The intersection problem for small G-designs

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Abstract

A G-design of order n is a pair (P, B) where P is the vertex set of the complete graph K_n and B is an edge-disjoint decomposition of K_n into copies of the simple graph G. Following design terminology, we call these copies "blocks". Given a particular graph G, the intersection problem asks for which k is it possible to find two G-designs (P, B_1) and (P, B_2) of order n, with $|B_1 \cap B_2| = k$, that is, with precisely k common blocks. Here we complete the solution of this intersection problem for several G-designs where G is "small", so that now it is solved for all connected graphs G with at most four vertices or at most four edges.

1 Introduction and preliminaries

Let G be a simple graph which is some subgraph of K_n, the complete undirected graph on n vertices. A *G-design of order n* is a pair (V, B) where V is the vertex set of K_n and B is an edge-disjoint decomposition of K_n into copies of the simple graph G. Following design terminology, we refer to these copies of G as blocks. Thus, for example, a Steiner triple system is a K_3-design and a balanced incomplete block design with block size four and index $\lambda = 1$ is a K_4-design. The number of blocks, $|B|$, is $\binom{n}{2}/|E(G)|$ where $E(G)$ is the edge-set of G; this number clearly must be an integer.

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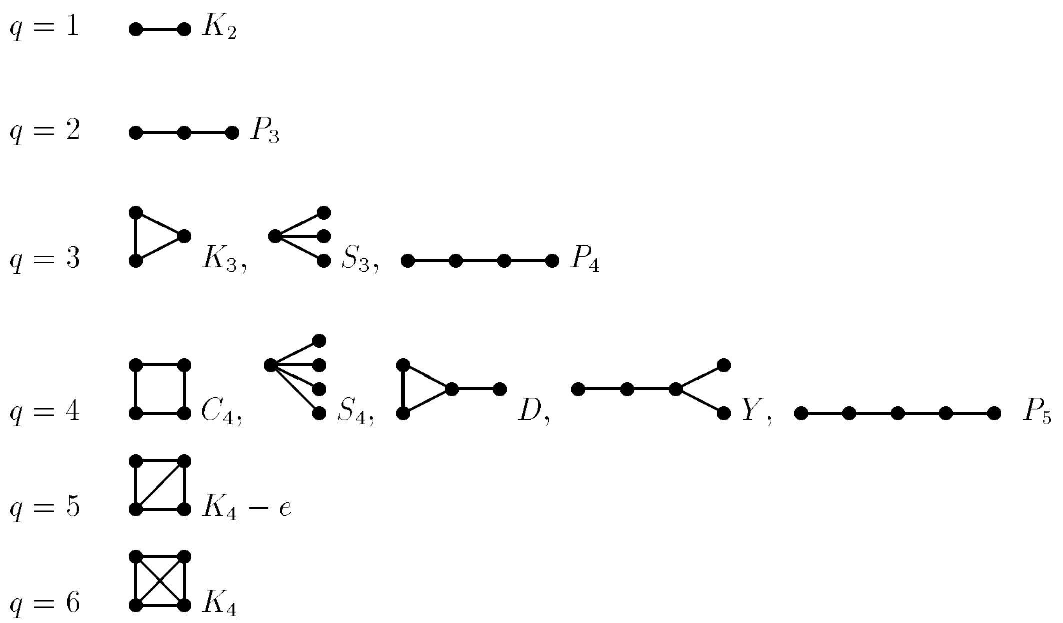
The intersection problem for G-designs asks for what values of k is it possible to find two G-designs (V, B_1) and (V, B_2), of the same order $|V|$ and based on the same set V, with $|B_1 \cap B_2| = k$; that is, having precisely k common blocks. This problem was first considered for Steiner triple systems or K_3-designs (see [9]), and since then the intersection problem has been considered for many different types of combinatorial structures; see [3] for a recent survey.

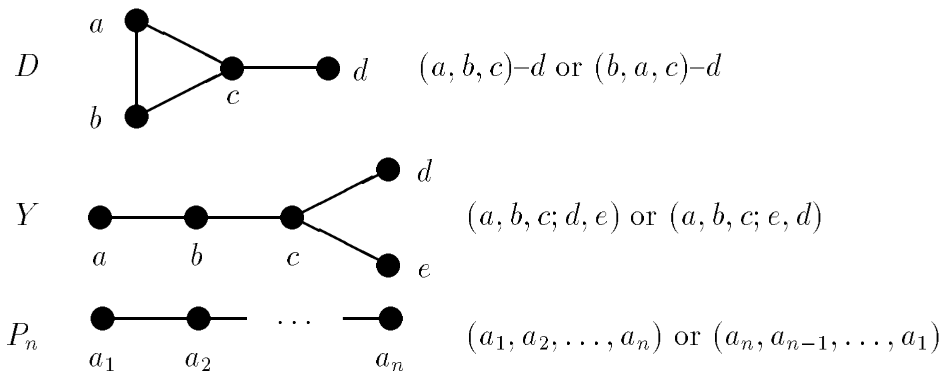
A (p, q) graph is one with p vertices and q edges. We list below all non-trivial connected simple (p, q) graphs with $\min(p, q) \leq 4$.

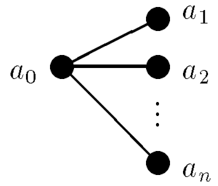


Clearly a K_2-design is unique; each block is an edge! And so for this design we cannot find two distinct designs, let alone a pair of designs intersecting in a specified number of blocks! So we leave this trivial case.

As mentioned above, the intersection problem for K_3-designs was dealt with in [9]. The intersection problem for C_4-designs appears in [4], for $(K_4 - e)$-designs in [5] and for K_4-designs (with a few exceptions) in [7].

The remaining cases, namely the graphs P_3, P_4, P_5, S_3, S_4, D and Y, we deal with below. We use the notation of [2] for names of these graphs, and the following diagram shows how we label the blocks.



S_n 

$(a_0 : a_{i_1}, a_{i_2}, \ldots, a_{i_n})$ where $\{i_1, i_2, \ldots, i_n\} = \{1, 2 \ldots, n\}$

In what follows we let $IG(n)$ denote the set of integers k for which there exist two G-designs (P, B_1) and (P, B_2) with $|P| = n$ and $|B_1 \cap B_2| = k$. Also if G is a graph with q edges, let

$$JG(n) = \begin{cases} \{0, 1, 2, \ldots, \frac{1}{q}\binom{n}{2} - 2, \frac{1}{q}\binom{n}{2}\} & \text{if } q|\binom{n}{2}; \\ \emptyset & \text{otherwise.} \end{cases}$$

In other words, $JG(n)$ denotes the intersection numbers one expects to achieve with a G-design of order n.

We also modify this notation slightly and let $IG(H)$ and $JG(H)$ denote respectively the achievable and expected intersection numbers for a G-decomposition of the graph H.

We also need the following definition. If S is a set of positive integers and h is some positive integer, then $h * S$ denotes the set of all integers which can be obtained by adding any h elements of S together (repetitions of elements of S allowed). For example, $2 * \{0, 1, 3\} = \{0, 1, 2, 3, 4, 6\}$.

Subsequently we shall need to decompose certain bipartite and tripartite graphs into edge-disjoint copies of the graphs G. Consider the following example.

Example **1.1** *Decompositions of $K_{4,4}$ into copies of P_5.*

Let $K_{4,4}$ have vertex set $\{1_1, 2_1, 3_1, 4_1\} \cup \{1_2, 2_2, 3_2, 4_2\}$, and let $P = \{A, B, C, D\}$ where

$$A = (1_2, 1_1, 2_2, 2_1, 3_2), \quad B = (1_2, 4_1, 4_2, 3_1, 3_2),$$
$$C = (1_1, 3_2, 4_1, 2_2, 3_1), \quad D = (1_1, 4_2, 2_1, 1_2, 3_1).$$

These cover the 16 edges of $K_{4,4}$, and so form a P_5-decomposition of $K_{4,4}$.

Now C and D cover the same edges as

$$C' = (1_2, 2_1, 4_2, 1_1, 3_2), D' = (1_2, 3_1, 2_2, 4_1, 3_2),$$

while B, C and D together cover the same edges as

$$B^* = (2_2, 4_1, 3_2, 1_1, 4_2), C^* = (1_2, 2_1, 4_2, 3_1, 3_2), D^* = (2_2, 3_1, 1_2, 4_1, 4_2).$$

Moreover, the permutation (1 2) applied to the subscripts of blocks A, B, C and D yields a different P_5-decomposition of $K_{4,4}$ having no blocks in common with P; call these blocks $\overline{P}$.

Thus we see that $|P \cap \overline{P}| = 0$, $|P \cap \{A, B^*, C^*, D^*\}| = 1$, $|P \cap \{A, B, C', D'\}| = 2$, $|P \cap P| = 4$. (Clearly it is not possible to have two decompositions which have all but one block in common.) We record these intersection numbers for P_5-decompositions of $K_{4,4}$ as

$$IP_5(K_{4,4}) = \{0, 1, 2, 4\}.$$

□

More generally, if $\mathcal{K}$ is a collection of graphs, then a $\mathcal{K}$-decomposition of the graph H, $(V, \mathcal{B})$, is an edge-disjoint decomposition of H with vertex set V into a set of subgraphs $\mathcal{B}$, with each subgraph isomorphic to some graph in $\mathcal{K}$. If $\mathcal{K} = \{G\}$, then we call this a G-decomposition of H, and if also $H = K_n$, then it is a G-design of order n.

The following lemma will be most useful in the rest of this paper.

LEMMA **1.1** *Let G be a graph with q edges and suppose $(V, \mathcal{B})$ is a $\{K_m, H\}$-decomposition of K_n. If $IG(m) = JG(m)$ and $IG(H) \supseteq \{0, r\}$ with $|E(H)| = qr$ and $qr \leq \binom{m}{2}$, then $IG(n) = JG(n)$.*

Proof. First a G-design of order n can be constructed by replacing each of the blocks $B \in \mathcal{B}$ that is isomorphic to K_m by a G-design of order m, and replacing each of the blocks $B \in \mathcal{B}$ that is isomorphic to H by a G-decomposition of H.

Secondly, if $q | \binom{m}{2}$, then for any positive integer x,

$$x * JG(m) = \left\{0, 1, 2, \ldots, \frac{x}{q}\binom{m}{2} - 2, \frac{x}{q}\binom{m}{2}\right\},$$

and for all $x \geq r + 2$,

$$\{0, 1, 2, \ldots, x - 2, x\} + \{0, r\} = \{0, 1, 2, \ldots, x + r - 2, x + r\}.$$

Thus if $\mathcal{B}$ contains α blocks isomorphic to K_m and β blocks isomorphic to H, then

$$IG(v) \supseteq \alpha * JG(m) + \beta * \{0, r\} = \{0, 1, 2, \ldots, z - 2, z\}$$

where $z = \alpha \frac{1}{q}\binom{m}{2} + \beta r$. But now $\alpha\binom{m}{2} + \beta qr = \binom{n}{2}$. Thus $z = \frac{1}{q}\binom{n}{2}$, as required. Hence $IG(n) = JG(n)$. □

In what follows, the graph H in Lemma 1.1 will usually be a complete bipartite or tripartite graph.

2 Paths on 3, 4 and 5 vertices

2.1 The path P_3

Note that a P_3-design of order n contains $n(n-1)/4$ blocks and so we must have $n \equiv 0$ or $1 \pmod 4$.

EXAMPLE **2.1** $IP_3(K_{2,2}) = \{0, 2\}$.

Take designs (P, B_i), $i = 1, 2$, where the vertex set of $K_{2,2}$ is $P = \{a, b\} \cup \{c, d\}$, and $B_1 = \{(a, c, b), (a, d, b)\}$, $B_2 = \{c, a, d), (c, b, d)\}$. Since $|B_1 \cap B_2| = 0$ we have $IP_3(K_{2,2}) = \{0, 2\}$. $\square$

EXAMPLE **2.2** $IP_3(4) = \{0, 1, 3\}$.

We use designs (P, B_i), $i = 1, 2, 3$, where $P = \{a, b, c, d\}$ and

$$\begin{aligned} B_1 = &\ \{(a, b, c), (a, c, d), (a, d, b)\}, \\ B_2 = &\ \{(a, b, c), (d, a, c), (b, d, c)\}, \\ B_3 = &\ \{(a, b, d), (a, d, c), (a, c, b)\}. \end{aligned}$$

Here $|B_1 \cap B_2| = 1$, $|B_1 \cap B_3| = 0$ and of course $|B_1 \cap B_1| = 3$. The result follows. $\square$

EXAMPLE **2.3** $IP_3(5) = \{0, 1, 2, 3, 5\}$.

Take the vertex set of K_5 to be $\{\infty, 1, 2, 3, 4\}$. Let

$$\begin{aligned} X &= \{(1, \infty, 2), (3, \infty, 4)\},\ X' = \{(1, \infty, 3), (2, \infty, 4)\}, \\ Y &= \{(1, 2, 3), (1, 3, 4), (2, 4, 1)\}, \\ Y' &= \{(1, 2, 4), (1, 4, 3), (1, 3, 2)\}, \\ Y'' &= \{(1, 2, 3), (4, 1, 3), (2, 4, 3)\}. \end{aligned}$$

Then X and X' cover the same edges (so they form a *trade* or are *mutually balanced sets*), as do Y, Y' and Y''; also the blocks $X \cup Y$ cover K_5. We tabulate the intersection numbers:

blocks	intersection size
$(X \cup Y), (X' \cup Y')$	0
$(X \cup Y), (X' \cup Y'')$	1
$(X \cup Y), (X \cup Y')$	2
$(X \cup Y), (X' \cup Y)$	3
$(X \cup Y), (X \cup Y)$	5

$\square$

EXAMPLE **2.4** $IP_3(K_{1,2n}) = \{0, 1, 2, \dots, n-2, n\}$.

The verification of this is immediate. $\square$

Now let $n = 4m$, and take the vertex set of K_n to be $\{(i, j) \mid 1 \le i \le 2m,\ \ j = 1, 2\}$. Take K_4 blocks $\{(2i-1, j), (2i, j) \mid j = 1, 2\}$, for $1 \le i \le m$, and $K_{2,2}$ blocks $\{(a, 1), (a, 2)\} \cup \{(b, 1), (b, 2)\}$ where $1 \le a < b \le 2m$ and $\{a, b\} \ne \{2i-1, 2i\}$ for $1 \le i \le m\}$. The result is a $\{K_4, K_{2,2}\}$-decomposition of K_{4m} and consequently by Lemma 1.1 we have $IP_3(4m) = JP_3(4m)$.

Now let $n = 4m + 1$, and let the vertex set of K_n be $\{1, 2, \dots, 4m, \infty\}$. We may use P_3-designs of order $4m$ on $\{1, 2, \dots, 4m\}$ and use Example 2.4 to find P_3-decompositions of $K_{1,4m}$ on $\{\infty\} \cup \{1, 2, \dots, 4m\}$. Thus

$$\begin{aligned} IP_3(4m+1) &\supseteq IP_3(4m) + IP_3(K_{1,4m}) \\ &= \{0, 1, 2, \dots, m(4m+1) - 2, m(4m+1)\} = JP_3(4m+1). \end{aligned}$$

We have now proved

THEOREM **2.1** *The intersection numbers for P_3-designs are given by $IP_3(n) = JP_3(n) = \{0, 1, \ldots, b-2, b\}$ where $b = n(n-1)/4$, the total number of blocks in a P_3-design of order n.* □

2.2 The path P_4

A P_4-design of order n contains $n(n-1)/6$ blocks so that $n \equiv 0$ or $1 \pmod 3$, $n \geq 4$. So let $n = 3m$ or $3m+1$. First we give some necessary examples.

EXAMPLE **2.5** $IP_4(4) = \{0, 2\}$.

Let $V = \{1, 2, 3, 4\}$, $B_1 = \{(1, 2, 3, 4), (2, 4, 1, 3)\}$, $B_2 = \{(1, 4, 3, 2), (3, 1, 2, 4)\}$. Then (V, B_1), (V, B_2) are both P_4-designs, and $|B_1 \cap B_2| = 0$; the result follows. □

EXAMPLE **2.6** $IP_4(K_{3,3}) \supseteq \{0, 3\}$.

Let $K_{3,3}$ have vertex set $V = \{1, 2, 3\} \cup \{4, 5, 6\}$. Two disjoint decompositions are $B_1 = \{(1, 4, 2, 5), (2, 6, 3, 4), (3, 5, 1, 6)\}$, $B_2 = \{(2, 5, 3, 6), (3, 4, 1, 5), (1, 6, 2, 4)\}$. The result follows. □

EXAMPLE **2.7** $IP_4(6) = \{0, 1, 2, 3, 5\}$.

Let K_6 have vertex set $V = \{0, 1, 2, 3, 4, 5\}$, and let $A = \{(0, 1, 2, 3), (3, 0, 5, 2), (0, 4, 3, 1)\}$, $B = \{(0, 2, 4, 5), (3, 5, 1, 4)\}$, and $C = \{(0, 4, 3, 1), (3, 5, 1, 4)\}$. Then $(V, A \cup B)$ is one P_4-design of order 6. Note that the blocks A trade with $A' = \{(1, 0, 5, 2), (4, 0, 3, 2), (4, 3, 1, 2)\}$, and the blocks C trade with $C' = \{(0, 4, 1, 3), (1, 5, 3, 4)\}$. Let $X = A \cup B$, and let α denote the permutation (14)(35) and β the permutation (15)(34). The following table lists intersection numbers.

blocks	intersection size
$X,\ X\alpha$	0
$X,\ X\beta$	1
$X,\ \{A', B\}$	2
$X,\ ((X \setminus C) \cup C')$	3
$X,\ X$	5

□

EXAMPLE **2.8** $IP_4(7) = \{0, 1, 2, 3, 4, 5, 7\}$.

Let K_7 have vertex set $V = \{0, 1, 2, 3, 4, 5, 6\}$. Let $A = \{(0, 1, 3, 6), (1, 2, 4, 0)\}$, $B = \{(2, 3, 5, 1), (3, 4, 6, 2), (4, 5, 0, 3)\}$ and $C = \{(5, 6, 1, 4), (6, 0, 2, 5)\}$. Then (V, X), where $X = (A \cup B \cup C)$, is a P_4-design of order 7. Moreover, A, B and C trade with $A' = \{(2, 1, 3, 6), (1, 0, 4, 2)\}$, $B' = \{(1, 5, 0, 3), (6, 4, 5, 3), (6, 2, 3, 4)\}$ and $C' = \{(4, 1, 6, 0), (0, 2, 5, 6)\}$ respectively. Let α denote the permutation (06)(13). The following table lists intersection numbers.

blocks	intersection size
$X,\ A' \cup B' \cup C'$	0
$X,\ X\alpha$	1
$X,\ A \cup B' \cup C'$	2
$X,\ A' \cup B \cup C'$	3
$X,\ A \cup B' \cup C$	4
$X,\ A' \cup B \cup C$	5
$X,\ X$	7

$\square$

EXAMPLE **2.9** $IP_4(9) = \{0, 1, 2, \ldots, 9, 10, 12\}$.

Take a P_4-design of order 6, on $\{0, 1, 2, 3, 4, 5\}$, and adjoin elements H, J and K, and also the blocks

$$\begin{aligned} X &= \{(0, H, 1, J), (2, H, 3, J), (4, H, 5, J), (1, K, 0, J), \\ &\qquad (3, K, 2, J), (5, K, J, H), (H, K, 4, J)\}. \end{aligned}$$

Now using $IP_4(6)$ we have $\{7, 8, 9, 10, 12\} \subseteq IP_4(9)$. Also applying the permutation $(H\,J)$ to the set X changes all the blocks in X, so again using $IP_4(6)$ we have $\{0, 1, 2, 3, 5\} \subseteq IP_4(9)$.

Thus it remains to show that 4 and 6 are in $IP_4(9)$. To do this, let D denote the design with blocks $X \cup A \cup B$ where A and B are as in Example 2.7 above. Then $|D \cap D\gamma| = 4$ where γ is the permutation $(0\,3)(1\,2)$. Finally, let

$$T = \{(0, 2, 4, 5), (3, 0, 5, 2), (0, 4, 3, 1), (3, 5, 1, 4)\}$$

which has trade

$$T' = \{(3, 1, 4, 5), (0, 3, 5, 1), (3, 4, 2, 0), (4, 0, 5, 2)\}.$$

Then $|D\gamma \cap ((D \setminus T) \cup T')| = 6$, which completes the intersection numbers for designs of order 9. $\square$

Now let $n = 3m + 1$ and let the vertex set of K_n be $V = \{(i, j) \mid 1 \leq i \leq m,\ j = 1, 2, 3\} \cup \{\infty\}$. There is a $\{K_7, K_4, K_{3,3}\}$-decomposition of K_n with: one K_7 block $\{\infty\} \cup \{(i, j) \mid i = 1, 2;\ j = 1, 2, 3\}$; K_4 blocks $\{\infty\} \cup \{(i, j) \mid j = 1, 2, 3\}$, for $3 \leq i \leq m$; $K_{3,3}$ blocks $\{(i, j) \mid j = 1, 2, 3\} \cup \{(i', j) \mid j = 1, 2, 3\}$, for all $1 \leq i < i' \leq m$, excluding $\{i, i'\} = \{1, 2\}$. Then using Examples 2.8, 2.5, 2.6 and a slight generalization of Lemma 1.1, it follows that $IP_4(3m + 1) = JP_4(3m + 1) = \{0, 1, 2, \ldots, t - 2, t\}$ where $t = m(3m + 1)/2$, the total number of blocks in a P_4-design of order $3m + 1$.

Next let $n = 3m$. The cases m even and m odd are treated separately. When m is even let $n = 6M$ and let the vertex set of K_n be $\{(i, j) \mid 1 \leq i \leq 2M;\ j = 1, 2, 3\}$. There is a $\{K_6, K_{3,3}\}$-decomposition of K_n with K_6 blocks $\{(2i - 1, j), (2i, j) \mid j =$

$1,2,3\}$ for $1 \le i \le M$ and $K_{3,3}$ blocks $\{(i_1,j) \mid j=1,2,3\} \cup \{(i_2,j) \mid j=1,2,3\}$ for all $1 \le i_1 < i_2 \le 2M$ excluding $\{i_1,i_2\} = \{2i-1,2i\}$, $1 \le i \le M$.

The result $IP_4(6M) = JP_4(6M)$ then follows from Examples 2.7, 2.6 and Lemma 1.1.

When m is odd let $n = 6M = 3$, and let the vertex set of K_n be $\{(i,j) \mid 1 \le i \le 2M+1,\ j=1,2,3\}$. There is a $\{K_9, K_6, K_{3,3}\}$-decomposition of K_n with: one K_9 block $\{(i,j) \mid i,j = 1,2,3\}$; K_6 blocks $\{(2i,j),(2i+1,j) \mid j=1,2,3\}$ for $i = 2,\dots,M$; $K_{3,3}$ blocks $\{(a,j) \mid j=1,2,3\} \cup \{(b,j) \mid j=1,2,3\}$ for all pairs $\{a,b\}$ with $a \ne b$ and with a and b not *both* in $\{1,2,3\}$ or in $\{2i,2i+1\}$, $2 \le i \le M$.

Then from Examples 2.9, 2.7, 2.6 and Lemma 1.1, we have $IP_4(6M+3) = JP_4(6M+3)$.

We have now proved

THEOREM **2.2** *The intersection numbers for P_4-designs are given by $IP_4(n) = \{0,1,\dots,b-2,b\}$ where $b = n(n-1)/6$.* □

2.3 The path P_5

The graph P_5 has 4 edges, and so a suitable decomposition of K_n will contain $n(n-1)/8$ blocks; consequently we must have $n \equiv 0$ or $1 \pmod 8$. The only ingredients needed are decompositions of $K_{4,4}$, K_8 and K_9, and of course their intersection numbers too.

Now let the vertex set of K_n be $V = \{(i,j) \mid 1 \le i \le 2m,\ 1 \le j \le 4\}$ or $V \cup \{\infty\}$, according as $n = 8m$ or $8m+1$.

In the former case there is a $\{K_8, K_{4,4}\}$-decomposition of K_n with K_8 blocks $\{(2i-1,j),(2i,j) \mid 1 \le j \le 4\}$ for $1 \le i \le m$, and $K_{4,4}$ blocks $\{(a,j) \mid 1 \le j \le 4\} \cup \{(b,j) \mid 1 \le j \le 4\}$ for all $1 \le a < b \le 2m$ and $\{a,b\} \ne \{2i-1,2i\}$ for $1 \le i \le m$. In the latter case there is a $\{K_9, K_{4,4}\}$-decomposition of K_n; the K_9 blocks have $\{\infty\}$ adjoined to each of the K_8 blocks above, otherwise blocks are the same as when $n = 8m$.

In Example 1.1 we showed that $IP_5(K_{4,4}) = \{0,1,2,4\}$. We also need the following two examples.

EXAMPLE **2.10** $IP_5(8) = \{0,1,2,3,4,5,7\}$.

On the vertex set $Z_7 \cup \{\infty\}$, developing the base block $\beta = (\infty,0,1,3,6)$ modulo 7 generates a P_5-decomposition of K_7. For each $i \in Z_7$ the blocks $A_i = \{\beta+i, \beta+i+1\}$ trade with $A_i' = \{(6,3,1,0,4)+i, (0,\infty,1,2,4)+i\}$, and $B = \{\beta+4,\beta+5,\beta+6\}$ trades with $B' = \{(3,0,5,\infty,4),(\infty,6,5,2,0),(0,6,1,4,5)\}$. We observe that A_0, A_2 and A_4 are mutually disjoint and that B is disjoint from A_0 and A_2. Consequently $IP_5(8) = \{0,1,2,3,4,5,7\}$. □

Example **2.11** $IP_5(9) = \{0, 1, 2, 3, 4, 5, 6, 7, 9\}$.

On the vertex set Z_9, a P_5-design is generated by developing the base block $\beta = (0, 1, 3, 7, 4)$ (modulo 9). For each $i \in Z_9$ the blocks $A_i = \{\beta + i, \beta + i + 2\}$ trade with $A_i' = \{(0, 5, 3, 7, 4) + i, (2, 3, 1, 0, 6) + i\}$ and the blocks $B_i = \{\beta + i, \beta + i + 1, \beta + i + 2\}$ trade with $B_i' = \{(0, 1, 2, 3, 5) + i, (1, 3, 7, 4, 2) + i, (6, 0, 5, 8, 4) + i\}$.

Moreover, the blocks $C = \{\beta + 5, \beta + 7, \beta + 8\}$ trade with the blocks $C' = \{(7, 8, 3, 6, 5), (3, 0, 8, 6, 2), (8, 1, 5, 2, 0)\}$. The following table lists the disjoint trades which may be used in order to achieve the required intersection values.

trades	intersection achieved
B_0, B_3, B_6	0
A_0, A_1, A_4, A_5	1
C, A_0, A_1	2
B_0, B_3	3
A_0, B_6	4
A_0, A_1	5
B_0	6
A_0	7
nothing	9

□

Now applying Lemma 1.1 yields the following result for P_5-designs.

Theorem **2.3** *The intersection numbers for P_5-designs are given by $IP_5(n) = \{0, 1, \ldots, b-2, b\}$ where $b = n(n-1)/8$.* □

3 Stars with 3 and 4 edges

3.1 S_3-designs

The number of blocks in an S_3-design of order n is $n(n-1)/6$, and so $n \equiv 0$ or $1 \pmod 3$, and $n \geq 6$. (S_3 involves four vertices, and it is easy to see that K_4 has no S_3-decomposition.)

We start with the following example.

Example **3.1** $IS_3(K_{3,3}) = \{0, 3\}$.

Let $K_{3,3}$ have vertex set $\{1, 2, 3\} \cup \{4, 5, 6\}$. The following two S_3-decompositions are disjoint.

$$\begin{aligned} D_1 &= \{(1 : 4, 5, 6), (2 : 4, 5, 6), (3 : 4, 5, 6)\}, \\ D_2 &= \{(4 : 1, 2, 3), (5 : 1, 2, 3), (6 : 1, 2, 3)\}. \end{aligned}$$

Moreover, it is straightforward to see that $2 \notin IS_3(K_{3,3})$. □

One slight difficulty in this case (and, indeed, for S_m-designs in general) is that the expected full set of intersection numbers for a design of order 6 (or $2m$ in general) cannot be achieved. In the case of S_3-designs, each block involves 4 vertices, and it is impossible to find a trade consisting of two blocks when the design is of order 6. The smallest trade involves *seven* vertices, such as $\{(x : a, b, c), (x : d, e, f)\}$ trading with $\{(x : a, b, d), (x : c, e, f)\}$. We do however achieve the other expected intersection numbers, as the following example shows.

EXAMPLE **3.2** $IS_3(6) = \{0, 1, 2, 5\}$.

Let $V = \{0, 1, 2, 3, 4, 5\}$ and take

$$B = \{(0 : 5, 1, 2),\ (1 : 5, 2, 3),\ (2 : 5, 3, 4),\ (3 : 5, 4, 0),\ (4 : 5, 0, 1)\}.$$

Let $\alpha = (0\,1\,2)$, $\beta = (3\,4\,5)$ and $\gamma = (0\,1)$ be permutations on V. The result then follows from the table below.

blocks	intersection
$B \cap B\alpha$	0
$B \cap B\beta$	1
$B \cap B\gamma$	2
$B \cap B$	5

$\square$

Three more necessary examples follow.

EXAMPLE **3.3** $IS_3(7) = \{0, 1, 2, 3, 4, 5, 7\}$.

Take the vertex set $\{0, 1, 2, 3, 4, 5, 6\}$, and blocks $B \cup \{(6 : 0, 1, 2),\ (6 : 3, 4, 5)\} = B \cup Y$ where B is as in Example 3.2. The permutations α, β and γ of Example 3.2 fix Y. Hence $\{2, 3, 4, 7\} \subseteq IS_3(7)$. Moreover, Y trades with $Y' = \{(6 : 0, 1, 3),\ (6 : 2, 4, 5)\}$, and so $0 \in IS_3(7)$. Also $|(B \cup Y) \cap (B\beta \cup Y')| = 1$ and $|(B \cup Y) \cap (B \cup Y')| = 5$, so the result follows. $\square$

EXAMPLE **3.4** $IS_3(9) = \{0, 1, \ldots, 10, 12\}$.

Let the vertex set be Z_9, and take blocks B as follows.

block	in subset(s)	block	in subset(s)
$(0 : 1, 3, 6)$	X	$(6 : 1, 2, 7)$	Y, T
$(1 : 2, 4, 7)$	Y	$(7 : 2, 0, 8)$	T
$(2 : 0, 5, 8)$		$(8 : 0, 1, 6)$	X, T
$(3 : 1, 4, 6)$	X	$(3 : 2, 7, 8)$	
$(4 : 2, 5, 7)$	Y	$(4 : 0, 8, 6)$	X
$(5 : 0, 3, 8)$	Z	$(5 : 1, 6, 7)$	Y, Z

The set X trades with $X' = \{(1:0,3,8),(0:3,4,8),(6:0,3,8),(4:3,6,8)\}$; the set Y trades with $Y' = \{(2:1,4,6),(1:4,5,6),(7:1,4,6),(5:4,6,7)\}$; the set Z trades with $Z' = \{(5:0,3,7),(5:8,1,6)\}$; and the set T trades with $T' = \{(6:1,2,8),(7:0,2,6),(8:0,1,7)\}$. Also Z and T are disjoint. The intersection values now follow from the table below, where numbers in parentheses are permutations on Z_9.

blocks	intersection
$B \cap B(6\,7\,8)$	0
$B \cap B(4\,7\,5\,8)$	1
$B \cap B(4\,5)(7\,8)$	2
$B \cap B(7\,8)$	3
$B \cap ((B \setminus (X \cup Y)) \cup X' \cup Y')$	4
$B \cap B(4\,5)$	5
$B \cap ((B \setminus (X \cup Z)) \cup X' \cup Z')$	6
$B \cap ((B \setminus (X \cup T)) \cup X' \cup T')$	7
$B \cap ((B \setminus X) \cup X')$	8
$B \cap ((B \setminus T) \cup T')$	9
$B \cap ((B \setminus Z) \cup Z')$	10
$B \cap B$	12

□

Example **3.5** $IS_3(10) = \{0,1,\ldots,13,15\}$.
Take Z_{10} and blocks B of Example 3.4 above, together with $P = \{(9:0,1,2),(9:3,4,5),(9:6,7,8)\}$. The blocks in P are fixed by the above permutations (except for $(4\,7\,5\,8)$) and by the trades on B, so $\{3,4,6,7,8,9,10,11,12,13,15\} \subseteq IS_3(10)$. Also P trades with $P' = \{(9:0,1,3),(9:2,4,6),(9:5,7,8)\}$, and so in particular $\{0,1,5\} \subseteq IS_3(10)$ also. Finally we see that $2 \in IS_3(10)$, using $1 \in IS_3(9)$ and the trade $\{(9:0,1,2),(9:3,4,5)\}$ with $\{(9:0,1,3),(9:2,4,5)\}$. This completes the example. □

In the general situation we deal with four cases: $n = 6m$, $n = 6m+1$, $n = 6m+3$ and $n = 6m+4$. In each case the vertex set is $V = \{(i,j) \mid 1 \le i \le 2m,\ j = 1,2,3\}$, or $V \cup \{\infty\}$, or $V' = V \cup \{(2m+1,j) \mid j = 1,2,3\}$ or $V' \cup \{\infty\}$ (respectively).

First, when $n = 6m+1$, there is a $\{K_7, K_{3,3}\}$-decomposition of K_n with K_7 blocks $\{\infty\} \cup \{(2i-1,j),(2i,j) \mid j = 1,2,3\}$ for $1 \le i \le m$ and $K_{3,3}$ blocks $\{(a,j) \mid j = 1,2,3\} \cup \{(b,j) \mid j = 1,2,3\}$, for all $1 \le a < b \le 2m$, excluding $\{a,b\} = \{2i-1,2i\}, 1 \le i \le m$.

¿From Lemma 1.1 it follows that $IS_3(6m+1) = JS_3(6m+1)$.

Secondly, when $n = 6m+4$, we use a $\{K_{10}, K_7, K_{3,3}\}$-decomposition of K_n, with one K_{10} block and $m-1$ K_7 blocks. Once again Lemma 1.1 then yields $IS_3(6m+4) = JS_3(6m+4)$.

Thirdly, when $n = 6m$, in order to achieve the intersection number "$b-2$", with all but two blocks in common, since $5-2 = 3 \notin IS_3(6)$, we use a $\{K_9, K_6, K_{3,3}\}$-decomposition of K_n with two K_9 blocks and $m-3$ K_6 blocks. This assumes that $m \ge 3$, so $n \ge 18$; the case of order 12, therefore, must be considered separately.

Then, for $m \geq 3$, as before we obtain $IS_3(6m) = JS_3(6m)$.

Fourthly, when $n = 6m+3$, we use a $\{K_9, K_6, K_{3,3}\}$-decomposition of K_n with one K_9 block and $m-1$ K_6 blocks, and obtain $IS_3(6m+3) = JS_3(6m+3)$.

It now remains to consider the case of order 12.

EXAMPLE **3.6** $IS_3(12) = \{0, 1, \ldots, 20, 22\}$.

First, all intersection numbers except 20 (that is, $(b-2)$) can be achieved with the following construction using two designs of order 6 and four lots of decompositions of $K_{3,3}$. Let A, B, C and D each stand for a set of three vertices. Then on sets $\{A, B\}$ and $\{C, D\}$, place S_3-designs of order 6, and on the sets $\{A\} \cup \{C\}$, $\{A\} \cup \{D\}$, $\{B\} \cup \{C\}$, and $\{B\} \cup \{D\}$, place S_3-decompositions of $K_{3,3}$. The result is an S_3-design of order 12, and we see that

$$IS_3(12) \supseteq 2 * IS_3(6) + 4 * IS_3(K_{3,3})$$

which includes all required intersection numbers except 20.
Secondly, in order to obtain this intersection number, note that in the above construction, one of the four decompositions of $K_{3,3}$ is on the sets $\{A\} \cup \{C\}$ while another is on the sets $\{A\} \cup \{D\}$; so there will be two blocks of the form $(x : u, v, w)$ and $(x : r, s, t)$. These may be traded with $(x : u, v, t)$ and $(x : r, s, w)$; so we have $20 \in IS_3(12)$ as required. □

The results in this subsection have shown

THEOREM **3.1** *The intersection numbers for S_3-designs are given by $IS_3(n) = \{0, 1, \ldots, b-2, b\}$ where $n \equiv 0$ or 1 (mod 3), $n \geq 6$ and $b = n(n-1)/6$, except that $3 \notin IS_3(6)$.* □

3.2 S_4-designs

Since the number of blocks in an S_4-design of order n is $n(n-1)/8$, we must have $n \equiv 0$ or 1 (mod 8). First note that once we have intersection numbers $IS_4(8m)$, we can easily obtain $IS_4(8m+1)$. For in order to construct an S_4-design of order $8m+1$ from one of order $8m$ we may simply adjoin one new vertex, say x, and $2m$ new blocks of the form $\{(x : a, b, c, d) \mid a, b, c, d \in V\}$ where V is the vertex set of the design of order $8m$. Moreover, by judicious interchange of the $2m$ elements, we see that we may construct two S_4-designs of order $8m+1$ so that

$$IS_4(8m+1) \supseteq IS_4(8m) + \{0, 1, 2, \ldots, 2m-2, 2m\}.$$

Now consider the following examples.

EXAMPLE **3.7** $IS_4(K_{4,4}) \supseteq \{0, 4\}$.

Imitate the construction in Example 3.1 above, but taking four vertices rather than three in each partite set. $\square$

Example **3.8** $IS_4(8) = \{0, 1, 2, 3, 4, 7\}$.

With vertex set $\{0, 1, 2, 3, 4, 5, 6, 7\}$, let blocks B be as follows.

$$(0 : 1, 2, 3, 7), (1 : 2, 3, 4, 7), (2 : 3, 4, 5, 7), (3 : 4, 5, 6, 7),$$
$$(4 : 5, 6, 0, 7), (5 : 6, 0, 1, 7), (6 : 0, 1, 2, 7).$$

The following table shows the intersection values achieved by applying the given permutations to the vertices.

permutation	intersection size
$(0\,1\,2\,3)$	0
$(\infty\,0\,1\,2)$	1
$(0\,1\,2)$	2
$(\infty\,0)$	3
$(0\,1)$	4
identity	7

$\square$

Example **3.9** $IS_4(9) = \{0, 1, 2, 3, 4, 5, 6, 7, 9\}$.

As indicated in the remark preceding Example 3.7,

$$\begin{aligned} IS_4(9) &\supseteq IS_4(8) + \{0, 2\} \\ &= \{0, 1, 2, 3, 4, 7\} + \{0, 2\} \\ &= \{0, 1, 2, 3, 4, 5, 6, 7, 9\}. \end{aligned}$$

$\square$

Example **3.10** $IS_4(16) = \{0, 1, \ldots, 28, 30\}$.

First, all intersection numbers except 28 (that is, $(b - 2)$) can be achieved with the following construction using two designs of order 8 and four lots of decompositions of $K_{4,4}$. Let A, B, C and D each stand for a set of four vertices. Then on sets $\{A, B\}$ and $\{C, D\}$, place S_4-designs of order 8, and on the sets $\{A\}\cup\{C\}$, $\{A\}\cup\{D\}$, $\{B\}\cup\{C\}$, and $\{B\} \cup \{D\}$, place S_4-decompositions of $K_{4,4}$. The result is an S_4-design of order 16, and we see that

$$IS_4(16) \supseteq 2 * IS_4(8) + 4 * IS_4(K_{4,4})$$

which includes all required intersection numbers except 28.
Secondly, in order to obtain this intersection number, take another design of order 16 with vertex set $Z_{15} \cup \{\infty\}$ and 30 blocks as follows:

$$(i : i + 1, i + 2, i + 3, i + 4), \quad (i : i + 5, i + 6, i + 7, \infty), \quad i \in Z_{15}.$$

The two blocks $(0:1,2,3,4)$, $(0:5,6,7,\infty)$ trade with $(0:5,6,7,4)$, $(0:1,2,3,\infty)$, changing just two blocks, and thus showing that $28 \in IS_4(16)$ as required. $\square$

Again, using the remark at the start of this subsection, using the above example it is easy to obtain $IS_4(17) = \{0, 1, \ldots, 32, 34\}$.

Now the general construction for order $8m$ uses a $\{K_{16}, K_8, K_{4,4}\}$-decomposition of K_{8m} with one K_{16} block and $m-2$ K_8 blocks. Explicitly, let the vertex set be $\{(i,j) \mid 1 \le i \le 2m,\ 1 \le j \le 4\}$, and let the K_{16} block be $\{(i,j) \mid 1 \le i,j \le 4\}$, the K_8 blocks be $\{(2i-1,j),(2i,j) \mid 1 \le j \le 4\}$ for $3 \le i \le m$, and the $K_{4,4}$ blocks be $\{(a,j) \mid 1 \le j \le 4\} \cup \{(b,j) \mid 1 \le j \le 4\}$ for all $a \ne b$ where a and b are not both first components of elements in the same K_{16} or K_8 blocks. Then $IS_4(8m) = JS_4(8m)$.

The only difference for order $8m+1$ is that, since $IS_4(9)$ includes all intersection numbers expected, including "$b-2$", we may merely use a $\{K_9, K_{4,4}\}$-decomposition of K_{8m+1}, in order to achieve $IS_4(8m+1) = JS_4(8m+1)$.

We have now proved

Theorem **3.2** *The intersection numbers for S_4-designs are given by $IS_4(n) = \{0, 1, \ldots, b-2, b\}$ where $n \equiv 0$ or 1 (mod 8), except that $5 \notin IS_4(8)$.* $\square$

4 D, a triangle with pendant edge

Once again, since D has four edges, we find that a D-design of order n contains $n(n-1)/8$ blocks and so $n \equiv 0$ or $1 \pmod 8$. However, since D contains an odd cycle (a triangle!) there is no D-decomposition of any *bi*partite graph, so in this case we require a D-decomposition of a *tri*partite graph.

Example **4.1** *$ID(K_{2,2,2}) \supseteq \{0,3\}$, and $ID(K_{4,4,4}) \supseteq \{0,3,6,9,12\}$.*

For $K_{2,2,2}$, take the vertex sets $\{1,1'\} \cup \{2,2'\} \cup \{3,3'\}$. Then disjoint D-decompositions are given by $\{(1,3,2)\text{–}1',\ (3,2',1')\text{–}3',\ (1,2',3')\text{–}2\}$ and $\{(1',3',2')\text{–}1,\ (3',2,1)\text{–}3,\ (1',2,3)\text{–}2'\}$. Thus $\{0,3\} \subseteq ID(K_{2,2,2})$.

Now let the vertex sets for $K_{4,4,4}$ be $\{A,D\} \cup \{B,E\} \cup \{C,F\}$, where each letter here is itself a set of two points. Then we may take four decompositions of $K_{2,2,2}$ on the four sets $A \cup B \cup F$, $A \cup E \cup C$, $D \cup B \cup C$ and $D \cup E \cup F$, yielding 12 blocks for a D-decomposition of $K_{4,4,4}$. Then using the intersection values for $ID(K_{2,2,2})$ we obtain $ID(K_{4,4,4}) \supseteq \{0,3,6,9,12\}$. $\square$

For the general construction, we take the vertex set $V = \{(i,j) \mid 1 \le i \le 2m,\ 1 \le j \le 4\}$ if $n = 8m$, or $V \cup \{\infty\}$ if $n = 8m+1$.

Then if $2m \equiv 0$ or $2 \pmod 6$, $2m \ge 6$, we may use a GDD with group size 2 and block size 3 on $\{1, 2, \ldots, 2m\}$, while if $2m \equiv 4 \pmod 6$, $2m \ge 10$, we may use a GDD with one group of size 4 and the rest of size 2, and block size 3 on $\{1, 2, \ldots, 2m\}$. These exist; see for instance Lemma 2.1 in [1], or the general result in [8]. Then for each group $\{x_1, \ldots, x_g\}$ of the GDD, place a D-design on the set

$\{(x_i, j) \mid 1 \le i \le g,\ 1 \le j \le 4\}$ or on this set together with ∞. Since the group sizes are 2 or 4, this means we require D-designs of orders 8, 9, 16 and 17. And for each block $\{a, b, c\}$ of the GDD, place a D-decomposition of $K_{4,4,4}$ on $\{(a, j) \mid 1 \le j \le 4\} \cup \{(b, j) \mid 1 \le j \le 4\} \cup \{(c, j) \mid 1 \le j \le 4\}$.

It now remains to deal with orders 8, 9, 16 and 17.

Example **4.2** $ID(8) = \{0, 1, \ldots, 5, 7\}$.

Take the vertex set $\{\infty\} \cup Z_7$, and blocks $B = \{(i, 1+i, 3+i)\text{–}\infty \mid i \in Z_7\}$. Note the following trades.

$$\begin{aligned}
&X = \{(1,2,4)\text{–}\infty,\ (3,4,6)\text{–}\infty\} \text{ trades with } X' = \{(1,2,4)\text{–}3,\ (\infty,4,6)\text{–}3\},\\
&Y = \{(2,3,5)\text{–}\infty,\ (4,5,0)\text{–}\infty\} \text{ trades with } Y' = \{(2,3,5)\text{–}4,\ (\infty,5,0)\text{–}4\},\\
&Z = \{(5,6,1)\text{–}\infty,\ (0,1,3)\text{–}\infty\} \text{ trades with } Z' = \{(5,6,1)\text{–}0,\ (\infty,1,3)\text{–}0\},\\
&A = \{(0,1,3)\text{–}\infty,\ (2,3,5)\text{–}\infty,\ (5,6,1)\text{–}\infty\} \text{ trades with}\\
&A' = \{(0,3,1)\text{–}\infty,\ (2,5,3)\text{–}\infty,\ (6,1,5)\text{–}\infty\}.
\end{aligned}$$

Here X, Y and Z are pairwise disjoint, and A is also disjoint from X. Thus we achieve the following intersection values, where α below denotes the permutation $(1\,\infty)$ applied to B.

trades	blocks changed	intersection achieved
$B\alpha$	7	0
X, Y, Z	6	1
X, A	5	2
X, Y	4	3
A	3	4
X	2	5
nothing	0	7

□

Example **4.3** $ID(9) = \{0, 1, \ldots, 7, 9\}$.

With vertex set Z_9, let $D = \{(i, i+1, i+4)\text{–}(i+6) \mid i \in Z_9\}$. The following trades are disjoint:

$$\begin{aligned}
&X = \{(1,2,5)\text{–}7,\ (4,5,8)\text{–}1\} \text{ trades with } X' = \{(8,4,5)\text{–}7,\ (2,5,1)\text{–}8\},\\
&Y = \{(2,3,6)\text{–}8,\ (5,6,0)\text{–}2\} \text{ trades with } Y' = \{(3,6,2)\text{–}0,\ (0,5,6)\text{–}8\},\\
&Z = \{(0,1,4)\text{–}6,\ (3,4,7)\text{–}0,\ (6,7,1)\text{–}3\} \text{ trades with}\\
&Z' = \{(3,1,7)\text{–}6,\ (0,7,4)\text{–}3,\ (6,4,1)\text{–}0\}.
\end{aligned}$$

Now denote permutations by $\alpha = (0\,1)$, $\beta = (1\,2\,5)$, $\gamma = (1\,2\,3\,4)$, and let $T = \{(7,8,2)\text{–}4,\ (8,0,3)\text{–}5\}$. The following table then completes this example.

blocks	intersection size
$D \cap D\gamma$	0
$D \cap D\beta$	1
$D \cap \{X' \cup Y' \cup Z' \cup T\}$	2
$D \cap D\alpha$	3
$D \cap \{X \cup Y' \cup Z' \cup T\}$	4
$D \cap \{X' \cup Y' \cup Z \cup T\}$	5
$D \cap \{X \cup Y \cup Z' \cup T\}$	6
$D \cap \{X' \cup Y \cup Z \cup T\}$	7
$D \cap D$	9

□

EXAMPLE **4.4** $ID(16) = \{0, 1, \ldots, 28, 30\}$.

With vertex set $Z_{15} \cup \{\infty\}$, a design is given by

$$\{(i, 1+i, 6+i)\text{–}(8+i),\ \ (i, 3+i, 7+i)\text{–}\infty\} \text{ where } i \in Z_{15}.$$

Now blocks A_i trade with A_i' for $0 \leq i \leq 6$ where

$$\begin{aligned} A_i &= \{(i, 3+i, 7+i)\text{–}\infty,\ \ (7+i, 10+i, 14+i)\text{–}\infty\} \text{ and} \\ A_i' &= \{(i, 3+i, 7+i)\text{–}(10+i),\ \ (7+i, \infty+i, 14+i)\text{–}(10+i)\}. \end{aligned}$$

Disjoint from these trades are the following five trades, B_i with B_i', for $0 \leq i \leq 4$, where

$$B_i = \{(i, 1+i, 6+i)\text{–}(8+i),\ (5+i, 6+i, 11+i)\text{–}(13+i),\ (10+i, 11+i, 1+i)\text{–}(3+i)\}$$

and

$$B_i' = \{(i, 6+i, 1+i)\text{–}(3+i),\ (5+i, 11+i, 6+i)\text{–}(8+i),\ (10+i, 1+i, 11+i)\text{–}(13+i)\}$$

(addition in Z_{15}). Thus we have trades on $2a + 3b$ blocks, where $0 \leq a \leq 7$ and $0 \leq b \leq 5$. This means that we may trade $2a + 3b = c$ blocks for $2 \leq c \leq 29$. Thus $\{1, 2, \ldots, 28\} \subseteq ID(16)$. And trivially $30 \in ID(16)$. Finally, to show $0 \in ID(16)$, let

$$X = \{(6, 9, 13)\text{–}\infty,\ (13, 1, 5)\text{–}\infty,\ (14, 2, 6)\text{–}\infty\}$$

which trades with

$$X' = \{(14, 2, 6)\text{–}9\Phi,\ (1, 15, 13)\text{–}9,\ (13, 6, \infty)\text{–}15\}.$$

Thus trading $\{B_i\}_{i=0}^{4} \cup \{A_i\}_{i=0}^{5} \cup \{X\}$ will change *all* the blocks, so $0 \in ID(16)$. This concludes the example. □

EXAMPLE **4.5** $ID(17) = \{0, 1, \ldots, 32, 34\}$.

Let the vertex set be Z_{17}. Then a design is given by

$$D = \{(i, i+3, i+8)\text{–}(i+12),\ (i, i+1, i+7)\text{–}(i+9) \mid i \in Z_{17}\}.$$

Let permutations on Z_{17} be given by

$$\begin{array}{lll} \alpha_0 = (0\,1\,2\,3\,4\,5\,6\,7\,8), & \alpha_1 = (0\,1\,2\,3\,4\,5\,6\,7\,8\,9\,10), & \\ \alpha_2 = (0\,1)(2\,3\,4\,5\,6), & \alpha_3 = (0\,1)(2\,3\,4\,5), & \alpha_4 = (0\,1\,2\,3\,4). \end{array}$$

Then $|D \cap D\alpha_i| = i$, $0 \le i \le 4$, so $\{0,1,2,3,4\} \subseteq ID(17)$. For the remaining intersection values we consider trades as follows.
The set $A_i = \{(1,4,9)\text{–}13,\ (13,16,4)\text{–}8\} + i \pmod{17}$ trades with $A_i' = \{(16,4,13)\text{–}9,\ (9,1,4)\text{–}8\} + i \pmod{17}$, $0 \le i \le 4$. Disjoint from this are the blocks

$$B_i = \{(1,2,8)\text{–}10,\ (9,10,16)\text{–}1\} + i \pmod{17}$$

trading with

$$B_i' = \{(8,2,1)\text{–}16,\ (9,16,10)\text{–}8\} + i \pmod{17},$$

$0 \le i \le 7$. Also let

$$C_i = \{(0,3,8)\text{–}12,\ (12,15,3)\text{–}7,\ (11,12,1)\text{–}3\} + i,$$

which trades with

$$C_i' = \{(0,8,3)\text{–}7,\ (12,15,3)\text{–}1,\ (1,11,12)\text{–}8\} + i,$$

for $0 \le i \le 4$.

$$\begin{array}{llll} \text{Note that} & C_0 & \text{is disjoint from } A_i, & i = 0,1,2,3, \\ & C_1 & \text{is disjoint from } A_i, & i = 1,2,3,4, \\ & C_2 & \text{is disjoint from } A_i, & i = 0,2,3,4, \\ & C_3 & \text{is disjoint from } A_i, & i = 0,1,3,4, \\ & C_4 & \text{is disjoint from } A_i, & i = 0,1,2,4. \end{array}$$

Thus we may obtain trades of sizes 2, 3, ...,28, 29, yielding $\{5, 6, \ldots, 31, 32\} \subseteq ID(17)$. Finally, $34 \in ID(17)$ trivially. This completes the example. □

Now combining the results of this section we have

Theorem 4.1 *The intersection numbers for D-designs are given by* $ID(n) = \{0, 1, \ldots, b-2, b\}$ *where* $b = n(n-1)/8$. □

5 The graph Y

A Y-design of order n contains $n(n-1)/8$ blocks, and so $n \equiv 0$ or $1 \pmod 8$. The only ingredients we need are Y-designs of orders 8 and 9, a Y-decomposition of $K_{4,4}$, and their intersection numbers. (In fact, it suffices to use $IY(K_{4,4}) \supseteq \{0, 4\}$.)

EXAMPLE **5.1** $IY(K_{4,4}) \supseteq \{0, 4\}$.

Let the vertex set be $\{1, 2, 3, 4\} \cup \{5, 6, 7, 8\}$. Then two disjoint decompositions are given by

$$\{(4, 7, 1; 5, 6),\ (1, 8, 2; 6, 7),\ (2, 5, 3; 7, 8),\ (3, 6, 4; 5, 8)\}$$

and

$$\{(8, 3, 5; 1, 2),\ (5, 4, 6; 2, 3),\ (6, 1, 7; 3, 4),\ (7, 2, 8; 1, 4)\}.$$

□

EXAMPLE **5.2** $IY(8) = \{0, 1, 2, 3, 4, 5, 7\}$.

With vertex set $\{\infty\} \cup Z_7$, take blocks $D = A \cup B \cup C$ where

$$\begin{aligned} A &= \{(0, 1, 3; 6, \infty),\ (1, 2, 4; 0, \infty)\},\ B = \{(2, 3, 5; 1, \infty),\ (3, 4, 6; 2, \infty)\}, \\ C &= \{(4, 5, 0; 3, \infty),\ (5, 6, 1; 4, \infty),\ (6, 0, 2; 5, \infty)\}. \end{aligned}$$

Blocks A trade with $A' = \{(6, 3, 1; 0, 2),\ (3, \infty, 4; 0, 2)\}$,
blocks B trade with $B' = \{(1, 5, 3; 2, 4),\ (5, \infty, 6; 2, 4)\}$ and
blocks C trade with $C' = \{(4, 5, 0; 3, 2),\ (4, 1, 6; 5, 0),\ (5, 2, \infty; 1, 0)\}$.
Now let α denote the permutation $(0\,1)$ and β the permutation $(0\,1\,2)$. We obtain the following intersection numbers, which completes the result.

blocks	intersection
$D \cap D\beta$	0
$D \cap D\alpha$	1
$D \cap \{A \cup B' \cup C'\}$	2
$D \cap \{A' \cup B' \cup C\}$	3
$D \cap \{A \cup B \cup C'\}$	4
$D \cap \{A \cup B' \cup C\}$	5
$D \cap D$	7

□

EXAMPLE **5.3** $IY(9) = \{0, 1, \ldots, 7, 9\}$.

Let the vertex set be Z_9, and blocks be $D = \{(0 + i, 1 + i, 3 + i; 6 + i, 7 + i) \mid i \in Z_9\}$ (addition mod 9). The blocks

$$A_i = \{(i - 1, i, 2 + i; 5 + i, 6 + i),\ (i, 1 + i, 3 + i; 6 + i, 7 + i)\},\ 1 \le i \le 4,$$

trade with

$$A_i' = \{(5 + i, 2 + i, i; i - 1, i + 1),\ (2 + i, 6 + i, 3 + i; 1 + i, 7 + i)\},\ 1 \le i \le 4.$$

Also the blocks

$$\begin{aligned} B_i \ = \ & \{(3i - 3, 3i - 2, 3i; 3i + 3, 3i + 4), \\ & (3i - 2, 3i - 1, 3i + 1; 3i + 4, 3i + 5),\ (3i - 1, 3i, 3i + 2; 3i + 5, 3i + 6)\}, \end{aligned}$$

$1 \leq i \leq 3$, trade with the blocks

$$\begin{aligned} B_i' &= \{(3i+3, 3i, 3i-2; 3i-3, 3i-1), \\ &\quad (3i, 3i+4, 3i+1; 3i-1, 3i+5),\ (3i+2, 3i+5, 3i+1; 3i+4, 3i-1)\}, \end{aligned}$$

$1 \leq i \leq 3$. Thus we obtain the required intersection numbers:

blocks	intersection
$D \cap \{B_1' \cup B_2' \cup B_3'\}$	0
$D \cap \{\{(8, 0, 2; 5, 6)\} \cup \{A_i' \mid 1 \leq i \leq 4\}\}$	1
$D \cap \{A_1 \cup A_2' \cup A_3' \cup B_3'\}$	2
$D \cap \{A_1' \cup A_2' \cup A_3' \cup B_3\}$	3
$D \cap \{A_1 \cup A_2 \cup A_3' \cup B_3'\}$	4
$D \cap \{A_1' \cup A_2' \cup A_3 \cup B_3\}$	5
$D \cap \{A_1 \cup A_2 \cup A_3 \cup B_3'\}$	6
$D \cap \{A_1' \cup A_2 \cup A_3 \cup B_3\}$	7
$D \cap D$	9

□

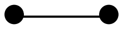
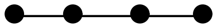
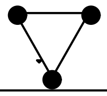
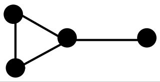
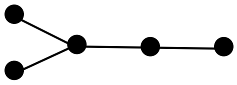
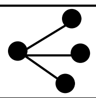
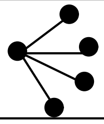
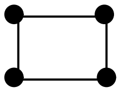
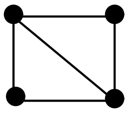
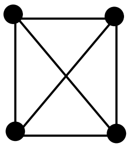
Thanks to Lemma 1.1 we now have

THEOREM **5.1** *The intersection numbers for Y-designs are given by $IY(n) = \{0, 1, \ldots, b-2, b\}$ where $b = n(n-1)/8$.* □

6 Summary

The following table summarises the intersection results for G-designs where G is a connected graph on at most four vertices or at most four edges.

In the table, b denotes the number of blocks in a G-design of order n, and the impossible intersection values are $b - x$ where x is as given. A reference is listed if the result is not in this paper.

G	b	x	*Comments*	*Ref*
K_2 	$n(n-1)/2$	*all except* b	*unique design!*	
P_3 	$n(n-1)/4$	*1*	$n \equiv 0, 1$ *(mod 4)*	
P_4 	$n(n-1)/6$	*1*	$n \equiv 0, 1$ *(mod 3)*, $n \geq 4$	
P_5 	$n(n-1)/8$	*1*	$n \equiv 0, 1$ *(mod 8)*	
K_3 	$n(n-1)/6$	*1, 2, 3, 5*	$n \equiv 1, 3$ *(mod 6)*, $5, 8 \notin IK_3(9)$.	*[9]*
D 	$n(n-1)/8$	*1*	$n \equiv 0, 1$ *(mod 8)*	
Y 	$n(n-1)/8$	*1*	$n \equiv 0, 1$ *(mod 8)*	
S_3 	$n(n-1)/6$	*1*	$n \geq 6$, $n \equiv 0, 1$ *(mod 8)*, $3 \notin IS_3(6)$	
S_4 	$n(n-1)/8$	*1*	$n \equiv 0, 1$ *(mod 8)*, $5 \notin IS_4(8)$	
C_4 	$n(n-1)/8$	*1*	$n \equiv 1$ *(mod 8)*	*[4]*
$K_4 - e$ 	$n(n-1)/10$	*1, 2*	$n \equiv 0, 1$ *(mod 5)*, $n \geq 6$; $7, 8 \notin I(11)$	*[5]*
K_4 	$n(n-1)/12$	*1, 2, 3, 4, 5, 7*	$n \equiv 1, 4$ *(mod 12)*; $7, 9, 10, 11, 14 \notin I(16)$; *several unknown values for* $n = 25, 28, 37$.	*[7]*

References

[1] P. Adams, E.J. Billington and C.C. Lindner, *k-perfect $3k$-cycle systems*, J. Combin. Math. Combin. Computing **15** (1994), 141–154.

[2] J.C. Bermond and J. Schönheim, *G-decomposition of K_n, where G has four vertices or less*, Discrete Math. **19** (1977), 113–120.

[3] E.J. Billington, *The intersection problem for combinatorial designs*, Congressus Numerantium (to appear).

[4] E.J. Billington, *The intersection problem for m-cycle systems*, J. Combinatorial Designs **1** (1993), 435–452.

[5] E.J. Billington, M. Gionfriddo and C.C. Lindner, *The intersection problem for $K_4 - e$ designs*, J. Stat. Plann. Inference (to appear).

[6] A.E. Brouwer, *Optimal packings of K_4's into a K_n*, J. Combinatorial Theory, series A **26** (1979), 278–297.

[7] C.J. Colbourn, D.G. Hoffman and C.C. Lindner, *Intersections of $S(2,4,v)$ designs*, Ars Combinatoria **33** (1992), 97–111.

[8] C.J. Colbourn, D.G. Hoffman and R. Rees, *A new class of group divisible designs with block size three*, J. Combinatorial Theory, series A **59** (1992), 73–89.

[9] C.C. Lindner and A. Rosa, *Steiner triple systems having a prescribed number of triples in common*, Canad. J. Math. **27** (1975), 1166–1175. Corrigendum: Canad. J. Math. **30** (1978), 896.