A note on spanning trees in near d-angulations

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Abstract A near d–angulation is a planar graph in which every region has degree d except for the boundary region. Let T be a spanning tree with all of its vertices of odd degree on the boundary. Then the interior regions can be 2–colored so that regions that share edges of T receive different colors and regions which share edges not in T receive the same color. The boundary region is given a third color. We prove that the number of regions of each color can be determined from only knowing the behavior on the boundary.

1 Introduction

The m by n grid graph $G_{m,n}$ is an m by n array of nodes with node (i, j) adjacent to to node (i', j') if and only if $|i' - i| + |j' - j| = 1$. Thus $G_{m,n}$ is a planar graph all of whose regions have degree 4 except for the boundary region which has degree $2(m + n) - 4$. The regions of degree 4 are called cells. Let P be any Hamiltonian path of $G_{m,n}$ with both ends on the boundary and let p be a path of length ℓ along the boundary connecting the ends of P, see Figure 1. Fisher, Collins and Krompart prove in [1] that the union of P and p enclose $(mn - 1 - \ell)/2$ cells of $G_{m,n}$. In Theorem 2.1 we generalize this result. An example can be seen in Figure 2.

2 A generalization

A coloring of a planar graph is a *k-coloring* if k colors have been used and no two adjacent regions have received identical colors. A coloring that does not satisfy this condition will be said to be *arbitrary.* Let H be a planar graph. Arbitrarily color the regions of H and denote by $r_\alpha(H)$ the number of regions of H that have been colored α. If H is a spanning subgraph of the planar graph G, then we say that the region R of G receives color α if R is contained in an α colored region of H.

Suppose the planar graph G with boundary region C has been arbitrarily colored and S is a subgraph of G. Then we denote by $n_\alpha(S)$ the number of edges of $C \setminus S$ that are incident to an α colored interior region plus the number of edges in $C \cap S$ that are not incident to an α colored interior region. The quantity $n_\alpha(S)$ can be easily computed by traversing once around the cycle C. Indeed in Figure 2 we see that $n_\alpha(T) = 9$ and $n_\beta(T) = 12$.

A planar graph all of whose regions have degree d except for the boundary region is called a *near d-angulation.* When $d = 3$, they are called *near triangulations* and when $d = 4$ they are *near quadrangulations.*

Theorem 2.1 *Let G be a near d-angulation with v vertices, e edges and r regions. Let C be the boundary region of G and let T be a spanning tree of G such that the vertices of odd degree in T lie on C. Then the subgraph $H = T \cup C$ of G can be 3-colored, say with colors α, β and γ, so that C is colored γ. Furthermore the number of regions in G that receive color α is*

$$r_\alpha(G) = \frac{n_\beta(T) - n_\alpha(T) + (d-2)(r-1)}{2(d-2)} \tag{1}$$

$$= \frac{v - 1 - n_\alpha(T)}{(d-2)} \tag{2}$$

Proof: Let $\tilde{H}$ be the graph whose vertices are the regions in H and in which two regions are adjacent if they share at least one edge. Then the conditions on T guarantee that $\tilde{H} \setminus C$ contains no odd cycles. Thus after coloring C with color γ, the remaining vertices of $\tilde{H}$ can be 2–colored with colors β and α. This gives the desired 3–coloring of H.

Let $e_{\alpha\beta}$ be the number of edges of H that are simultaneously incident to a region colored α and to a region colored β, similarly define $e_{\alpha\gamma}$ and $e_{\beta\gamma}$.

Let h be a region of H, $h \neq C$, and let x_h be the number of regions of G that are contained in h. Then

$$\deg(h) = dx_h - 2(x_h - 1) = (d-2)x_h + 2.$$

Summing over all α colored regions h of H gives

$$e_{\alpha\beta} + e_{\alpha\gamma} = (d-2)r_\alpha(G) + 2r_\alpha(H) \tag{3}$$

and summing over all β colored regions h of H gives

$$e_{\alpha\beta} + e_{\beta\gamma} = (d-2)r_\beta(G) + 2r_\beta(H). \tag{4}$$

The ends of T lie on the boundary region C since their degree is 1 which is odd. Hence if h is a region in H it has a unique edge in C that is not in T. Consequently there are $r_\alpha(H)$ edges of $C \setminus T$ that are incident to an α colored region of G. Also the number of edges in $C \cap T$ that are not incident to an α colored region is $e_{\beta\gamma} - r_\beta(H)$. This gives

$$n_\alpha(T) = r_\alpha(H) + e_{\beta\gamma} - r_\beta(H) \tag{5}$$

and similarly we have

$$n_\beta(T) = r_\beta(H) + e_{\alpha\gamma} - r_\alpha(H) \tag{6}$$

Using the fact that $r_\alpha(G) + r_\beta(G) + 1 = r$ each pair of the above equations can be solved for the quantity $e_{\alpha\beta} - e_{\beta\gamma}$. Equating the results one can solve for $r_\alpha(G)$. Observe that $d(r-1) + \deg(C) = 2e$ since the number $r-1$ of interior regions times d gives all the edges counted twice, except those comprising C, which are counted once. So add $\deg(C)$ to count them twice. Equation 2 can now be obtained from Equation 1 by using Euler's formula and the additional fact: $n_\alpha(T) + n_\beta(T) = \deg(C)$. $\Box$

Corollary 2.2 *Let G be a near d-angulation on v vertices. Let T be any Hamiltonian path of G with both ends on the boundary and let p be a path of length ℓ along the boundary connecting the ends of T. Then T and p enclose $(v-1-\ell)/(d-2)$ regions of G.*

Proof: T is a spanning tree with all of it vertices of odd degree on the boundary. Color the regions enclosed by $T \cup p$ of $H = C \cup P$ with color α; where C is the boundary region of G. Then $n_\alpha(T) = \ell$, the length of p. The result now follows from Theorem 2.1. $\Box$

When $d = 4$ and G is the grid graph $G_{m,n}$ Corollary 2.2 gives the Fisher, Collins, Krompart result.

Acknowledgment

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References

[1] D.C. Fisher, K.L. Collins and L.B. Krompart, A note on Hamiltonian paths of grid graphs, Preprint 1995.

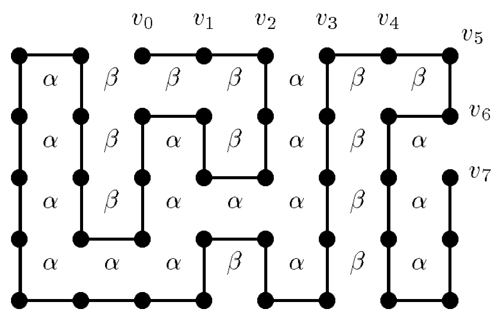

Figure 1: The solid line is a Hamilton path P. A path of length 7 connecting the ends of P is given by $p = v_0v_1v_2v_3v_4v_5v_6v_7$. Then $(8 \cdot 5 - 1 - 7)/2 = 16$ cells are labeled α.

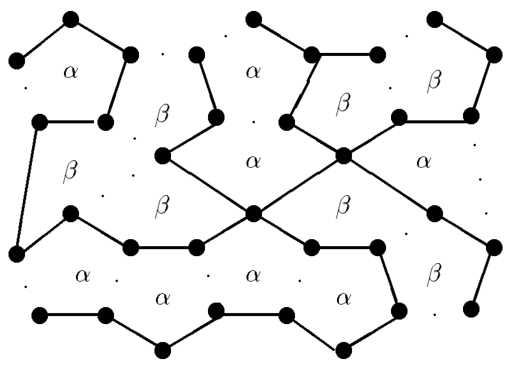

Figure 2: The solid line is a spanning tree T with vertices of odd degree on the boundary. There are $(12 - 9 + 3 \cdot 15)/6 = 8$ regions labeled α.