\sigma^2 T = \sigma(\sigma T) = \sigma(T) = T$ and $\sigma^2 T \subseteq \sigma^2 B \Rightarrow T \subseteq \sigma^2 B$. Thus, by Lemma 2, $B = \sigma^2 B$. $\square$

Lemma 3 $2 \leq m, n$.

Proof. Suppose $m = 1$. Consider the t-element set $T = K_{1,t}$ on $\{a_1, b_1, \ldots, b_t\}$. Let $B \in \mathcal{B}$ be a block containing T. Since $B \neq T$, B contains an additional edge, say $e = a_1 b_{t+1}$. Therefore, $K_{1,t+1}$ on $\{a_1, b_1, \ldots, b_{t+1}\}$ is a subset of B.

(i) $n = t + 1$.

In this case, $B \supseteq K_{1,t+1} = K_{m,n}$, contrary to the assumption that $B \neq K_{m,n}$.

(ii) $n = t + 2$.

Here, $B = K_{1,t+1}$, since B is not complete. The orbit of B covers T exactly twice, producing the trivial design t–$(t + 2, t + 1, 2)$.

(iii) $n \geq t + 3$.

Applying the automorphism $\sigma = (b_{t+1}, b_i, b_j), t+2 \leq i, j \leq n$ fixes T. By Corollary 2.1, $a_1 b_j \in B$ for $t + 2 \leq j \leq n$. Thus, $K_{1,n} = K_{m,n} \subseteq B$. Consequently, B is the complete bipartite graph.

□

2.1 $t \leq n$

Lemma 4 *Every bigraphical 1-wise balanced design of index 2 is listed in Appendix A1.*

Proof. Let xy be in block B. B must contain at least one other edge. We will consider the three cases (i) deg $x > 1$, (ii) deg $y > 1$ or (iii) deg $x = \deg y = 1$.

(i) Deg $x > 1$.

If $n \geq 4$, then using 3-cycles on the right forces B to be a $K_{r,n}$, for some $r < m$. The orbit of $K_{r,n}$, with $1 \leq r < m$ covers xy, $\binom{m-1}{r-1}$ times. Then either $r = 1$ and $K_{1,n}$ is a block whose orbit covers $T = \{xy\}$ once and so the blocks is repeated and the design D_3 is obtained; or there is an additional block with deg $x = 1$ that contains T; or $r = 2$ and $m = 3$ and the design is D_4.

If $n < 4$, then all such designs were computed in Lemma 1. Consequently, the design is in Appendix A1 and is thus D_1, D_3, D_4, D_6, D_7, or D_9.

(ii) Deg $y > 1$.

By similar arguments as in (i), the design is D_1, D_2, D_5, D_6, D_8 or the design contains the block $K_{m,1}$ and an additional block with deg $y = 1$.

(iii) Deg $x = \deg y = 1$ for any xy in any block B.

If $m, n \geq 4$, then applying a 3-cycle forces B to to contain a point with degree greater than 1, contrary to the hypothesis.

Thus, either $m = 3$, $n = 3$ and the design is D_{12}, $m = 2$, $n = 3$ and the design is D_{11}, or $m = 2$, $n = 2$ and a one-factor is a block, yielding designs D_8, D_9 and D_{10}.

Finally, for $m, n \geq 2$, a combination of the blocks $K_{1,n}$ and $K_{m,1}$ from (i) and (ii) yields design D_1. □

Lemma 5 *If $t = 2$, then $n \leq 4$.*

Proof. Suppose $n \geq 5$. Let $T = \{a_1b_1, a_2b_2\}$ be a matching of size 2, and let B be a block that contains T. Since $B \neq T$, B contains an additional edge, e. There are, up to symmetry, four types of edges: (i)a_1b_2, (ii) a_1b_3, (iii) a_3b_1 and (iv) a_3b_3.

(i) $a_1b_2 \in B$.

The automorphisms $\sigma = (b_1, b_i, b_j), 3 \leq i, j \leq n$ fix the t-element set $\{a_1b_2, a_2b_2\}$. Thus, $a_1b_j \in B$ for $1 \leq j \leq n$. Applying 3-cycles on the right then forces $K_{2,n}$ on $\{a_1, a_2, b_1, \ldots, b_n\}$ to be a subset of B. Consequently, $m \geq 3$ and if $m = 3$, then $B = K_{2,n}$ is a block. But, this is the only orbit of blocks that covers $T = \{a_1b_1, a_2b_2\}$ and it does so only once. If $m \geq 4$, 3-cycles on the left force $K_{m,n} \subseteq B$.

(ii) $a_1b_3 \in B$.

The automorphisms $\sigma = (b_3, b_i, b_j), 4 \leq i, j \leq n$ fix the t-element set $\{a_1b_1, a_2b_2\}$. Thus, $a_1b_j \in B$ for $3 \leq j \leq n$. Applying the automorphism $\gamma = (b_1, b_3, b_2)$ fixes at least two edges, since $n \geq 5$. Thus $a_1b_2 \in B$. Finish as in case (i).

(iii) $a_3b_1 \in B$.

The automorphisms $\sigma = (b_2, b_i, b_j), 3 \leq i, j \leq n$ fix the t-element set $\{a_1b_1, a_3b_1\}$. Thus, $a_2b_j \in B$ for $3 \leq j \leq n$. Applying the automorphism $\gamma = (b_1, b_3, b_2)$ fixes at least two edges. Thus $a_1b_2 \in B$. Finish as in case (i).

(iv) $a_3b_3 \in B$.

The automorphisms $\sigma = (b_3, b_i, b_j), 4 \leq i, j \leq n$ fix the t-element set $\{a_1b_1, a_2b_2\}$. Thus, $a_3b_j \in B$ for $3 \leq j \leq n$. Applying the automorphism $\gamma = (b_1, b_3, b_2)$ fixes at least two edges. Thus $a_1b_2 \in B$. Finish as in case (i).

Thus, if $m \neq 3$, then $n \leq 4$, and if $m = 3$, $K_{2,n}$ is the unique block whose orbit covers the t-element set $T = \{a_1b_1, a_2b_2\}$ once. So $K_{2,n}$ must be a repeated block in the design.

But the orbit of $K_{2,n}$ covers the t-element set $\{a_1b_1, a_1b_2\}$ twice. Therefore, if $m = 3$, then $n \leq 4$.

□

Corollary 5.1 *Every bigraphical 2-wise balanced design of index 2 is listed in Appendix A2.*

Proof. Lemma 5 establishes that $m \leq n \leq 4$. Thus by Lemma 1, every bigraphical 2-wise balanced design of index 2 is listed in Appendix A2.

□

Lemma 6 *If $t = 3$, then $n \leq 5$.*

Proof. Suppose $n \geq 6$.

Let $m = 3$. Consider the t-element set $T = K_{3,1} = \{a_1b_1, a_2b_1, a_3b_1\}$. Let B be a block that contains T. $B \neq T$, so B contains an additional edge, e. There is, up to symmetry, one type of edge: $e = a_1b_2$. The automorphisms $\sigma = (b_2, b_i, b_j), 3 \leq i, j \leq n$ fix T. Thus, by Corollary 2.1, $K_{1,n}$ on $\{a_1, b_1, \ldots, b_n\} \subseteq B$. Applying 3-cycles on the right forces $K_{m,n} \subseteq B$.

Let $m \neq 3$. Let $T = \{a_1b_1, a_2b_2, a_2b_3\}$, and let B be a block that contains T. Then $B \neq T$, so B contains an additional edge, e. There are, up to symmetry, seven types of edges: (i)a_1b_2, (ii) a_1b_4, (iii) a_2b_1, (iv) a_2b_4, (v) a_3b_1, (vi) a_3b_2 and (vii) a_3b_4.

(i) $a_1b_2 \in B$.

The automorphisms $\sigma = (b_3, b_i, b_j), 4 \leq i, j \leq n$ fix the t-element set $\{a_1b_1, a_1b_2, a_2b_2\}$. So, by Corollary 2.1, $a_2b_j \in B$ for $2 \leq j \leq n$. Applying the automorphism $\gamma = (b_1, b_2, b_3)$ fixes at least three edges. Thus, $K_{1,n}$ on $\{a_2, b_1, \ldots, b_n\} \subseteq B$. Applying 3-cycles on the right then forces $K_{2,n}$ on $\{a_1, a_2, b_1, \ldots, b_n\}$ to be a subset of B. If $m = 2$, then $K_{2,n} = K_{m,n} \subseteq B$. If $m \geq 4$, applying 3-cycles on the left forces $K_{m,n} \subseteq B$, a contradiction.

(ii) $a_1b_4 \in B$.

The automorphisms $\sigma = (b_4, b_i, b_j), 5 \leq i, j \leq n$ fix T. So, $a_1b_j \in B$ for $4 \leq j \leq n$. The automorphism $\gamma = (b_1, b_3, b_2)$ fixes at least three edges. Thus $a_1b_2 \in B$. Finish as in case (i).

(iii) $a_2b_1 \in B$.

Applying the automorphism $\gamma = (b_1, b_3, b_2)$ fixes three edges. Thus $a_1b_2 \in B$. Finish as in case (i).

(iv) $a_2b_4 \in B$.

The automorphisms $\sigma = (b_4, b_i, b_j), 5 \leq i, j \leq n$ fix T. So, $a_2b_j \in B$ for $2 \leq j \leq n$. The automorphism $\gamma = (b_1, b_3, b_2)$ fixes at least three edges. Thus $a_1b_2 \in B$. Finish as in case (i).

(v) $a_3b_1 \in B$.

The automorphisms $\sigma = (b_3, b_i, b_j), 4 \leq i, j \leq n$ fix the t-element set $\{a_1b_1, a_2b_2, a_3b_1\}$. So, $a_2b_j \in B$ for $2 \leq j \leq n$. The automorphism $\gamma = (b_1, b_3, b_2)$ fixes at least three edges. Thus $a_1b_2 \in B$. Finish as in case (i).

(vi) $a_3b_2 \in B$.

The automorphisms $\sigma = (b_3, b_i, b_j), 4 \leq i, j \leq n$ fix the t-element set $\{a_1b_1, a_2b_2, a_3b_2\}$. So, $a_2b_j \in B$ for $2 \leq j \leq n$. The automorphism $\gamma = (b_1, b_3, b_2)$ fixes at least three edges. Thus $a_1b_2 \in B$. Finish as in case (i).

(vii) $a_3b_4 \in B$.

The automorphism $\beta = (b_4, b_5, b_6)$ fixes T. Thus, $K_{1,3}$ on $\{a_3, b_4, b_5, b_6\} \subseteq B$. Applying the automorphism $\gamma = (b_1, b_3, b_2)$ fixes three edges, and so $a_1b_2 \in B$. Finish as in case (i).

□

Corollary 6.1 *Every bigraphical 3-wise balanced design of index 2 is listed in Appendix A3.*

Proof. Lemma 6 establishes that $m \leq n \leq 5$. Thus by Lemma 1, every bigraphical 3-wise balanced design of index 2 is listed in Appendix A3.

□

Lemma 7 *If $m \neq 3$, then $n \neq t$. If $m = 3$ and $n = t$, then $K_{2,n}$ is the unique block whose orbit covers the t-element set $K_{1,t}$ twice.*

Proof. Suppose $n = t$. Consider the t-element set $T = K_{1,t} = K_{1,n}$ on $\{a_1, b_1, \ldots, b_n\}$. Let B be a block that contains T. $B \neq T$, so B contains an additional edge, e. There is, up to symmetry, one type of edge: $e = a_2b_1$. Applying 3-cycles on the right forces $K_{2,n}$ on $\{a_1, a_2, b_1, \ldots, b_n\} \subseteq B$. Therefore, $m \geq 3$. If $m = 3$, then $B = K_{2,n}$ is a block whose orbit covers the t-element set $T = K_{1,t}$ twice. Adding an additional edge to $K_{2,n}$ will force B to be the complete bipartite graph. Thus, $K_{2,n} = B$ is the unique block that covers T. If $m \geq 4$, applying 3-cycles on the left forces $K_{m,n} \subseteq B$.

□

Lemma 8 *If $n \geq t+1$ and $m \neq 3$, then $t < 4$. If $n \geq t+1$ and $m = 3$, then $K_{2,n}$ is the only block covering a t-element set isomorphic to $T = \{a_1b_j : 1 \leq j \leq t-1\} \cup \{a_2b_{t-1}\}$.*

Proof. We have $m \geq 2$ by Lemma 3. Assume $n \geq t+1$. Suppose $t \geq 4$ and let B be a block containing the t-element set

$$T = \{a_1b_j : 1 \leq j \leq t-1\} \cup \{a_2b_{t-1}\}.$$

Then $B \neq T$, so B contains an additional edge, e. There are, up to symmetry, six types of edges: (i)a_1b_t, (ii) a_2b_1, (iii) a_2b_t, (iv) a_3b_1, (v) a_3b_{t-1} and (vi) a_3b_t.

(i) $a_1 b_t \in B$.

The automorphisms $\sigma = (b_{t-1}, b_i, b_j)$,with $1 \le i, j \le t, i, j \ne t-1$ fix the t-element set $K_{1,t}$ on $\{a_1, b_1, \ldots, b_t\}$. Thus $K_{2,t}$ on $\{a_1, a_2, b_1, \ldots, b_t\} \subseteq B$. Now apply the automorphisms $\gamma = (b_{t-1}, b_t, b_j), t+1 \le j \le n$. γ fixes $2(t-2)$ edges. Since $t \ge 4$, $2(t-2) \ge t$, so by Corollary 2.1, $K_{2,n}$ on $\{a_1, a_2, b_1, \ldots, b_n\} \subseteq B$. If $m = 2$, then $K_{2,n} = K_{m,n} \subseteq B$. If $m = 3$, then $B = K_{2,n}$ is a block whose orbit covers the t-element set T once. If $m \ge 4$, applying 3-cycles on the left forces $K_{m,n} \subseteq B$.

(ii) $a_2 b_1 \in B$.

The automorphism $\beta = (b_2, b_{t+1}, b_t)$ fixes $(t-1)+1 = t$ edges. Thus $a_1 b_t \in B$. Finish as in case (i).

(iii) $a_2 b_t \in B$.

Applying the automorphism $\sigma = (b_{t-1}, b_2, b_1)$ fixes $(t-1)+1 = t$ edges. Thus $a_2 b_1 \in B$. Finish as in case (ii).

(iv) $a_3 b_1 \in B$.

The automorphism $\beta = (b_2, b_{t+1}, b_t)$ fixes $(t-1)+1 = t$ edges. Thus $a_1 b_t \in B$. Finish as in case (i).

(v) $a_3 b_{t-1} \in B$.

The automorphism $\gamma = (b_1, b_{t+1}, b_t)$ fixes $(t-1)+1 = t$ edges. Thus $a_1 b_t \in B$. Finish as in case (i).

(vi) $a_3 b_t \in B$.

Applying the automorphism $\sigma = (b_{t-1}, b_2, b_1)$ fixes $(t-1)+1 = t$ edges. Thus $a_2 b_1 \in B$. Finish as in case (ii).

□

Corollary 8.1 *If $n \geq t$ and $m \neq 3$, then $t < 4$.*

Proof. $n \neq t$ by Lemma 7, and, by Lemma 8, if $n \geq t+1$ and $m \neq 3$, then $t < 4$. $\square$

Corollary 8.2 *If $m = 3$ and $n \geq t+1$, then $t < 4$.*

Proof. Assume $m = 3$ and $n \geq t+1$. By Lemma 8, if $t \geq 4$, then the orbit of the block $K_{2,n}$ covers the t-element set $T = \{a_1 b_j : 1 \leq j \leq t-1\} \cup \{a_2 b_{t-1}\}$ once and no other blocks cover this t-element set. Thus, any design with $m = 3$ and $n \geq t+1$ must have $K_{2,n}$ as a repeated block. But the orbit of $K_{2,n}$ covers the t-element set $K_{1,t}$ twice, so it cannot be repeated. $\square$

Definition: *Derived design.*

The *derived design* of a t–$(v, \mathcal{K}, \lambda)$ design $(X, \mathcal{B})$ with respect to $S \subseteq X$ is $(X', \mathcal{B}')$ where $X' = X \setminus S$ and $\mathcal{B}' = \{B \setminus S : S \subseteq B \in \mathcal{B}\}$. If $|S| = j \leq t$, it is a $(t-j)$–$(v-j, \mathcal{K}', \lambda)$ design; where $\mathcal{K}' = \{k - j : k \in \mathcal{K}\}$.

Lemma 9 *If $m = 3$ and $n = t$, then $t \leq 3$.*

Proof. Assume $m = 3$ and $n = t$. Suppose $t \geq 8$. Consider the t-element set $T = K_{1,q} \dot{\cup} K_{1,n-q} = \{a_1 b_j : 1 \leq j \leq q\} \cup \{a_2 b_j : q+1 \leq j \leq n\}$, where $q = \lceil t/2 \rceil$. We will show that the only block that covers T is $K_{2,n} = \{a_i b_j : 1 \leq i \leq 2, 1 \leq j \leq n\}$. Let B be a block that contains T. $B \neq T$, so B contains an additional edge, e. There are, up to symmetry, four types of edges: (i)$a_1 b_t$, (ii) $a_2 b_1$, (iii) $a_3 b_1$ and (iv) $a_3 b_t$.

(i) $a_1 b_t \in B$.

The automorphisms $\sigma = (b_t, b_i, b_j), q+1 \leq i, j \leq t-1$ fix T. Thus, by Corollary 2.1, $K_{1,n}$ on $\{a_1, b_1, \ldots, b_n\} \subseteq B$. Applying 3-cycles on the right forces $K_{2,n} = B$.

(ii) $a_2 b_1 \in B$.

The automorphism (b_1, b_{t-1}, b_t) fixes the t-element set $a_2 b_1 \cup \{a_1 b_j : 2 \leq j \leq q\} \cup \{a_2 b_j : q+1 \leq j \leq n\}$. Hence, $a_1 b_t \in B$. Finish as in case (i).

(iii) $a_3b_1 \in B$.

The automorphisms $\gamma = (b_1, b_i, b_j), 2 \leq i, j \leq q$ fix T. Thus, by Corollary 2.1, $K_{2,q} = \{a_ib_j : i \in \{1,3\}, 1 \leq j \leq q\} \subseteq B$. Now applying the automorphism $\beta = (b_q, b_{t-1}, b_t)$ fixes $2(q-1)+(n-q)-2+1 = n+q-3 = t+q-3$ edges. Since $t \geq 8$, $q = \lceil t/2 \rceil \geq 4$. Therefore, β fixes a t-element set. So, by Corollary 2.1, $a_1b_t \in B$. Finish as in case (i).

(iv) $a_3b_t \in B$.

The automorphisms $\gamma = (b_t, b_i, b_j), q+1 \leq i, j \leq t-1$ fix T. Thus, by Corollary 2.1, $K_{2,t-q} = \{a_ib_j : i \in \{2,3\}, q+1 \leq j \leq t\} \subseteq B$. Now applying the automorphism $\beta = (b_t, b_2, b_1)$ fixes $2(t-q-1)+(q-2)+1 = 2t-q-3 = t+t-q-3$ edges. Since $t \geq 8$, $t-q = t - \lceil t/2 \rceil \geq 4$. Therefore, β fixes a t-element set, and $a_1b_t \in B$. Finish as in case (i).

Thus, the orbit of the block $K_{2,n}$ covers T once and no other blocks cover this t-element set, so $K_{2,n}$ must be a repeated block. But the orbit of $K_{2,n}$ covers the t-element set $K_{1,t}$ twice, contradicting $\lambda = 2$.

Suppose a bigraphical 7–$(3 \cdot 7, \mathcal{K}, 2)$ design exists. Then the derived design with respect to $K_{3,1}$, which is a bigraphical 4–$(3 \cdot 6, \mathcal{K}', 2)$ design, would also exist. This contradicts Corollary 8.2.

Suppose $n = t = 6$ and consider the t-element set $T = K_{3,2} = \{a_ib_j : i \in \{1,2,3\}, j \in \{1,2\}\}$. Let B be a block that contains T. $B \neq T$, so B contains an additional edge, e. There is, up to symmetry, one type of edge: $e = a_1b_3$. Applying the automorphism $\sigma = (b_3, b_i, b_j), 4 \leq i, j \leq 6$ fixes T. Thus, by Corollary 2.1, $K_{1,n} \subseteq B$. Applying 3-cycles on the right forces $K_{m,n} \subseteq B$.

Finally, by Lemma 1 we see that there are no designs with $m = 3$ and $n = t = 4$ or 5

and thus the result follows. $\square$

2.2 The Main Theorem

Main Theorem: *The only bigraphical t–wise balanced designs of index 2 are those listed in the Appendix.*

Proof. If not, choose a nontrivial unlisted bigraphical t–$(m \cdot n, \mathcal{K}, 2)$ design $(X, \mathcal{B})$ with (m, n) as small as possible. We may assume from Lemmas 1 through 9 that:

$$t \geq 4, \qquad 3 \leq m \leq n, \qquad t \geq n+1$$

Thus the derived design with respect to a $K_{1,n}$ is a bigraphical $(t-n)$–$((m-1) \cdot n, \mathcal{K}', 2)$ design $(X', \mathcal{B}')$ and must by the choice of m and n appear in the Appendix.

Suppose $t \geq n+2$. Then $t - n \geq 2$ and thus $(X', \mathcal{B}')$ appears in Appendix A2, A3, or A5. Hence $n \leq 5$. Thus by Lemma 1 no such design can exist.

If $t = n+1$, then the derived design with respect to $K_{1,n}$ is a design in Appendix A1. There are several cases to consider.

Case A $n \geq 5$.

There are then several subcases:

Case A.1 *The derived design with respect to $K_{1,n}$ is D_1 from Appendix A1.*

Then the graphs $B_1 = K_{2,n}$ and B_2 in Figure 1(a) are nonrepeated blocks. The orbits of these blocks cover the t-element set T_1 in Figure 1(b) twice.

Suppose $m \geq 5$ and consider the t-element set

$$T = K_{1,n-1} \dot{\cup} K_{2,1} = \{a_1 b_j : 1 \leq j \leq n-1\} \cup \{a_2 b_n, a_3 b_n\}.$$

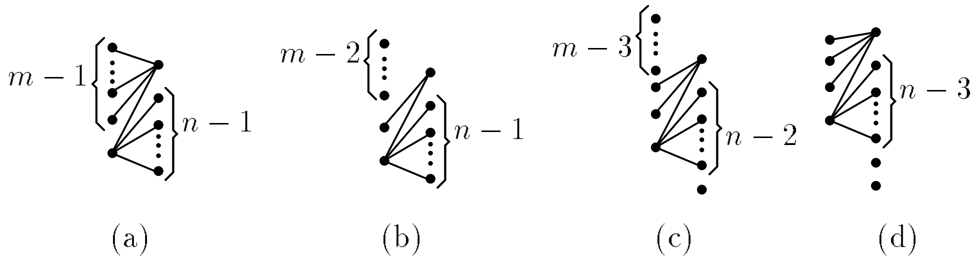


Figure 1.

Note that T is covered once by the orbit of B_2. Let B be a block that contains T. Since $B \neq T$, B contains an additional edge, e. There are, up to symmetry, four types of edges: (i)a_1b_n, (ii) a_2b_1, (iii) a_4b_1 and (iv) a_4b_n.

(i) $a_1b_n \in B$.

Then the orbit of B covers T_1, contradicting $\lambda = 2$.

(ii) $a_2b_1 \in B$.

Applying 3-cycles on the right that fix T forces (in particular)

$$K_{1,n}\dot{\cup}e = \{a_2b_j : 1 \leq j \leq n\} \cup \{a_3b_n\} \subseteq B.$$

Thus, the orbit of B covers T_1, contradicting $\lambda = 2$.

(iii) $a_4b_1 \in B$.

Switching a_4 with isolated vertices on the left covers T more than once. This forces all of the isolated vertices on the left to be incident to b_1, for otherwise the orbit of B would cover T too many times. Applying 3-cycles on the right that fix T forces $K_{m-2,n-1}\dot{\cup}K_{2,1} \subseteq B$. The automorphism (a_1, a_2, a_3) fixes $(m-3)(n-1)+1 \geq 2(n-1)+1 = 2(t-2)+1 \geq t$ edges, since $t \geq 4$ and $m \geq 5$. In particular, this forces $K_{1,n} \cup \{e\} = \{a_3b_j : 1 \leq j \leq n\} \cup \{a_1b_1\} \subseteq B$. Thus, the orbit of B covers T_1, contradicting $\lambda = 2$.

(iv) $a_4 b_n \in B$.

Switching a_4 with isolated vertices on the left covers T more than once, contradicting $\lambda = 2$. This forces all of the isolated vertices on the left to be incident to b_n. This forces $B_3 = K_{1,n-1} \dot{\cup} K_{m-1,1}$ to be a block in the design.

If B is any block not in the same orbit as B_2 and B_3, it cannot cover any t–element set already covered twice by these blocks, so if x and y are vertices on the left and right respectively their degrees in B must satisfy

$$\begin{aligned} \deg(x) + \deg(y) &\leq t - 1 \text{ if } xy \notin B; \\ \deg(x) + \deg(y) &\leq t + 1 \text{ if } xy \in B. \end{aligned}$$

Let B be a block containing the t–element set T in Figure 1(c). T has already been covered once by the orbit of B_2 and therefore must be covered exactly once more. Since $B \neq T$, B contains an additional edge e. Because of the degree restrictions, e cannot be incident to an isolated vertex on the left, a_m or b_1. Thus, up to symmetry, there are two types of edges: (i)$a_{m-1}b_2$ and (ii) $a_{m-1}b_n$.

(i) $a_{m-1}b_2 \in B$.

Applying 3-cycles on the right that fix T forces $K_{2,n-1} \cup \{e\}$ to be contained in B. The automorphisms (a_{m-2}, a_i, a_j) for $1 \leq i, j \leq m-3$ fix $2(n-1) = 2(t-2) \geq t$ edges. Thus, $a_m b_1 \in B$, $\deg(a_m) \geq n - 1$ and $\deg(b_1) = m$. Since $(n-1) + m \geq (t-2) + 5 = t + 3$, this contradicts the degree restrictions.

(ii) $a_{m-1}b_n \in B$.

The automorphisms (a_{m-2}, a_i, a_j) for $1 \leq i, j \leq m - 3$ fix a t-element set. Thus, $K_{1,n-2} \dot{\cup} K_{m-1,1}$ is contained in B, which contradicts the degree restrictions.

Suppose $m = 4$ and consider the t-element set T in Figure 1(d). T has already been covered once by the orbit of B_2 and therefore must be covered exactly once more. Since $B \neq T$, B contains an additional edge e. Thus, up to symmetry, there are three types of edges: (i)a_1b_2, (ii)a_1b_n and (iii) a_4b_n.

(i) $a_1b_2 \in B$.

The automorphisms (a_1, a_i, a_j) for $i, j \in \{2, 3\}$ fix T. Thus $a_2b_2, a_3b_2 \in B$. Switches on the right that fix T force $K_{4,n-2} \subseteq B$. Switching the vertex b_{n-2} with the isolated vertices on the right forces $B = K_{m,n}$.

(ii) $a_1b_n \in B$.

The automorphisms (a_1, a_i, a_j) for $i, j \in \{2, 3\}$ fix T. Thus $a_2b_n, a_3b_n \in B$. Switching b_n and b_{n-1} forces $K_{3,3} \cup K_{1,n-2} \subseteq B$, for otherwise the orbit of B covers T too many times. The automorphisms (b_1, b_i, b_j) for $2 \leq i, j \leq n-2$ fix $3\cdot2+(n-2) = 6+(t-3) \geq t$ edges. Thus $B \supseteq K_{m-1,n} \cup K_{1,n-2}$. The automorphism (b_{n-2}, b_{n-1}, b_n) fixes $n(m-1) = (t-1)3 \geq t$ edges and thus $B = K_{m,n}$.

(iii) $a_4b_n \in B$.

Switches on the right with the isolated point forces the block to be B_2, but B is a nonrepeated block and so T can only be covered once.

Suppose $m = 3$ and consider the t-element set

$$T = K_{2,1}\dot{\cup}K_{1,n-1} = \{a_1b_1, a_2b_1\} \cup \{a_3b_j : 2 \leq j \leq n\}.$$

Note that T has already been covered once by the orbit of B_2, which is forced to be a nonrepeated block. Let B be a block that contains T. Since $B \neq T$, B contains an additional edge, e. There are, up to symmetry, two types of edges: (i)a_1b_2 and (ii) a_3b_1.

(i) $a_1b_2 \in B$.

3-cycles on the right that fix T force B to cover the t-element set $K_{1,n}\dot{\cup}\{e\}$, which has already been covered by the orbits of B_1 and B_2.

(ii) $a_3b_1 \in B$.

This is B_2. This is the only block whose orbit covers T, so it must be repeated. This contradicts the derived design being D_1.

Case A.2 *The derived design with respect to $K_{1,n}$ is D_2 from Appendix A1.*

Then the graph in Figure 1(a) is a repeated block. Call it B_1. If B is any block not in the same orbit as B_1 it cannot cover any t–element set already covered by the orbit of B_1, so if x and y are vertices on the left and right respectively their degrees in B must satisfy

$$\begin{aligned} \deg(x) + \deg(y) &\leq t - 1 \text{ if } xy \notin B; \\ \deg(x) + \deg(y) &\leq t \text{ if } xy \in B. \end{aligned}$$

Let B be a block containing the t–element set T in Figure 2(a). Since $B \neq T$, B contains an additional edge e. Because of the degree restrictions and since $n \geq 5$, e cannot be incident to a_{m-1}, a_m, b_{n-1} or b_n. Thus, up to symmetry, $e = a_1b_1$. Applying 3-cycles on the right that fix T forces $K_{2,n-2}$ to be contained in B. The automorphism (b_1, b_2, b_{n-1}) fixes $2(n-4)+3+1 = 2(t-5)+4 = 2t-6 \geq t$ edges. Thus, there is an edge incident to b_{n-1}, which is a contradiction.

Case A.3 *The derived design with respect to $K_{1,n}$ is D_3 from Appendix A1.*

Then the graph $B_1 = K_{2,n}$ is a repeated block. If B is any block not in the same orbit as B_1, it cannot cover any t–element set already covered, so if x and x' are vertices on the

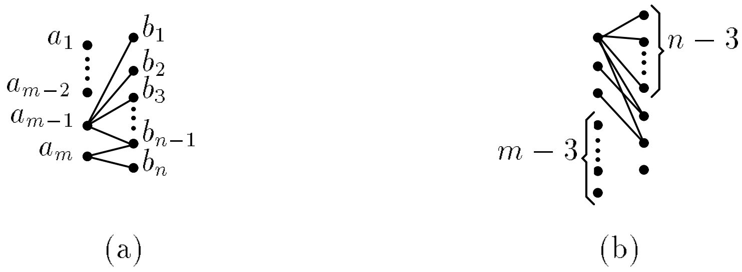


Figure 2.

left, their degrees in B must satisfy

$$\deg(x) + \deg(x') \quad \leq \quad t-1.$$

Suppose $n \geq 6$, and let B be a block containing the t–element set T in Figure 2(b). Since $B \neq T$, B contains an additional edge e. Observe that the above degree restrictions imply that the degree of a_i is at most 1, for all $i \neq 1$, since $\deg(a_1) = t-2$. Thus, e cannot be incident to a_1, a_2 or a_3. So, without loss, e must be incident to a_4. But then applying 3-cycles on the right that fix a t-element set forces $\deg(a_3) > 1$. Consequently, $n = 5$ and $t = 6$.

If $m = 5$, then the derived design with respect to a $K_{5,1}$ is a bigraphical $1-(5\cdot 4, \mathcal{K}', 2)$ design. This design cannot be extended, as can be shown by the simple backtracking program used in Lemma 1.

If $m = 4$, then the derived design with respect to a $K_{4,1}$ is a bigraphical $2-(4\cdot 4, \mathcal{K}', 2)$ design, which cannot be extended as can be shown by the simple backtracking program used in Lemma 1.

If $m = 3$, then the derived design with respect to a $K_{3,1}$ is a bigraphical $3-(3\cdot 4, \mathcal{K}', 2)$ design. This design is not listed in Appendix A3.

If $m = 2$, then the derived design with respect to a $K_{2,1}$ is a bigraphical $4-(2\cdot 4, \mathcal{K}', 2)$ design. Since $n = t$, this contradicts Lemma 7.

Case A.4 *The derived design with respect to* $K_{1,n}$ *is* D_4 *from Appendix A1.*

Then the graph $B_1 = K_{3,n}$ is a nonrepeated block. The orbit of B_1 covers twice every t-element set whose degree sequence on the left is (r, t-r, 0, 0). Thus if B is any block not in the same orbit as B_1, it cannot further cover any t–element set with left degree sequence (r, t-r, 0, 0). So if x and x' are vertices on the left, their degrees in B must satisfy

$$\deg(x) + \deg(x') \quad \leq \quad t - 1.$$

Let B be a block containing the t–element set T in Figure 2(b). Note that T is already covered once by the orbit of B_1. Since $B \neq T$, B contains an additional edge e. Because of the degree restrictions, e cannot be incident to a_1, a_2 or a_3. Thus e must be incident to a_4, and furthermore the degree of a_4 must be 1. There are, up to symmetry, three types of edges: (i)a_4b_1, (ii) a_4b_{n-1} and (iii) a_4b_n.

(i) $a_4b_1 \in B$.

Since T is already covered once, switching the vertices b_1 and b_2 forces the edge $a_4b_2 \in B$. This contradicts the above degree restriction.

(ii) $a_4b_{n-1} \in B$.

The automorphism (b_{n-2}, b_2, b_1) fixes a t-element set. Thus, a_2 has degree 2. This contradicts the degree restriction.

(iii) $a_4b_n \in B$.

The automorphism (b_{n-2}, b_2, b_1) fixes a t-element set. Thus, a_2 has degree 2. This contradicts the degree restriction.

Case A.5 *The derived design with respect to* $K_{1,n}$ *is* D_5 *from Appendix A1.*

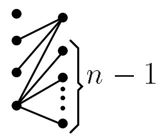


Figure 3.

In this case the graph B_1 in Figure 3 is a nonrepeated block and $m = 4$. Note that the orbit of B_1 covers the t-element set T_1 in Figure 1(c) once and T_2 in Figure 1(b) twice. Let B be a block containing the t–element set T in Figure 2(a). Since $B \neq T$, B contains an additional edge e. e cannot be incident to a_3, because then the orbit of B would cover T_2 again. If e were incident to a_4, then 3-cycles that fix T would force the orbit of B to cover T_2 again. Thus, without loss, e must be incident to a_1. There are, up to symmetry, three types of edges: (i)a_1b_1, (ii) a_1b_{n-1} and (iii) a_1b_n.

(i) $a_1b_1 \in B$.

The automorphisms (b_1, b_r, b_s) for $2 \leq r, s \leq n-2$ fix T. Thus $\{a_1b_s : 1 \leq s \leq n-2\}$ is contained in B. The automorphisms (b_{n-1}, b_i, b_j) for $1 \leq i, j \leq n-2$ are now seen to fix a t-element subset of B forcing $a_4b_i \in B$ for all $1 \leq i \leq n$. Thus B covers T_2, contradicting $\lambda = 2$.

(ii) $a_1b_{n-1} \in B$.

Then the orbit of B covers T_1 twice, unless $a_1b_n \in B$. But then the automorphism (b_{n-1}, b_2, b_1) fixes a t-element subset of B, forcing $a_1b_1 \in B$, which was shown in part (i) to lead to a contradiction.

(iii) $a_1b_n \in B$.

The automorphism (b_{n-1}, b_2, b_1) fixes a t-element subset of B, which forces $a_4b_1 \in B$, again leading to a contradiction.

Case B $n = 4$.

There are 3 cases.

Case B.1 *The derived design with respect to* $K_{1,4}$ *is a bigraphical* 1–$(2 \cdot 4, \mathcal{K}', 2)$ *design from Appendix A1.*

In this case the design $(X, \mathcal{B})$ is a 5–$(3 \cdot 4, \mathcal{K}, 2)$ design. The derived design with respect to $K_{1,4}$ must be D_1, D_2 or D_3 from Appendix A1, with parameters $m = 2$ and $n = 4$. The derived design with respect to $K_{3,1}$ must be one of the designs $D_{13}, D_{14}, \ldots, D_{19}$ from Appendix A2. There are thus 21 cases to consider. In all but three cases it is easy to show that $(X, \mathcal{B})$ cannot exist by an argument similar to the following example.

Example case: The derived design with respect to $K_{1,4}$ is D_3 from Appendix A1 and the derived design with respect to $K_{3,1}$ is D_{18} from Appendix A2. The graph B_1 in Figure 1(a) is then forced to be a repeated block in $(X, \mathcal{B})$. Note that the orbit of B_1 covers the t-element set T in Figure 1(b) once. So, since B_1 is a repeated block, T is covered twice. Another repeated block forced to be in $(X, \mathcal{B})$ is $B_2 = K_{2,4}$. The orbit of B_2 also covers T, which contradicts $\lambda = 2$.

The remaining three cases are handled as follows:

(i) The derived design with respect to $K_{1,4}$ is D_1 from Appendix A1. The derived design with respect to $K_{3,1}$ is D_{15} from Appendix A2.

Then $m = 3, n = 4$ and, in particular, the graphs B_1 in Figure 1(a), $B_2 = K_{2,4}$ and $B_3 = K_{3,3}$ are nonrepeated blocks. Let B be the block containing the t-element set $T = K_{1,3} \cup K_{1,2} = \{a_1 b_i : 1 \le i \le 3\} \cup \{a_2 b_j : j \in \{3, 4\}\}$. Note that T has already been covered once by B_2. Since $B \ne T$, B contains an additional edge e. e cannot be incident to a_1 or a_2, because then B would cover a t-element set already covered twice by the orbits of B_1, B_2 or B_3. There are, up to symmetry, three types of edges: (a)$a_3 b_1$, (b) $a_3 b_3$ and (c) $a_3 b_4$.

(a) $a_3b_1 \in B$.

Since T has already been covered once, switching b_1 and b_2 forces B to contain a t-element set already covered by B_2 and B_3.

(b) $a_3b_3 \in B$.

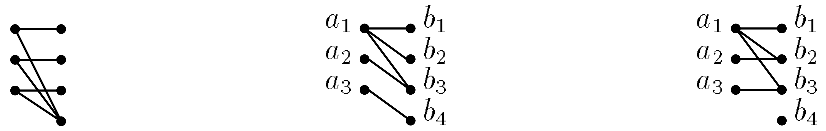
Then B covers a t-element set already covered by B_1 and B_3.

(c) $a_3b_4 \in B$.

The automorphisms (b_1, b_i, b_j) for $i, j \in \{2, 3\}$ fix a t-element set. This forces B to contain a t-element set already covered by B_2 and B_3.

(ii) The derived design with respect to $K_{1,4}$ is D_2 from Appendix A1 and the derived design with respect to $K_{3,1}$ is D_{18} from Appendix A2.

Then $m = 3, n = 4$ and the graphs B_1 in Figure 1(a), $B_2 = K_{3,2}$ and B_3 in Figure 4(a) are all repeated blocks. If B is any block not in the same orbit as B_1, B_2 or B_3, it cannot cover any t-element set covered by the orbits of B_1, B_2 or B_3, since these t-element sets have already been covered twice.



(a) (b) (c)

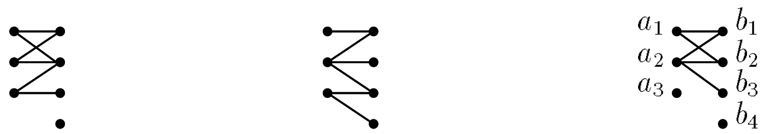
Figure 4.

Let B be a block containing the t-element set T in Figure 4(b). Since $B \neq T$, B contains an additional edge e. e cannot be incident to a_1 or a_3, because then B would cover a t-element set already covered twice by the orbits of B_1 or B_3. Thus e must be incident to a_2. There are, up to symmetry, two types of edges: a_2b_4 or a_2b_1.

If $a_2b_4 \in B$, then the automorphisms $\sigma = (b_3, b_2, b_1)$ and σ^2 fix a t-element set. This forces the t-element set in Figure 1(b) to be covered by B. But this t-element set has already been covered twice by the orbit of the repeated block B_1. Thus a_2b_4 is not contained in B and hence $B_4 = T \cup \{a_2b_1\}$ is a block. Note that the orbit of B_4 covers T twice.

Next, consider the t-element set T_1 in Figure 4(c). Let B be a block containing this t-element set. Since $B \neq T_1$, B contains an additional edge e. e cannot be incident to a_1, b_2, b_3 or b_4 because then B would cover a t-element set already covered twice by the orbits of B_1, B_2 or B_3. Thus, without loss, $e = a_2b_1 \in B$. Note that a_2b_1 and a_3b_1 are not both in B, because then B would cover a t-element set already covered twice by B_2. The t-element set $T_2 = K_{2,2}\dot{\cup}\{e'\} = \{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}\dot{\cup}\{a_3b_3\}$ is covered at least twice by the orbit of B, but T_2 has already been covered once by B_4.

(iii) The derived design with respect to $K_{1,4}$ is D_2 from Appendix A1. and the derived design with respect to $K_{3,1}$ is D_{19} from Appendix A2.



(a) (b) (c)

Figure 5.

Then $m = 3, n = 4$ and the graph B_1 in Figure 1(a) is a repeated block and B_2 in Figure 5(a) is a nonrepeated block. Let B be a block containing the t-element set T in Figure 4(c). Since $B \neq T$, B contains an additional edge e. e cannot be incident to a_1, b_1, b_2 or b_3 because then B would cover a t-element set already covered twice by the orbit of B_1 or B_2. Thus $e = a_2b_4$ or $e = a_3b_4$ is an edge of B. Both cannot be edges, for otherwise a t-element set covered twice by B_1 will again be covered by B. This forces the graph in Figure 5(b) to be a nonrepeated block, B_3, because its orbit covers T exactly twice.

Next, consider the t-element set T_1 in Figure 5(c). Let B be a block containing this t-element set. Since $B \neq T$, B contains an additional edge e. e cannot be incident to a_1, a_3, b_1, b_2 or b_4 because then B would cover a t-element set already covered twice by the orbits of B_1, B_2 or B_3. Thus $e = a_1 b_3$, and $B = B_4 = K_{2,3}$ is a repeated block. Thus, the orbits of the blocks B_1, B_2, B_3 and B_4 form the 5–$(12, 6, 2)$ design D_{52} listed in Appendix A5.

Case B.2 *The derived design with respect to* $K_{1,4}$ *is a bigraphical* 1–$(3 \cdot 4, \mathcal{K}', 2)$ *design from Appendix A1.*

In this case the design $(X, \mathcal{B})$ is a 5–$(4 \cdot 4, \mathcal{K}, 2)$ design. The derived design with respect to $K_{1,4}$ must be $D_1, D_2, \ldots, D_5$ from Appendix A1, with parameters $m = 3$ and $n = 4$. The derived design with respect to $K_{4,1}$ must be the reverse of $D_1, D_2, \ldots, D_5$ from Appendix A1, with parameters $m = 4$ and $n = 3$. Due to symmetry, there are 15 cases to consider. In all but six cases it is easy to show that $(X, \mathcal{B})$ cannot exist.

The remaining six cases are handled as follows:

(i) The derived design with respect to $K_{1,4}$ is D_1 from Appendix A1 and the derived design with respect to $K_{4,1}$ is the reverse of D_1 from Appendix A1. The simple backtrack search used in Lemma 1 shows that this case is impossible.

(ii) The derived design with respect to $K_{1,4}$ is D_2 from Appendix A1 and the derived design with respect to $K_{4,1}$ is the reverse of D_2 from Appendix A1. The simple backtrack search used in Lemma 1 shows that this case is impossible.

(iii) The derived design with respect to $K_{1,4}$ is D_3 from Appendix A1 and the derived design with respect to $K_{4,1}$ is the reverse of D_3 from Appendix A1. The backtrack search used in Lemma 1 produces the design D_{53} in Appendix A5.

(iv) The derived design with respect to $K_{1,4}$ is D_5 from Appendix A1 and the derived design with respect to $K_{4,1}$ is the reverse of D_3 from Appendix A1. Then $m = 4, n = 4$ and the graph $B_1 = K_{1,4} \cup K_{2,1}$ is a nonrepeated block and $B_2 = K_{4,2}$ is a repeated block. Let B be a block containing the t-element set $T = K_{3,1} \dot{\cup} K_{1,2} = \{a_i b_1 : 1 \leq i \leq 3\} \cup \{a_4 b_j : j \in \{2, 3\}\}$. Since $B \neq T$, B contains an additional edge e. e cannot be incident to b_1, b_2 or b_3, because then B would cover a t-element set already covered twice by the orbit of B_1 or B_2. Thus e must be incident to b_4. There are, up to symmetry, two types of edges: (a)$a_1 b_4$ or (b) $a_4 b_4$.

(a) $a_1 b_4 \in B$.

The automorphism $\sigma = (a_1, a_2, a_3)$ forces B to cover a t-element set already covered twice by the orbit of B_2.

(b) $a_4 b_4 \in B$.

Then $B = K_{3,1} \dot{\cup} K_{1,3}$. Since this is the only block that covers T, it must be a repeated block. Then the t-element set $T_1 = K_{2,1} \dot{\cup} K_{1,3}$ would be covered twice more. But T_1 has already been covered once by the orbit of B_1.

(v) The derived design with respect to $K_{1,4}$ is D_5 from Appendix A1 and the derived design with respect to $K_{4,1}$ is the reverse of D_4 from Appendix A1.

Then $m = 4, n = 4$ and the graphs $B_1 = K_{1,4} \cup K_{2,1}$ and $B_2 = K_{4,3}$ are nonrepeated blocks. The orbit of B_2 covers twice every t-element set whose degree sequence on the right is (r, 5-r, 0, 0). Thus if B is any block not in the same orbit as B_2, it cannot further cover any t–element set with right degree sequence (r, 5-r, 0, 0). Let B be a block containing the t–element set T in Figure 6.

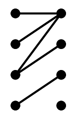


Figure 6.

Note that T is already covered once by the orbit of B_2. Since $B \neq T$, B contains an additional edge, e. e cannot be incident to b_1, b_2 or b_3, because then B would cover a t-element set having right degree sequence (r, 5-r, 0, 0). Thus e must be incident to b_4. There are, up to symmetry, three types of edges: (a)a_1b_4, (b)a_3b_4 or (c) a_4b_4.

(a) $a_1b_4 \in B$.

Switching a_1 and a_2 forces the orbit of B to cover a t-element set with right degree sequence (r, 5-r, 0, 0).

(b) $a_3b_4 \in B$.

This covers a t-element set that has been covered by the orbits of B_1 and B_2.

(c) $a_4b_4 \in B$.

The automorphism $\sigma = (a_3, a_2, a_1)$ forces B to cover a t-element set with right degree sequence (r, 5-r, 0, 0).

(vi) The derived design with respect to $K_{1,4}$ is D_5 from Appendix A1 and the derived design with respect to $K_{4,1}$ is the reverse of D_5 from Appendix A1. The simple backtracking search used in Lemma 1 shows that this case is impossible.

Case B.3 *The derived design with respect to* $K_{1,4}$ *is a bigraphical* 1–$(4 \cdot 4, \mathcal{K}, 2)$ *design from Appendix A1.*

Then $t = 5$, $m = 5$ and $n = 4$. The reverse of this contradicts Lemma 7.

Case C $n = 3$.

There are 2 cases.

Case C.1 *The derived design with respect to* $K_{1,3}$ *is a bigraphical* 1–$(2 \cdot 3, \mathcal{K}', 2)$ *design from Appendix A1.*

In this case the design $(X, \mathcal{B})$ is a 4–$(3 \cdot 3, \mathcal{K}, 2)$ design. The derived design with respect to $K_{1,3}$ must be D_1, D_2, D_3, D_6, D_7 or D_{11} from Appendix A1, with parameters $m = 2$ and $n = 3$. The derived design with respect to $K_{3,1}$ must be the reverse of D_1, D_2, D_3, D_6, D_7 or D_{11} from Appendix A1, with parameters $m = 3$ and $n = 2$. Due to symmetry, there are 21 cases to consider. In all but two cases it is easy to show that $(X, \mathcal{B})$ cannot exist.

The remaining two cases are handled as follows:

(i) The derived design with respect to $K_{1,3}$ is D_2 from Appendix A1 and the derived design with respect to $K_{3,1}$ is the reverse of D_1 from Appendix A1.

Then $m = n = 3$, and the graph B_1 in Figure 1(a) is a repeated block. Any $t-$element set already covered by the orbit of B_1 cannot be covered by any other block. Let B be a block containing the t-element set $T = \{a_1b_1, a_1b_2, a_2b_2, a_2b_3\}$. Since $B \neq T$, B contains an additional edge. But any edge added will cover a t-element set already covered by B_1.

(ii) The derived design with respect to $K_{1,3}$ is D_{11} from Appendix A1 and the derived design with respect to $K_{3,1}$ is the reverse of D_{11} from Appendix A1.

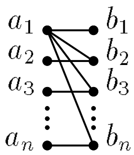


Figure 7.

In this case $m = n = 3$, and the graph B_1 in Figure 7 is a forced block. Note that the orbit of B_1 covers the t-element set $T_1 = \{a_1b_1, a_2b_1, a_2b_3, a_3b_3\}$ once and T_2 in Figure 1(b) twice.

Let B be a block containing $T = K_{1,2} \dot{\cup} K_{2,1} = \{a_1b_1, a_1b_2, a_2b_3, a_3b_3\}$. Since $B \neq T$, B contains an additional edge e. There are, up to symmetry, two types of edges: (a)a_1b_3 and (b) a_2b_1.

(a) $a_1b_3 \in B$.

This covers T_2, which has already been covered twice by the orbit of B_1.

(b) $a_2b_1 \in B$.

This covers T_1 twice more, which forces B to be the cycle of length 6. This is the only block that covers T, so it must be repeated. But then its orbit covers T_1 too many times.

Case C.2 *The derived design with respect to* $K_{1,3}$ *is a bigraphical* $1\text{–}(3 \cdot 3, \mathcal{K}', 2)$ *design from Appendix A1.*

Then $t = 4$, $m = 4$ and $n = 3$. The reverse of this contradicts Lemma 9.

Case D *Suppose* $n = 2$.

Then the derived design with respect to $K_{1,2}$ is a bigraphical $1\text{–}(2 \cdot 2, \mathcal{K}', 2)$ design from Appendix A1. Then $t = 3$, $m = 3$ and $n = 2$ and the reverse of this contradicts Lemma 7.□

ACKNOWLEDGMENTS

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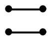
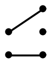
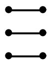
APPENDIX

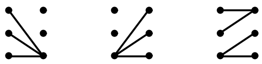
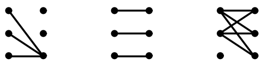
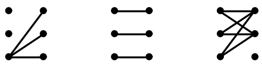
A1. The Bigraphical 1-wise balanced designs of index 2

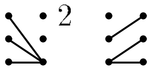
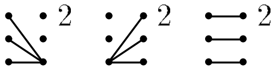
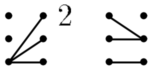
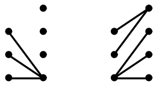
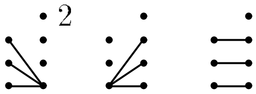
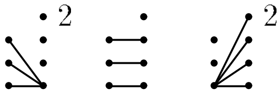
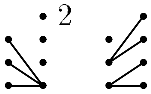
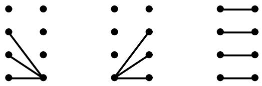
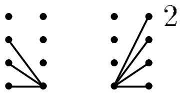
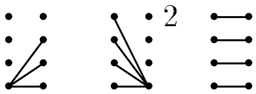
	Parameters	Bigraphical Representation[†]	
D_1:	$2 \leq m \leq n$ $1-(mn, \{m,n\}, 2)$	$K_{1,n}$	$K_{m,1}$
D_2:	$2 \leq m \leq n$ $1-(mn, m, 2)$	$K^2_{m,1}$	
D_3:	$2 \leq m \leq n$ $1-(mn, n, 2)$	$K^2_{1,n}$	
D_4:	$m = 3, n \geq 3$ $1-(3n, 2n, 2)$	$K_{2,n}$	
D_5:	$m = 3, n \geq 3$ $1-(3n, 2, 2)$	$K_{2,1}$	
D_6:	$m \in \{2,3\}, n = 3$ $1-(3m, 2m, 2)$	$K_{m,2}$	
D_7:	$m \in \{2,3\}, n = 3$ $1-(3m, 2, 2)$	$K_{1,2}$	
D_8:	$m = 2, n = 2$ $1-(4, 2, 2)$		

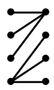
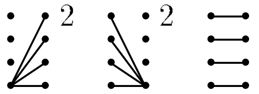
[†]A superscripted 2 denotes a repeated orbit of blocks.

D_9:	$m = 2, n = 2$ $1\text{–}(4, 2, 2)$	
D_{10}:	$m = 2, n = 2$ $1\text{–}(4, 2, 2)$	 2
D_{11}:	$m = 2, n = 3$ $1\text{–}(6, 2, 2)$	
D_{12}:	$m = 3, n = 3$ $1\text{–}(9, 3, 2)$	

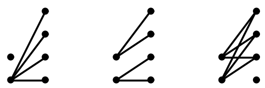
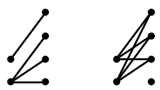
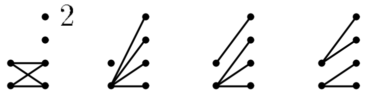
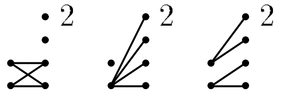
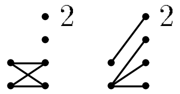
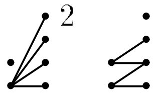
A2. The Bigraphical 2-wise balanced designs of index 2

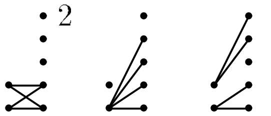
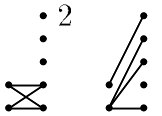
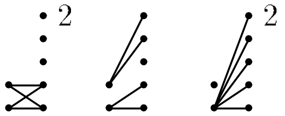
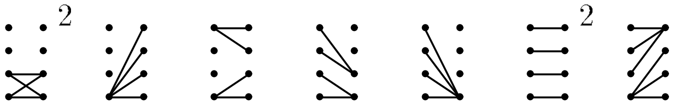
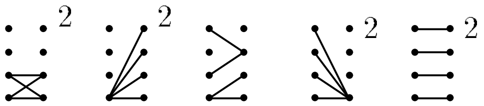
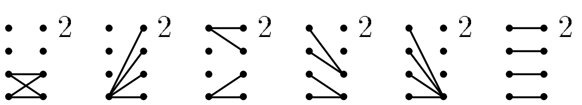
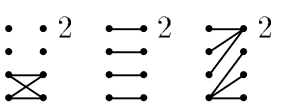
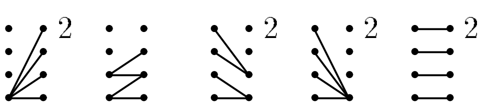
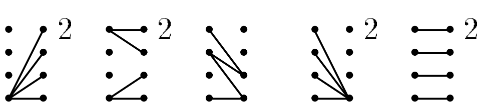
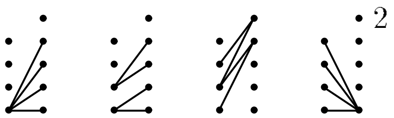
	Parameters	Bigraphical Representation†
D_{13}:	$m = 3, n = 3$ $2\text{–}(9, \{3, 4\}, 2)$	
D_{14}:	$m = 3, n = 3$ $2\text{–}(9, \{3, 6\}, 2)$	
D_{15}:	$m = 3, n = 3$ $2\text{–}(9, \{3, 6\}, 2)$	
D_{16}:	$m = 3, n = 3$ $2\text{–}(9, \{3, 4\}, 2)$	

D_{17}:	$m = 3, n = 3$ $2\text{–}(9, 3, 2)$	
D_{18}:	$m = 3, n = 3$ $2\text{–}(9, 3, 2)$	
D_{19}:	$m = 3, n = 3$ $2\text{–}(9, 3, 2)$	
D_{20}:	$m = 3, n = 4$ $2\text{–}(12, \{3, 5\}, 2)$	
D_{21}:	$m = 3, n = 4$ $2\text{–}(12, 3, 2)$	
D_{22}:	$m = 3, n = 4$ $2\text{–}(12, \{3, 4\}, 2)$	
D_{23}:	$m = 3, n = 4$ $2\text{–}(12, \{3, 4\}, 2)$	
D_{24}:	$m = 4, n = 4$ $2\text{–}(16, \{3, 4\}, 2)$	
D_{25}:	$m = 4, n = 4$ $2\text{–}(16, \{3, 4\}, 2)$	
D_{26}:	$m = 4, n = 4$ $2\text{–}(16, \{3, 4\}, 2)$	

D_{27}:	$m = 4, n = 4$ $2-(16, 6, 2)$	
D_{28}:	$m = 4, n = 4$ $2-(16, 4, 2)$	

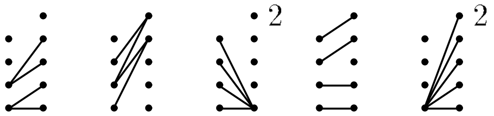
A3. The Bigraphical 3-wise balanced designs of index 2

	Parameters	Bigraphical Representation†
D_{29}:	$m = 2, n = 4$ $3-(8, \{4, 6\}, 2)$	
D_{30}:	$m = 2, n = 4$ $3-(8, \{4, 6\}, 2)$	
D_{31}:	$m = 2, n = 4$ $3-(8, 4, 2)$	
D_{32}:	$m = 2, n = 4$ $3-(8, 4, 2)$	
D_{33}:	$m = 2, n = 4$ $3-(8, 4, 2)$	
D_{34}:	$m = 2, n = 4$ $3-(8, 4, 2)$	

D_{35}:	$m=2, n=5$ $3-(10,4,2)$	
D_{36}:	$m=2, n=5$ $3-(10,4,2)$	
D_{37}:	$m=2, n=5$ $3-(10,\{4,5\},2)$	
D_{38}:	$m=4, n=4$ $3-(16,\{4,6\},2)$	
D_{39}:	$m=4, n=4$ $3-(16,4,2)$	
D_{40}:	$m=4, n=4$ $3-(16,4,2)$	
D_{41}:	$m=4, n=4$ $3-(16,\{4,6\},2)$	
D_{42}:	$m=4, n=4$ $3-(16,4,2)$	
D_{43}:	$m=4, n=4$ $3-(16,4,2)$	
D_{44}:	$m=4, n=5$ $3-(20,4,2)$	

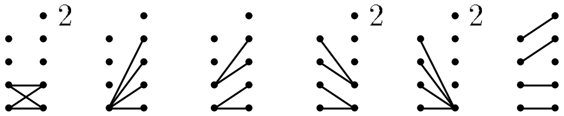
D_{45}: $m = 4, n = 5$

$3-(20, \{4, 5\}, 2)$



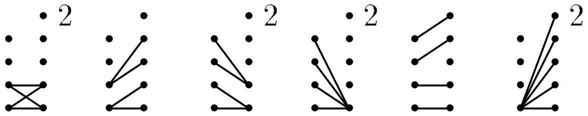
D_{46}: $m = 4, n = 5$

$3-(20, 4, 2)$



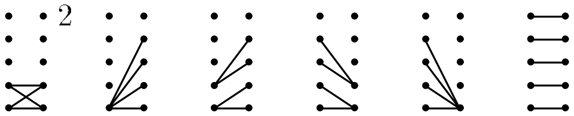
D_{47}: $m = 4, n = 5$

$3-(20, \{4, 5\}, 2)$



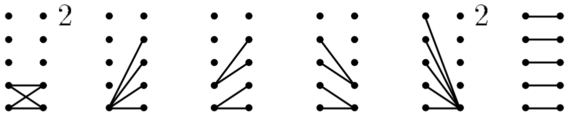
D_{48}: $m = 5, n = 5$

$3-(25, \{4, 5\}, 2)$



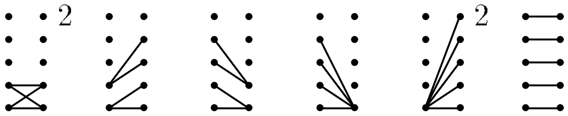
D_{49}: $m = 5, n = 5$

$3-(25, \{4, 5\}, 2)$



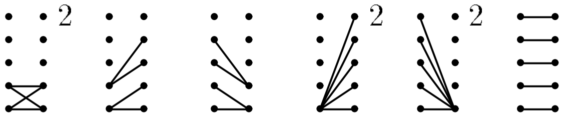
D_{50}: $m = 5, n = 5$

$3-(25, \{4, 5\}, 2)$



D_{51}: $m = 5, n = 5$

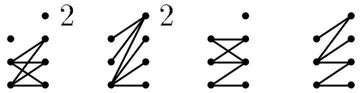
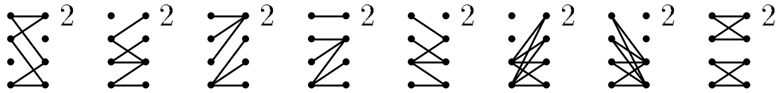
$3-(25, \{4, 5\}, 2)$



A4. There are no bigraphical 4–wise balanced designs of index 2.

A5. The Bigraphical 5-wise balanced designs of index 2

Parameters	Bigraphical Representation†

D_{52}:	$m = 3, n = 4$ $5-(12, 6, 2)$	
D_{53}:	$m = 4, n = 4$ $5-(16, \{6, 8\}, 2)$	

A6. There are no bigraphical t–wise designs of index 2, with t > 5.

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