An Infinite Family of (Simple) 6-Designs

Donald L. Kreher
Department of Mathematical Sciences
Michigan Technological University
Houghton, Michigan
U.S.A. 49931

June 14, 1996

Abstract

A simple 6-(22,8,60) designs is exhibited. It is then shown using Qui-rong Wu's generalization of a result of Luc Teirlinck that this design together with our 6-(14,7,4) design implies the existence of simple 6-(23+16m,8,4(m+1)(16m+17)) designs for all positive integers m. All the above mentioned designs are halvings of the complete design.

A t-(v,k,λ) design $(\mathcal{X},\mathcal{B})$ is a v-element set $\mathcal{X}$ of *points* and a collection $\mathcal{B}$ of k-element subsets of $\mathcal{X}$ called *blocks* such that every t-element subset of $\mathcal{X}$ is contained in precisely λ members of $\mathcal{B}$. It is *simple* if no two members of $\mathcal{B}$ are identical. All the designs in this paper will be simple designs. If every k-element subset of $\mathcal{X}$ is chosen to be in $\mathcal{B}$, then $(\mathcal{X},\mathcal{B})$ is a t-$(v,k,\overline{\lambda}(t,k,v))$ design where $\overline{\lambda}(t,k,v)=\binom{v-t}{k-t}$. In this case $(\mathcal{X},\mathcal{B})$ is said to be the *complete design*. A *large set* of t-$(v,k,\overline{\lambda}(t,k,v)/N)$ designs denoted by $\mathrm{LS}[N](t,k,v)$ is a partition, $[(\mathcal{X},\mathcal{B}_i)]_{i=1}^{N}$, of the complete design into N disjoint t-$(v,k,\overline{\lambda}(t,k,v)/N)$ designs. An $\mathrm{LS}[2](t,k,v)$ is called a *halving* of the complete design.

In Table I a simple 6-$(22,8,60)$ design is presented which is a halving of the complete design. This design was constructed while de Caen was visiting me using the methods described in [2]

Let $(\mathcal{X},\mathcal{A})$ be a 6-$(22,8,60)$ design, for example the one given in table 1. Using the LS[2](6,7,14) constructed in [2]. Teirlinck constructed an $\mathrm{LS}[2](6,7,6+8m)$ for all positive integers $m>0$, [4]. In particular when $m=2$ a 6-$(22,7,8)$ design $(\mathcal{X},\mathcal{B})$ is obtained. This design may also be constructed using Theorem 1 below. Set $\mathcal{Y}=\mathcal{X}\cup\{\infty\}$ and $\mathcal{C}=\mathcal{A}\cup\{B\cup\{\infty\}:B\in\mathcal{B}\}$. Then it is easy to check that $(\mathcal{Y},\mathcal{C})$ is a 6-$(23,8,68)$ design. Moreover, $\overline{\lambda}(6,8,23)=136$ so it too is a halving of the complete design, i.e. an LS[2](6,8,23).

If on the disjoint point sets $\mathcal{X}$ and $\mathcal{Y}$:

$\alpha=[(\mathcal{X},\mathcal{A}_i)]_{i=1}^{N}$, is an $\mathrm{LS}[N](t_1,k_1,v_1)$ and $\beta=[(\mathcal{Y},\mathcal{B}_i)]_{i=1}^{N}$, is an $\mathrm{LS}[N](t_2,k_2,v_2)$,

where $\mathcal{A}_i\uplus\mathcal{B}_{f(i)}=\{A\cup B: A\in\mathcal{A}_i \text{ and } B\in\mathcal{B}_{f(i)}\}$, then we define

$$\mathcal{C}(\alpha,\beta,f)=\bigcup_{i=1}^{N}\mathcal{A}_i\uplus\mathcal{B}_{f(i)}.$$

where f is any permutation of the symbols $1,2,\ldots,N$. Observe that $\mathcal{C}(\alpha,\beta,f)$ is simple. That is $\mathcal{C}(\alpha,\beta,f)$ contains no repeated blocks.

Table 1: A 6-(22, 8, 60) design

Automorphism group generators

(1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19)
(2 5 17 8 10 18 12 7 6)(3 9 14 15 19 16 4 13 11)
(1,20)(2,19)(3,10)(4,7)(5,15)(6,16)(8,9)(11,18)(12,13)(14,17)

Orbit representatives

1 2 3 4 5 6 7 9	0 1 2 3 4 5 6 9	0 1 2 3 5 6 8 9	0 1 2 3 4 5 8 9
0 1 2 3 4 7 8 9	0 1 2 3 5 7 8 9	1 2 3 5 6 7 9 10	1 2 3 5 6 8 9 12
1 3 4 5 6 7 8 9	1 2 3 4 6 7 8 9	0 1 3 4 6 7 8 9	0 1 3 5 6 7 8 9
0 1 3 4 5 6 8 10	0 1 2 4 5 6 9 10	1 2 4 5 6 8 9 10	1 2 3 5 6 7 8 10
1 2 5 6 7 8 11 12	1 2 3 4 5 6 8 10	0 1 2 3 4 6 8 10	0 1 2 3 4 6 7 9
1 2 3 4 5 6 7 21	1 3 4 5 6 7 9 10	1 2 4 5 6 7 8 10	1 2 3 6 7 8 9 10
1 3 4 5 8 9 10 11	1 2 3 5 6 7 10 11	1 2 3 4 5 6 9 10	1 2 3 6 7 8 10 11
1 2 4 5 6 7 8 9	0 1 4 5 6 7 8 9	0 1 3 4 5 6 7 9	1 2 3 4 6 8 9 10
1 2 3 4 6 7 8 10	1 2 3 5 7 8 9 11	0 1 2 3 4 6 8 9	0 1 2 3 4 5 6 21
1 2 3 6 7 8 10 12	0 1 2 3 5 6 7 21	0 1 2 3 4 6 7 21	1 2 3 4 5 7 9 21
0 1 2 3 4 7 8 21	0 1 2 4 5 7 8 21	1 2 3 5 6 7 8 21	0 1 2 4 5 7 9 21
0 1 4 5 6 7 9 21	1 2 3 4 5 8 9 21	0 1 2 3 6 8 9 21	1 2 3 5 6 8 9 21
1 2 4 5 6 8 9 21	0 1 2 3 6 8 10 21	1 2 4 5 7 8 9 21	1 2 3 5 7 8 9 21
1 2 3 6 7 8 9 21	1 3 5 6 7 8 9 21	1 2 3 6 7 8 10 21	1 3 4 5 6 8 10 21

Developing the above 8-element sets into orbits under the group generated by the above generators produces a 6-(22,8,60) design. The group generated is isomorphic to $PSL_2 19$.

Lemma 1 *Let $0 < j \le t < v$ be integers, and f be any permutation of $\{1, 2, \dots, N\}$ If:*

$$\begin{aligned} \alpha &= [(\mathcal{X}, \mathcal{A}_i)]_{i=1}^N, \text{ is an } LS[N](t-j, t-j+1, v_1); \\ \beta &= [(\mathcal{Y}, \mathcal{B}_i)]_{i=1}^N, \text{ is an } LS[N](j-1, j, v_2) \text{ and} \\ \mathcal{X} \cap \mathcal{Y} &= \emptyset, \end{aligned}$$

then the number of blocks of $\mathcal{C}(\alpha, \beta, f)$ that contain a given t-element subset $T \subseteq \mathcal{X} \cup \mathcal{Y}$ is:

$$\begin{cases} \overline{\lambda}(t-j, t-j+1, v_1)/N & \text{if} \quad |T \cap \mathcal{Y}| = j; \\ \overline{\lambda}(j-1, j, v_2)/N & \text{if} \quad |T \cap \mathcal{X}| = t-j+1; \\ 0 & \text{otherwise.} \end{cases}$$

Proof: Let T be any t-element subset of $\mathcal{X} \cup \mathcal{Y}$. If T is in some block of $\mathcal{C}(\alpha, \beta, f)$, then $|\mathcal{X} \cap T| \le t-j+1$ and $|\mathcal{Y} \cap T| \le j$. There are two possibilities either $|\mathcal{X} \cap T| = t-j$ and $|\mathcal{Y} \cap T| = j$ or $|\mathcal{X} \cap T| = t-j+1$ and $|\mathcal{Y} \cap T| = j-1$.

If $|T \cap \mathcal{Y}| = j$, then $T \cap \mathcal{Y}$ is a block of exactly one of the designs in β, say $(\mathcal{Y}, \mathcal{B}_{f(i)})$. The remaining $t-j$ points of T are contained in $\overline{\lambda}(t-j, t+1-j, v_1)/N$ blocks of $(\mathcal{X}, \mathcal{A}_i)$. Consequently, T is contained in $\overline{\lambda}(t-j, t+1-j, v_1)/N$ blocks of $\mathcal{A}_i \uplus \mathcal{B}_{f(i)}$ and in no other blocks of $\mathcal{C}(\alpha, \beta, f)$.

When $|T \cap \mathcal{X}| = t-j+1$ the proof is similar. ∎

One application of this lemma is to establish the method of *large set recursion.*

Large set recursion: Given integers $t \leq v, w$; let $\mathcal{X} = \{x_1, x_2, \ldots, x_{v-t}\}$, $\mathcal{Y} = \{y_1, y_2, \ldots, y_{w-t}\}$ and $\mathcal{Z} = \{z_1, z_2, \ldots, z_t\}$. An $LS[N](t, t+1, v+w-t)$ on $\mathcal{X} \cup \mathcal{Y} \cup \mathcal{Z}$ is constructed as follows:

1. Take $\alpha_i = [(\mathcal{X}_i, \mathcal{A}_{ij})]_{j=1}^{N}$ to be an LS$[N](t-i, t-i+1, v-i)$, on $\mathcal{X}_i = \mathcal{X} \cup \mathcal{Z} \setminus \{z_1, z_2, \ldots, z_i\}$ for each $i = 0, 1, 2, \ldots, t$.
2. Take $\beta_i = [(\mathcal{Y}_i, \mathcal{B}_{ij})]_{j=1}^{N}$ to be the LS$[N](i-1, i, w-(t-i+1))$, on $\mathcal{Y}_i = \mathcal{Y} \cup \mathcal{Z} \setminus \{z_i, z_{i+1}, \ldots, z_t\}$ for each $i = 1, 2, \ldots, t+1$.
3. Take L to be any Latin square of order N and define f_i to be the permutation of $1, 2, \ldots, N$ given by row h of L, i.e $f_h(j) = L(h, j)$, for each $h = 1, 2, \ldots, N$.
4. Finally, for each $h = 1, 2, \ldots, N$, set
$$\mathcal{D}_h = A_{0h} \cup \left(\bigcup_{i=1}^{t} \mathcal{C}(\alpha_i, \beta_i, f_h) \right) \cup \mathcal{B}_{t+1,h}.$$

 Then $[(\mathcal{X} \cup \mathcal{Y} \cup \mathcal{Z}, \mathcal{D}_h)]_{h=1}^{N}$ is the required LS$[N](t, t+1, v+w-t)$.

The first proof that large set recursion works was implicit in [3] but it is not obvious. To correct this Teirlinck gave a self-contained proof of it in [4], but it was somewhat difficult to understand. Later, in [1], a more elementary and quite understandable proof was given. This paper also contains a surprising application to the support size problem. In [5] another proof is given. I should point out that all of these papers [1, 3, 4, 5] contain a construction that is a bit more general, than what I give here. For completeness I give yet another proof, that I hope the reader will find even more accessible.

To see that large set recursion works it is first shown that for each $h = 1, 2, \ldots N$, $\mathcal{D}_h$ is a t-$(v+w-t, t+1, (v+w-2t)/N)$ design.

Consider any t-element set $T = X \cup Y \cup Z$ where $X = T \cap \mathcal{X}$, $Y = T \cap \mathcal{Y}$ and $Z = T \cap \mathcal{Z}$. Set $x = |X|$, $y = |Y|$ and $z = |Z|$. Let

$$\begin{aligned}
\mu_0 &= \text{the number of blocks in } \mathcal{A}_{0h} \text{ that contain } T; \\
\mu_{t+1} &= \text{the number of blocks in } \mathcal{B}_{t+1,h} \text{ that contain } T; \textit{and} \\
\mu_i &= \text{the number of blocks in } \mathcal{C}(\alpha_i, \beta_i, f_h) \text{ that contain } T, \text{ for } i = 1, 2, \ldots, t.
\end{aligned}$$

Note that if $z_i \in T$, then T is not contained in any subset of $(\mathcal{X} \cup \mathcal{Z} \setminus \{z_1, \ldots, z_i\}) \cup (\mathcal{Y} \cup \mathcal{Z} \setminus \{z_i, \ldots, z_t\})$. Thus $z_i \in T$ implies $\mu_i = 0$. By the lemma it follows that for each $i = 1, 2, \ldots, t$ that:

$$\mu_i = \begin{cases} \frac{w-t}{N} & \text{if } |T \cap (\mathcal{X} \cup \mathcal{Z} \setminus \{z_1, \ldots, z_i\})| = t-i+1; \\ \frac{v-t}{N} & \text{if } |T \cap (\mathcal{Y} \cup \mathcal{Z} \setminus \{z_i, \ldots, z_t\})| = i; \\ 0 & \text{if } z_i \in T \text{ or otherwise.} \end{cases}$$

This expression can be simplified by observing that

$$|T\cap(\mathcal{X}\cup\mathcal{Z}\setminus\{z_1,\ldots,z_i\})| = x+z-|T\cap\{z_1,\ldots,z_i\}| = t-y-|T\cap\{z_1,z_2,\ldots,z_i\}|.$$

and that

$$|T\cap(\mathcal{Y}\cup\mathcal{Z}\setminus\{z_i,\ldots,z_t\})| = y+|T\cap\{z_1,z_2,\ldots,z_{i-1}\}|.$$

Thus, using $z_i\in T\Rightarrow \mu_i=0$,

$$\mu_i=\begin{cases}\frac{w-t}{N} & \text{if } |T\cap\{z_1,\ldots,z_{i-1}\})| = i-1-y;\\ \frac{v-t}{N} & \text{if } |T\cap\{z_1,\ldots,z_{i-1}\})| = i-y;\\ 0 & \text{if } z_i\in T \text{ or otherwise.}\end{cases}$$

If $y=0$, it is easy to see that $\mu_0=(v-t)/N$. Also, μ_{t+1} is $(w-t)/n$ if $x=0$ and is 0 otherwise. Also if any $\mu_i\neq 0$ for some $i=1,2,\ldots,t$, then $\mu_i=(w-t)/N$, since $|T\cap\{z_1,\ldots,z_{i-1}\}|\leq i-1$. Whence $\mu_i=(w-t)/N$ if and only if $|Z\cap\{z_1,z_2,\ldots,z_{i-1}\}|=i-1$ and $z_i\notin Z$. Clearly there is a unique such i. Thus in this case such a t-element set is contained in $\frac{v-t}{N}+\frac{w-t}{N}=\frac{v+w-2t}{N}$ blocks of $\mathcal{D}_h$. A similar argument shows that the t-element sets with $x=0$ are also contained in $(v+w-2t)/N$ blocks of $\mathcal{D}_h$.

If neither x nor y are 0, then it is obvious that $\mu_0=\mu_{t+1}=0$. Furthermore $y\neq 0$ and $x\neq 0$ implies that $1\leq y\leq t-z-1$. Let $i_1<i_2<\ldots<i_y<i_{y-1}<\ldots<i_{t-z}$ be the indices such that $z_{i_j}\notin T$, $j=1,2,\ldots,t-z$. Then $|T\cap\{z_1,\ldots,z_{i_y-1}\}|=(i_y-1)-(y-1)=i_y-y$. Hence $\mu_{i_y}=(v-t)/n$. Furthermore, $|T\cap\{z_1,\ldots,z_{i_{y+1}-1}\}|=(i_{y+1}-1)-y=i_{y+1}-1-y$. Hence $\mu_{i_{y+1}}=(w-t)/N$. It is also apparent by this argument that the remainder of the $\mu_i's$ are zero. Thus in this case such a t-element set is also contained in exactly $\frac{v-t}{N}+\frac{w-t}{N}=\frac{v+w-2t}{N}$ blocks of $\mathcal{D}_h$.

Therefore $\mathcal{D}_h$ is a t-$(v+w-t,t+1,(v+w-2t)/N)$ design, for each $h=1,2,\ldots N$. They are simple because each $\mathcal{C}(\alpha_i,\beta_i,f_h)$ is simple. Thus, in particular $|\mathcal{D}_h|=\overline{\lambda}(t,t+1,v+w-t)/N$, for each $h=1,2,\ldots N$. Consequently to show that $[(\mathcal{X}\cup\mathcal{Y}\cup\mathcal{Z},\mathcal{D}_h)]_{h=1}^N$ is an LS$[N](t,t+1,v+w-t)$ we need only show that $\mathcal{D}_h\cap\mathcal{D}_\ell=\emptyset$, for all $h\neq\ell$.

Suppose $\mathcal{D}_h\cap\mathcal{D}_\ell\neq\emptyset$, for some $h\neq\ell$. Then there is a block $B\in\mathcal{D}_h\cap\mathcal{D}_\ell$.

If $B\in\mathcal{A}_{0h}$, then $B\cap\mathcal{Y}=\emptyset$. Hence, $B\in\mathcal{A}_{0\ell}$, since these are the only blocks in $\mathcal{D}_\ell$ that contain no points of $\mathcal{Y}$. This is a contradiction, since $\mathcal{A}_{0h}$ and $\mathcal{A}_{0\ell}$ belong to the large set α and are therefore disjoint. Similarly $B\notin\mathcal{B}_{t+1,h}$.

If $B\in\mathcal{C}(\alpha_i,\beta_i,f_h)$ for some i, then $B\in\mathcal{A}_{ij}\uplus\mathcal{B}_{if_{h(j)}}$ for some j. In particular this means that $|B\cap\mathcal{X}\setminus\{z_1,z_2,\ldots,z_i\}|=t-i+1$ and $|B\cap\mathcal{Y}\setminus\{z_i,z_{i+1},\ldots,z_t\}|=i$. Therefore $B\in\mathcal{A}_{ij'}\uplus\mathcal{B}_{if_{\ell(j')}}$ for some j'. Hence $\mathcal{A}_{ij'}=\mathcal{A}_{ij}$ and $\mathcal{B}_{if_{\ell(j')}}=\mathcal{B}_{if_{h(j)}}$. Thus $j=j'$ and $\ell(j')=f_{h(j)}$, which implies $h=\ell$ since f_h and f_ℓ are rows of the Latin square L.

Consequently, $\mathcal{D}_1,\mathcal{D}_2,\ldots,\mathcal{D}_h$ are pairwise disjoint and thus $[(\mathcal{X}\cup\mathcal{Y}\cup\mathcal{Z},\mathcal{D}_h)]_{h=1}^N$ is an LS$[N](t,t+1,v+w-t)$ as was to be shown.

If $\alpha=[(\mathcal{X},\mathcal{A}_i)]_{i=1}^N$ is an LS$[N](t,k,v)$ and $S\subseteq\mathcal{X}$, $|S|=s\leq t$, then the *derived large set* of α with respect to S is

$$\alpha\downarrow S=[(\mathcal{X}\setminus\{S\},\mathcal{A}_i')]_{i=1}^N$$

where $\mathcal{A}'_i = \{K \setminus \{S\} : K \in \mathcal{A}_i \text{ and } K \supseteq S\}$ is an LS$[N](t-s, k-s, v-s)$. Note that $\overline{\lambda}(t,k,v) = \overline{\lambda}(t-s,k-s,v-s)$

Theorem 1 (Khosrovshahi and Ajoodani-Namini [4]) *If there is an LS$[N](t,t+1,v)$ and an LS$[N](t,t+1,w)$, then there is also an LS$[N](t,t+1,v+w-t)$.*

Proof: Let $\mathcal{X} = \{x_1, x_2, \ldots, x_{v-t}\}$, $\mathcal{Y} = \{y_1, y_2, \ldots, y_{w-t}\}$ and $\mathcal{Z} = \{z_1, z_2, \ldots, z_t\}$. Let α be an LS$[N](t,t+1,v)$ on $\mathcal{X} \cup \mathcal{Z}$ and let β be an LS$[N](t,t+1,v)$ on $\mathcal{Y} \cup \mathcal{Z}$. Then, for each i, $i = 0,1,2,\ldots,t$, define $\alpha_i = \alpha \downarrow S_i$; where $S_i = \{z_1, z_2, \ldots, z_i\}$ and $\beta = \beta \downarrow T_i$; where $T_i = \{z_t, z_{t-1}, \ldots, z_{t-i+1}\}$. Now apply large set recursion. ∎

The case when $v = w$ in Theorem 1 was first established by Teirlinck [3, 4]. A very useful generalization is the Corollary below. The first proof of this corollary was given by Qiu-rong Wu [5], I give another proof.

Corollary 1 (Qiu-rong Wu [5]) *If there are large sets with parameters LS$[N](t-1,t+1,v)$, LS$[N](t-1,t+1,w)$, LS$[N](t-1,t,v-1)$ and LS$[N](t-1,t,w-1)$, then there is also an LS$[N](t-1,t+1,v+w-t)$.*

Proof: Let $\mathcal{X} = \{x_1, x_2, \ldots, x_{v-t}\}$, $\mathcal{Y} = \{y_1, y_2, \ldots, y_{w-t}\}$ and $\mathcal{Z} = \{z_1, z_2, \ldots, z_t\}$. To obtain an LS$[N](t,t+1,v+w-t)$ large set recursion requires that for each i, $i = 0,1,2,\ldots,t$:

1. that there exist an LS$[N](t-i,t-i+1,v-i)$ $\alpha_i = [(\mathcal{X}_i, \mathcal{A}_{ij}]_{j=1}^N$ on $\mathcal{X}_i = \mathcal{X} \cup \mathcal{Z} \setminus \{z_1, z_2, \ldots, z_i\}$ for each $i = 0,1,2,\ldots,t$; and
2. that there exist an LS$[N](t-i,t-i+1,w-i)$ $\beta_i = [(\mathcal{X}_i, \mathcal{B}_{ij}]_{j=1}^N$ on $\mathcal{Y}_i = \mathcal{Y} \cup \mathcal{Z} \setminus \{z_t, z_{t-1}, \ldots, z_{t-i+1}\}$.

For each i, $i = 1,2,\ldots,t$ the large sets α_i and β_i can be obtained by taking the appropriate derived large set of the given LS$[N](t-1,t,v-1)$ and LS$[N](t-1,t,w-1)$. Unfortunately, the hypothesis do not grant us an LS$[N](t,t+1,v)$ for use as α_0 nor an LS$[N](t,t+1,w)$ for use as β_0. This is alleviated by observing that an LS$[N](t,t+1,v)$ is also an LS$[N](t-1,t+1,v)$ and an LS$[N](t,t+1,w)$ is also an LS$[N](t-1,t+1,v)$. Thus taking α_0 to be a given LS$[N](t-1,t+1,v)$ and taking β_{t+1}, to be a given LS$[N](t-1,t+1,w)$ constructs the required LS$[N](t-1,t+1,v+w-t)$. ∎

Finally I come to the main objective of this paper, Theorem 2. The above detail was included so that the reader wishing to study any of the designs in the infinite series might understand their actual construction rather than simply knowing they exist.

Theorem 2 *There is a 6-(23+16m,8,4(m+1)(16m+17)) design for all* $m \geq 0$.

Proof: In table 1 there is a 6-$(22,8,60)$ design and as noted before it can be extended to a 6-$(23,8,68)$ design. Furthermore it is a halving of the complete design. Thus there is a LS$[2](6,8,23)$. Also as noted earlier there is a LS$[2](6,7,22+8m)$ for all $m \geq 0$. Recursively applying the corollary to these designs gives a LS[2]$(6, 8, 23+16m)$ for all $m \geq 0$. $\overline{\lambda}(6, 8, 23 + 16m)/2 = 4(m + 1)(16m + 17)$. ∎

Acknowledgments: I am extremely grateful for support given by the National Security Agency, grant number MDAA904-92-H-3036. I also wish to thank professors Earl Kramer and Spyros Magliveras for their helpful comments.

References

[1] G.B. Khosrovshahi and S. Ajoodani-Namini, Combining t-Designs, *Journal of Combinatorial Theory (A)* **58** (1991) 26-34.

[2] D.L. Kreher and S.P. Radziszowski, The existence of simple 6-$(14, 7, 4)$ designs, *Journal of Combinatorial Theory (A)* **43** (1986) 237-243.

[3] L. Teirlinck, Non-trivial t-designs exist for all t, *Discrete Math.* **65** (1987) 345-356.

[4] L. Teirlinck, Locally trivial t-designs and t-designs without repeated blocks, *Discrete Math.* **77** (1989) 345-356.

[5] Qiu-rong Wu, A note on Extending t-Designs, *Australasian Journal of Combinatorics* **4** (1991) 229-235.