# The Steiner Grid graphical t-wise balanced designs

**Draft: 28 April 1996**

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**ABSTRACT**

**We will hopefully determine all grid graphical $t$-wise balanced designs.**

## 1 Introduction

A *$t$–wise balanced design* ($t$BD) of type $t$–$(v, \mathcal{K}, \lambda)$ is a pair $(X, \mathcal{B})$ where $X$ is a $v$–element set of points and $\mathcal{B}$ is a family of distinct subsets of $X$ called blocks such that: (1) $\mathcal{K}$ is a set of positive integers strictly between $t$ and $v$; (2) the size of every block is in $\mathcal{K}$; and (3) every $t$–element subset of $X$ is contained in exactly $\lambda$ blocks. Note that $X$ is not a block. Also if we give the parameters of a specific $t$BD, then we will choose a minimal $\mathcal{K}$. The index of a $t$BD is $\lambda$. All of the designs in this paper have index 1.

Let $G_{m,n}$ be the line graph of $K_{m,n}$. So $G_{m,n}$ is an $m$ by $n$ array of points with two points adjacent if and only if they either belong to the same row or the same column. Thus, $G_{m,n}$ has $m\binom{n}{2}+n\binom{m}{2}$ edges. We call this graph the *$m$ by $n$ grid graph.* The graph $G_{3,3}$ is given in Figure 1.

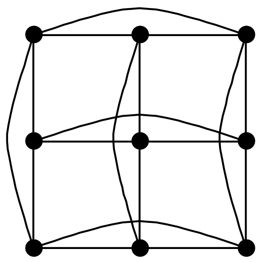


Figure 1: The graph $G_{3,3}$.

A $t$-wise balanced design $(X, \mathcal{B})$ of type $t$–$(m\binom{n}{2} + n\binom{m}{2}, \mathcal{K}, \lambda)$ is *grid graphical* if $X$ is the set of edges of the grid graph $G_{m,n}$ and whenever $B$ is a block and $\alpha$ is an automorphism of $G_{m,n}$ (that fixes the rows and the columns), then $\alpha(B)$ is also a block. That is $\mathcal{B}$ is a union of orbits under the action of the subgroup $\mathcal{G}$ of the automorphism

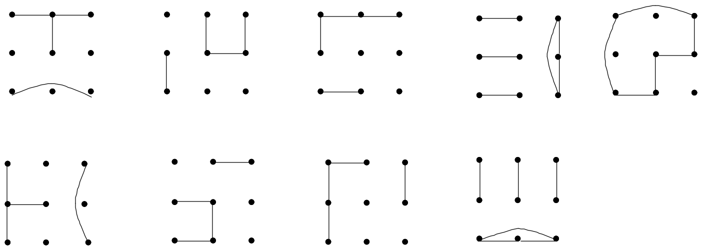


Figure 2: A grid graphical 3–(18, {4, 6}, 1) design.

group of $G_{m,n}$ that preserves the rows and preserves the columns. ($\mathcal{G} = \textbf{aut } G_{m,n}$ if $m \neq n$. ) Without loss we will always assume that $m \leq n$.

An Example of a grid graphical 3–(18, {4, 6}, 1) design is given in Figure 2.

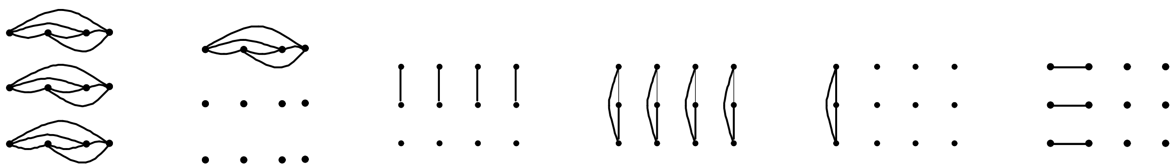
## 2 Notation and definitions

We first mention a few more definitions and notational conventions. The vertices of $G_{m,n}$ will be $\{g_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$ the $i$-th row is the set $R_i = \{g_{i,1}, g_{i,2}, \ldots, g_{i,m}\}$ and the $j$-th column is $C_i = \{g_{1,j}, g_{2,j}, \ldots, g_{n,j}\}$ . A *horizontal* edge has both of its ends in some row $R_i$ and a *vertical* edge has both ends in some column $C_j$. All of our subgraphs will contain only horizontal or vertical edges. The subgraph consisting of all and only horizontal edges is said to be *row complete* and we denote it by $RK_n$. Similarly we denote by $CK_m$ the *column complete* subgraph. A *row permutation* is denoted by a permutation $\pi$ of $\{R_1, R_2, \ldots, R_m\}$ and if $\pi : R_i \mapsto R_{i'}$, then $\pi(g_{i,j}) = g_{i',j}$ for all $j$. For example the *row transposition* $(R_1, R2)$ is the permutation $(g_{11}, g_{21})(g_{12}, g_{22})(g_{13}, g_{23}) \cdots (g_{1n}, g_{2n})$. *Column permutations* and *column transposition* are defined similarly. We restrict our attention to the group $\Gamma_{m,n}$ generated by the row and column permutations. This is the automorphism group of $G_{m,n}$ if $m \neq n$. (If $n = m$, then there are automorphisms that interchange rows with columns.) The *orbit of a subgraph* will mean the orbit under the group $\Gamma_{m,n}$. The *transpose of a subgraph* will be the subgraph obtained by interchanging the roles of the row and column vertices. Notice that transpose is not an automorphism in $\Gamma_{m,n}$, even when $m = n$. The notation $G_1 \dot{\cup} G_2$ will denote the disjoint union of subgraphs $G_1$ and $G_2$. The symbols $B$ or $B_i$ will always be blocks and the symbols $T$ and $T_i$ will always be a $t$-element subset of edges. The symbols $v$, $v'$, $v_i$ will invariably be vertical edges and $h$, $h'$, $h_i$ will horizontal edges.

Through out this paper suppose that $(X, \mathcal{B})$ is a nontrivial grid graphical $t$-wise balance $t$–$(m\binom{n}{2} + n\binom{m}{2}, \mathcal{K}, 1)$ design. Suppose $m \leq n$.

## 3 $t = 1$

**Lemma 3.1.** *If $4 \leq m \leq n$, then no block in a Steiner grid graphical 1BD contains both horizontal and vertical edges.*



(i) $RK_4$ (ii) $KR$ (iii) $RM$ (iv) $CK_4$ (v) $KC$ (vi) $CM$

Figure 3: Some orbits of 3 by 4 grid subgraphs.

*Proof.* Let $(X, \mathcal{B})$ be a nontrivial Steiner grid graphical 1BD and suppose $B \in \mathcal{B}$ contains the horizontal edge $h$ and the vertical edge $v$.

If $v$ and $h$ are adjacent then we may assume without loss that $h = \{g_{11}, g_{12}\}$ and $v = \{g_{11}, g_{21}\}$. The column transposition $\tau = (C_1, C_2)$ fixes $h$ and thus also $B$. Consequently $\tau(v) = \{g_{12}, g_{22}\} \in B$. Now the column transposition $\sigma = (C_2, C_j)$, $j = 3, 4, \ldots$ each fix $v$ and $\sigma(\tau(v)) = \{g_{1j}, g_{2j}\}$. Thus $B$ contains all of the vertical edges between rows 1 and 2. Since each of the column permutations and the row transposition $(R1_, R_2)$ fix this set of vertical edges $B$ is forced to be complete. Therefor $B$ cannot contain adjacent horizontal and vertical edges.

If $v$ and $h$ are nonadjacent, then we may assume without loss that $h$ is in row 1 and $v$ is in column 1. Applying, in this order, column permutations that fix column 1, row permutations that fix row 1, the column transposition $(C_1, C_2)$ and finally the row transposition $(R_1, R_2)$, will force $B$ to contain a horizontal edge adjacent to a vertical edge. This is contrary to the above. $\square$

The orbit of subgraphs containing the subgraph consisting of the all the edges in a single column will be denoted by $KC$ for *complete column*. The orbit of subgraphs containing the subgraph consisting of the $m$ vertical edges between a given pair of columns will be denoted by $CM$ for *column matching*. We define $KR$ and $RM$ similarly. See Figure 3 If $m = 4$ we denote the special graph consiting of two disjoint row matchings by $DRM$. The graph $DRM$ is given in Figure 4.

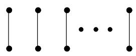


Figure 4: The graph $DRM$.

**Lemma 3.2.** *If $B$ is a block of a $m$ by $n$ Steiner grid graphical 1BD, then*

i. *for each $i = 1, 2, \ldots, m$, the $i$–th row $B \cap R_i$ is either complete, consists of a single edge or consists of two disjoint edges and $n = 4$;*

ii. *for each $j = 1, 2, \ldots, m$, the $j$–th column $B \cap C_j$ is either complete, consists of a single edge or consists of two disjoint edges and $m = 4$;*

*Proof.* If a row (column) of the block $B$ contains two horizontal (vertical) edges, then applying all column (row) transpositions that fix a horizontal (vertical) edge will force $B$ to be complete in such a row (column), unless $n = 4$ ($m = 4$) and the two edges are disjoint. □

Applying Lemma's 3.1. and 3.2. is is easy to obtain Theorem 3.3.

**Theorem 3.3.** *The only $m$ by $n$ Steiner grid graphical 1-wise balanced designs are those that appear in Table 1*

**Table 1** The Steiner grid graphical 1-wise balanced designs

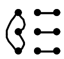
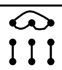
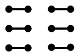
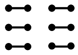
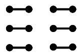
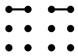
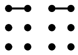
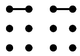
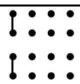
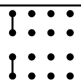
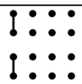
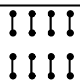
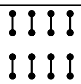
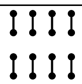
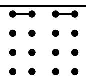
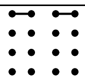
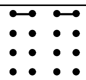
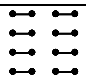
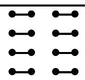
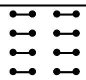
| | Parameters | Grid Graphical Representation |
|---|---|---|
| $D_1$: | $2 \le m \le n$<br>$1\!-\!(n\binom{m}{2}+m\binom{n}{2},,1)$ | $RK_n \quad CK_n$ |
| $D_2$: | $2 \le m \le n$<br>$1\!-\!(n\binom{m}{2}+m\binom{n}{2},\{m\binom{n}{2},n\},1)$ | $RK_n \quad RM$ |
| $D_3$: | $2 \le m \le n$<br>$1\!-\!(n\binom{m}{2}+m\binom{n}{2},\{m\binom{n}{2},\binom{m}{2}\},1)$ | $RK_n \quad KC$ |
| $D_4$: | $2 \le m \le n$<br>$1\!-\!(n\binom{m}{2}+m\binom{n}{2},\{m,n\binom{m}{2}\},1)$ | $CM \quad CK_n$ |
| $D_5$: | $2 \le m \le n$<br>$1\!-\!(n\binom{m}{2}+m\binom{n}{2},\{m,n\},1)$ | $CM \quad RM$ |
| $D_6$: | $2 \le m \le n$<br>$1\!-\!(n\binom{m}{2}+m\binom{n}{2},\{m,\binom{m}{2}\},1)$ | $CM \quad KC$ |
| $D_7$: | $2 \le m \le n$<br>$1\!-\!(n\binom{m}{2}+m\binom{n}{2},\{\binom{n}{2},n\binom{m}{2}\},1)$ | $KR \quad CK_n$ |
| $D_8$: | $2 \le m \le n$<br>$1\!-\!(n\binom{m}{2}+m\binom{n}{2},\{\binom{n}{2},n\},1)$ | $KR \quad RM$ |
| $D_9$: | $2 \le m \le n$<br>$1\!-\!(n\binom{m}{2}+m\binom{n}{2},\{\binom{n}{2},\binom{m}{2}\},1)$ | $KR \quad KC$ |
| $D_{10}$: | $4 = m \le n$<br>$1\!-\!(6n+4\binom{n}{2},\{4\binom{n}{2},2n\},1)$ | $RK_n \quad DRM$ |
| $D_{11}$: | $4 = m \le n$<br>$1\!-\!(6n+4\binom{n}{2},\{4,2n\},1)$ | $CM \quad DRM$ |
| $D_{12}$: | $4 = m \le n$<br>$1\!-\!(6n+4\binom{n}{2},\{\binom{n}{2},2n\},1)$ | $KR \quad DRM$ |
| $D_{13}$: | $m=3, n=3$<br>$1\!-\!(18,2,1)$ |  |
| $D_{14}$: | $m=3, n=3$<br>$1\!-\!(18,6,1)$ |  |
| $D_{15}$: | $m=3, n=3$<br>$1\!-\!(18,6,1)$ |  |

*Table 1* continued.

| | Parameters | Grid Graphical Representation | |
|---|---|---|---|
| $D_{16}$: | $m=3, n=4$ <br> $1-(30,\{4,6\},1)$ | $RM$ |  |
| $D_{17}$: | $m=3, n=4$ <br> $1-(30,\{6,12\},1)$ | $CK_m$ |  |
| $D_{18}$: | $m=3, n=4$ <br> $1-(30,\{3,6\},1)$ | $KC$ |  |
| $D_{19}$: | $m=3, n=4$ <br> $1-(30,\{2,4\},1)$ | $RM$ |  |
| $D_{20}$: | $m=3, n=4$ <br> $1-(30,\{2,6\},1)$ | $CK_m$ |  |
| $D_{21}$: | $m=3, n=4$ <br> $1-(30,\{2,3\},1)$ | $KC$ |  |
| $D_{22}$: | $m=4, n=4$ <br> $1-(48,\{2,4\},1)$ | $CM$ |  |
| $D_{23}$: | $m=4, n=4$ <br> $1-(48,\{2,18\},1)$ | $RK_m$ |  |
| $D_{24}$: | $m=4, n=4$ <br> $1-(48,\{2,6\},1)$ | $KR$ |  |
| $D_{23}$: | $m=4, n=4$ <br> $1-(48,\{4,8\},1)$ | $CM$ |  |
| $D_{24}$: | $m=4, n=4$ <br> $1-(48,\{8,18\},1)$ | $RK_m$ |  |
| $D_{25}$: | $m=4, n=4$ <br> $1-(48,\{6,8\},1)$ | $KR$ |  |
| $D_{26}$: | $m=4, n=4$ <br> $1-(48,\{2,4\},1)$ | $RM$ |  |
| $D_{27}$: | $m=4, n=4$ <br> $1-(48,\{2,18\},1)$ | $CK_m$ |  |
| $D_{28}$: | $m=4, n=4$ <br> $1-(48,\{2,6\},1)$ | $KC$ |  |
| $D_{29}$: | $m=4, n=4$ <br> $1-(48,\{4,8\},1)$ | $RM$ |  |
| $D_{30}$: | $m=4, n=4$ <br> $1-(48,\{8,18\},1)$ | $CK_m$ |  |
| $D_{31}$: | $m=4, n=4$ <br> $1-(48,\{6,8\},1)$ | $KC$ |  |

# 4 $t = 2$

**Lemma 4.1.** *If $2 \leq m \leq 5$, there does not exist an $m$ by 5 Steiner grid graphical 2BD.*

*Proof.* Suppose $(X, \mathcal{B})$ is a $m$ by $n = 5$ gridgraphical 2BD, $2 \leq m \leq 5$. Then $\mathcal{B}$ has a block $B$ which contains
$$T_1 = \{\{g_{11}, g_{12}\}, \{g_{22}, g_{23}\}\}.$$
It is easy to see that the only possible such block is B=$RK_5$.

Also $\mathcal{B}$ has a block $B$ which contains

$$T_2 = \{\{g_{11}, g_{12}\}, \{g_{11}, g_{21}\}\}.$$

It is easy to see that the only possible such blocks are the square:

$$B_1 = \{\{g_{11}, g_{12}\}, \{g_{11}, g_{21}\}, \{g_{21}, g_{22}\}, \{g_{12}, g_{22}\}\}$$

and the star:

$$B_2 = \{\{g_{11}, g_{12}\}, \{g_{11}, g_{13}\}, \{g_{11}, g_{14}\}, \{g_{11}, g_{15}\}, \{g_{11}, g_{21}\}\}.$$

But $|B \cap B_1| = 2$ and $|B \cap B_2| = 4$ contrary to index equal to 1. □

**Lemma 4.2.** *If $n \geq 4$, then there does not exist a Steiner grid graphical 2–wise balanced design.*

*Proof.* Let $T = \{\{g_{11}, g_{12}\}, \{g_{23}, g_{24}\}\}$. If $B$ is the block containing $T$, then $B$ contains an additional edge $e$. Either $e$ is vertical or horizontal.

Suppose that $e$ is a vertical edge is column $j$. If $1 \leq j \leq 4$ , then we apply the automorphisms

$$\begin{array}{l}(R_1, R_2)(C_1, C_2)(C_3, C_4),\\ (R_1, R_2)(C_1, C_3)(C_2, C_4), \text{and}\\ (R_1, R_2)(C_1, C_4)(C_2, C_3).\end{array}$$

They each fix $T$ and hence fix the block $B$, moreover they are transitive on columns $C_1$, $C_2$, $C_3$ and $C_4$. Consequently, there is a vertical edge in each of these four columns of the block $B$. Thus every column transpositions fixes at least two edges of $B$. Hence rows $R_1$ and $R_2$ are complete and each contains at least 2 edges. Therefore applying row transpositions we see that $B$ is forced to be complete. Consequently, there cannot be a vertical edge among columns $C_1$, $C_2$, $C_3$ and $C_4$.

If $e$ is a vertical edge not in the first four columns we may assume that $e$ is in column 5. If $n \geq 6$, then transposing columns $C_5$ and $C_6$, fixes $T$ and thus there is a vertical edge in column $C_6$. Now the column transposition $(C_1, C_5)$ will fix two edges and force $B$ to contain a vertical edge in column $C_1$ contrary to the above. Hence $n = 5$, but this is also impossible by Lemma 4.1.. Therefore $e$ must be horizontal.

If $m \geq 4$, then regardless of the row that $e$ is in there will be a row transposition that fixes 2 edges. Applying all row permutations we see that all rows have the "same set of edges". I.e. if $\{g_{i_1,j_1}, g_{i_1,j_2}\}$ is an edge in row $i_1$ then $\{g_{i_2,j_1}, g_{i_2,j_2}\}$ is an edge in row $i_2$ for all rows $i_1, i_2$. Now since $T$ spans more, than 3 columns we see by applying column permutations that $B$ is row complete.

Consequently the only block $B$ that can cover the two edge graph

$$\{\{g_{11}, g_{12}\}, \{g_{23}, a_{24}\}\}$$

is the row complete graph $RK_n$. Thus it must be a block.

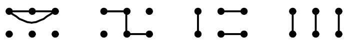
Transposing this argument we that the column complete graph $CK_m$ is also a block. It is now impossible to cover any two edge subgraph containing both a horizontal and a vertical edge. Thus $m = 2$ or $n = 3$.

$\square$

**Lemma 4.3.** *The Steiner grid graphical* $2$*–wise balanced design, is the* $2$–$(9, 3, 1)$ *design* $D_3 2$ *which appears in Table 2*

*Proof.* If there is a Steiner $m$ by $n$ grid graphical design, then we know by Lemma 4.2. that $n \leq 3$. An complete search by hand or computer is feasible and the result is easily obtained. $\square$

**Table 1** The Steiner grid graphical 2-wise balanced designs

| | Parameters | Grid Graphical Representation |
|---|---|---|
| $D_{32}$: | $m = 2, n = 3$ <br> $2$–$(9, 3, 1)$ |  |

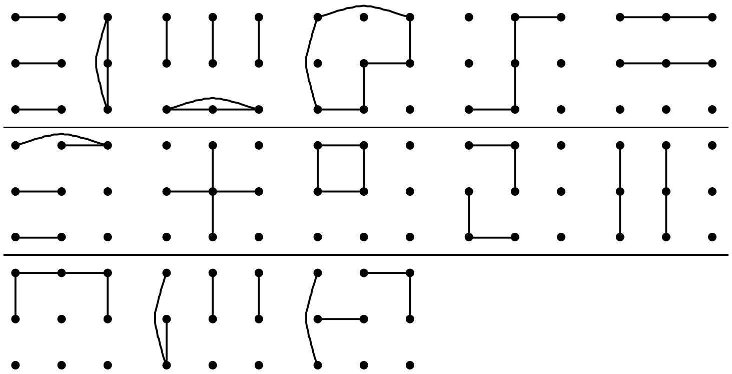
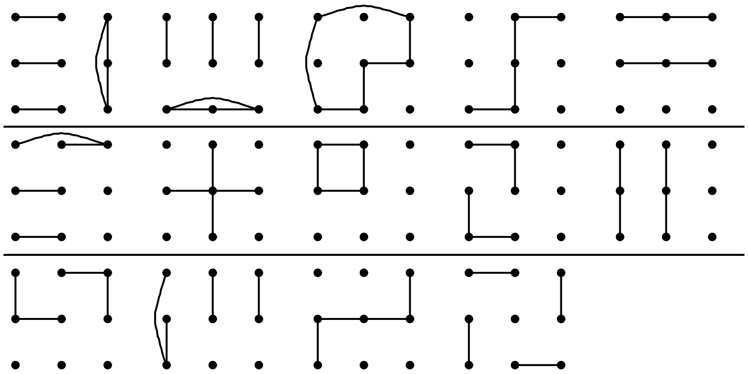
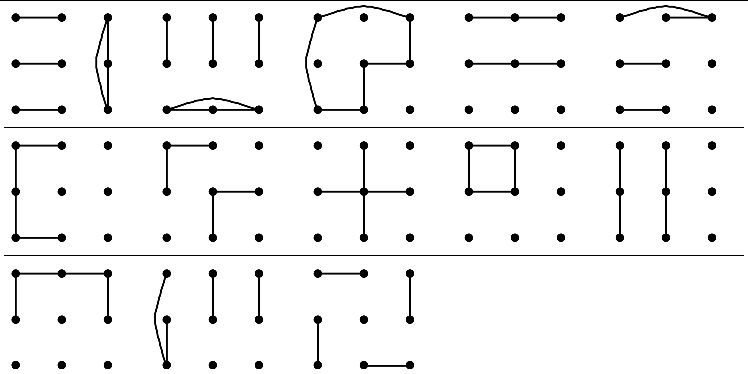
## 5 $t = 3$

**Lemma 5.1.** *If* $t = 3$ *and* $m \geq 3$*, then* $n \leq 5$*. The only Steiner grid graphical* $3$*–wise balanced designs*

*Proof.* Suppose that $t = 3$, $m \geq 3$ and $n \geq 5$. Let $T = \{\{g_{11}, g_{12}\}, \{g_{22}, g_{23}\}, \{g_{31}, g_{33}\}$. and observe that $\pi = (R_1, R_2, R_3)(C_1, C_2, C_3)$ is an automorphism of $T$. Let $B \in \mathcal{B}$ be the block that contains $T$. Then there is an edge $e \in B \setminus T$.

Suppose that $e$ is a vertical edge. If $e$ is in cloumns $C_1$, $C_2$, $C_3$. Then applying $\pi$ we see that there must be a vertical edge in each of columns $C_1$, $C_2$ and $C_3$. Every column transpositions (will now fix at least $\square$

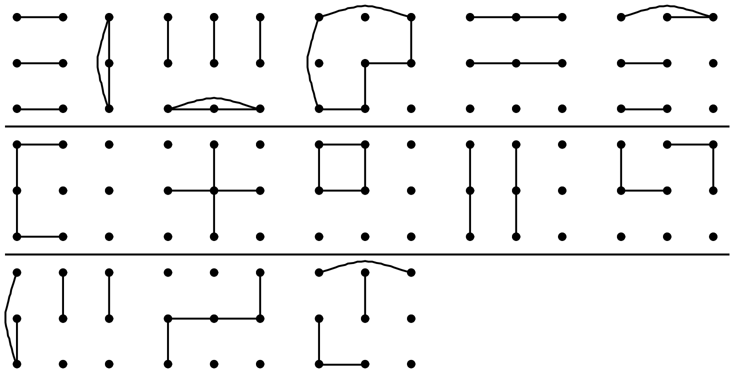
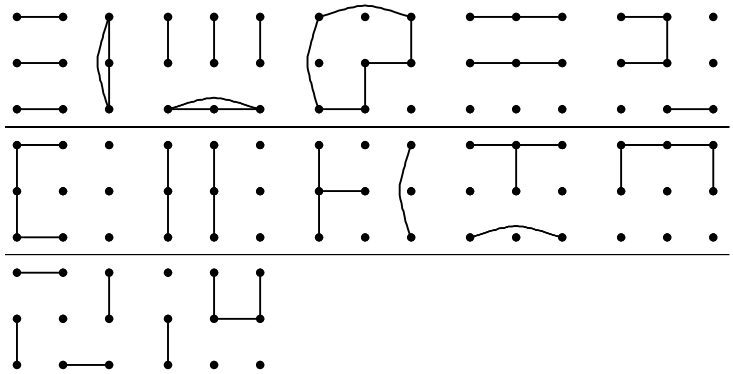
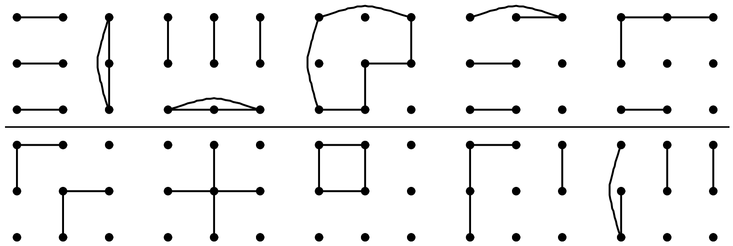
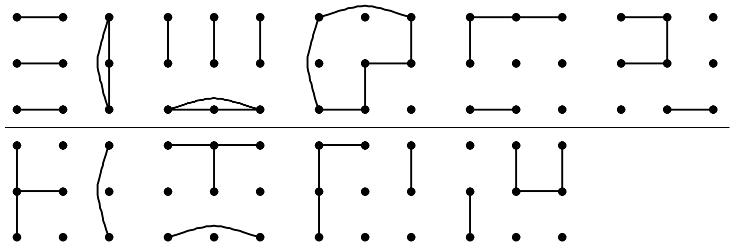
**Table 3** The Steiner grid graphical 3-wise balanced designs

| Parameters | Grid Graphical Representation |
|---|---|
| $D_{33}$: $m = 3, n = 3$ <br> $3\!-\!(18, \{4, 6\}, 1)$ |  |
| $D_{34}$: $m = 3, n = 3$ <br> $3\!-\!(18, \{4, 6\}, 1)$ |  |
| $D_{35}$: $m = 3, n = 3$ <br> $3\!-\!(18, \{4, 6\}, 1)$ |  |

cyclic graph (a) cyclic graph (b) cyclic graph (c) cyclic graph (d)

(a) (b) (c) (d)

*Table 3* continued.

| Parameters | Grid Graphical Representation |
|---|---|
| $D_{36}$: $m = 3, n = 3$ $3{-}(18, \{4, 6\}, 1)$ |  |
| $D_{37}$: $m = 3, n = 3$ $3{-}(18, \{4, 6\}, 1)$ |  |
| $D_{38}$: $m = 3, n = 3$ $3{-}(18, \{4, 6\}, 1)$ |  |
| $D_{39}$: $m = 3, n = 3$ $3{-}(18, \{4, 6\}, 1)$ |  |

**Lemma 5.2.** *Let $(X, \mathcal{B})$ be a $K_m \times K_n$ $t$–BD and write $t = \binom{x}{2} + y$, $0 \le y < t$. If $m \ge 4$ and $y \ge 4$, then there are four situations.*

i. *$x = y + 2 = m = n$ and the graph in Figure 3 (a) is a block.*

ii. $x = y + 1$, $n = y + 3$, $m = y + 1$ *and the graph in Figure 3 (b) is a block.*

iii. $x = y + 1$, $n = y + 3$, $m = y + 1$ *and the graph in Figure 3 (c) is a block.*

iv. $x = y + 1$, $n = y + 3$, $m = y + 1$ *and the graph in Figure 3 (d) is a block.*

*Proof.* Let $T$ be the $t$-edge subgraph of $K_m \times Kn$ defined by

$$T = K(\{g_{1,1}, g_{2,1}, g_{x,1}\}) \cup \{\alpha^i\{g_{1,1}, g_{1,2}\} : 0 \leq i < y\}$$

$\square$


## Acknowledgement

The authors are grateful to T. Norstrand for early discussions about this project.