Classroom Notes

Yet another Characterization of the Petersen graph

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Consider any simple graph on 10 vertices which has all its vertices of degree 3; this is known as a 3-regular or *cubic* graph of order 10. It is known that there are precisely 21 such graphs (19 are connected); see [2] for example. Since the sum of the degrees of all the vertices in any graph is equal to twice the number of edges, these graphs all have $\frac{1}{2} \cdot 3 \cdot 10$ or 15 edges. The Petersen graph is one of the 19 cubic graphs of order 10, and in fact it is the only one which has girth 5 (that is, has no cycles of length less than 5).

Noting that 15 is equal to $\binom{6}{2}$, let us consider labelling the 15 edges of a cubic graph of order 10 with the 15 unordered pairs chosen from some set of size 6. In the following, we shall take our set of size 6 to be $\{\infty, 0, 1, 2, 3, 4\}$. We would like to label the 15 edges so that the three edges incident with any vertex have disjoint labels, using up all six symbols. Another way of saying this is that we want the three labels on the edges incident with any of the 10 vertices to consist of a one-factor of K_6. We shall call such a labelling of a cubic graph of order 10 a *proper* labelling of the edges. A proper labelling of the Petersen graph was first given in [3].

There are several different characterizations of the Petersen graph; in this note we shall show the following.

Theorem. *The only cubic graph of order 10 which has a proper labelling of its edges is the Petersen graph.*

We begin by listing the 15 different one-factors of K_6; using the set $\{\infty, 0, 1, 2, 3, 4\}$ we see that these fall into three types, A, B or C, according as the pairs not involving ∞ have differences $1, 1$ or $2, 2$ or $1, 2$ (mod 5).

type A			type B			type C		
12	34	0∞	13	24	0∞	23	14	0∞
23	40	1∞	24	30	1∞	34	20	1∞
34	01	2∞	30	41	2∞	40	31	2∞
40	12	3∞	41	02	3∞	01	42	3∞
01	23	4∞	02	13	4∞	12	03	4∞

Considering the 10 vertices of the cubic graph that we are labelling, we see that precisely 10 of the above 15 one-factors are used; moreover, the 30 pairs in the 10 one-factors used must consist of *two* of each of the $15 = \binom{6}{2}$ pairs. Letting a of type A, b of type B and c of type C be used in the graph labelling, we see that

$$\begin{aligned} a+b+c &= 10, \\ 2a+c &= 10, \\ 2b+c &= 10. \end{aligned}$$

Noting that each of a, b, c lies between 0 and 5, the only solutions for a, b, c are

	a	b	c
P:	5	5	0
Q:	4	4	2
R:	3	3	4

It is easy to see that solution Q is not possible. For without loss of generality we could take, say, the first four one-factors of type A listed above. Considering pairs $12, 34$ and 40, this forces the two of type C to be 23 14 0∞ and 01 42 3∞. It is then not possible to have two one-factors containing 4∞, as only ones of type B remain to be chosen.

Type P is possible. Indeed, we may draw the graph uniquely, using the 5 one-factors of type A and 5 of type B, and it is the Petersen graph!

Finally, consider solution R. There is no loss of generality in assuming that the 4 one-factors of type C are the first four listed. This then forces the last one listed of type A, and the last one of type B, so that the pair 4∞ is in precisely two of the ten one-factors. So we have:

01	23	4∞	and	23	14	0∞
02	13	4∞		34	20	1∞
				40	31	2∞
				01	42	3∞.

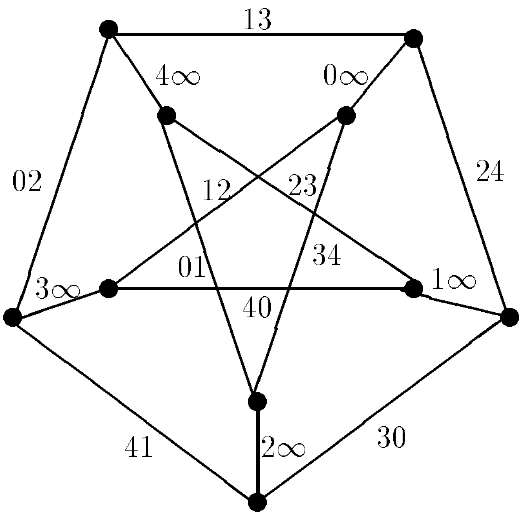

Figure 1.

Noting that the pairs 02 and 13 have occurred twice in the chosen one-factors, we see that 41 02 3∞ and 13 24 0∞ from type B are ruled out. This forces

24 30 1∞ and 30 41 2∞ from type B, and
12 34 0∞ and 40 12 3∞ from type A.

From these ten one-factors, the graph can be drawn uniquely; once again it is the Petersen graph. (see Figure 2).

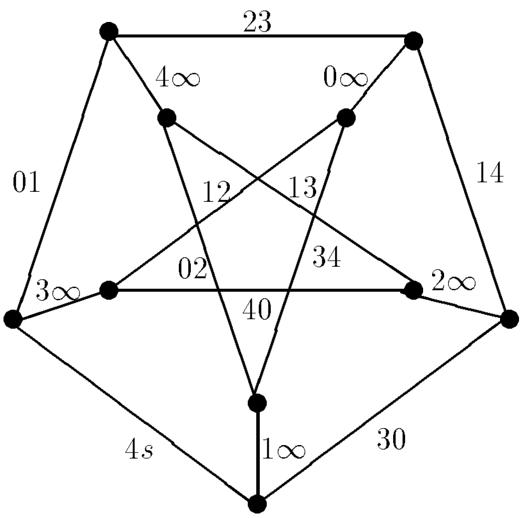

Figure 2.

This is isomorphic to the previous labelling; just apply the permutation (12).

Another proof was suggested by Dean Hoffman and relies on the fact that the Petersen graph is the unique cubic graph on 10 vertices with girth 5. Can the reader find it?

There is an interesting consequence of this result, which relies on the fact that the Petersen graph is not bipartite (for it contains a cycle of odd length!). We assume the reader is familiar with Room squares; if not, see Chapter 13 of [4] for instance. From a Room square of side n (where n is necessarily odd) we may form a bipartite graph on the vertices $\{r_1, \ldots, r_n\} \cup \{c_1, \ldots, c_n\}$ where there is an edge $\{r_i, c_j\}$ if and only if the (i, j) cell of the Room square is not empty. Moreover, we may label the edges of the bipartite graph with the entries from the Room square; thus if cell (i, j) contains the unordered pair ab, then edge $\{r_i, c_j\}$ of the bipartite graph is labelled ab. From the Room square properties, the labels on all edges incident with one vertex of the bipartite graph consist of a one-factor of K_{n+1}.

In our case, n is 5, and since the *only* graph we may properly label with one-factors of K_6 is the Petersen graph, which is not bipartite, there can be no Room square of side 5!

In [1] a Room graph of order s is defined to be a connected simple graph G with $2s$ vertices whose edges have been labelled by the pairs from an $(s+1)$-element set such that (i) each pair appears on exactly one edge; (ii) each edge is labelled by exactly one pair and (iii) the pairs labelling the edges incident with any one vertex form a one-factor of K_{s+1}. Thus our labelling of the Petersen graph is a Room graph of order 5. Room graphs were first introduced in [3] where a more general combinatorial object is defined. In [3] it was conjectured that a nonbipartite Room graph should exist for all $n \geq 5$. This conjecture was proven to be true in [1]. If we define $N(s)$ to be the number of Room graphs of order s, then we have shown in this paper that $N(5) = 1$. The value of $N(s)$ for any $s > 5$ is unknown.

The cover of the *Bulletin of the Institute of Combinatorics and its Applications* is graced with a lovely cubic graph known as the Heawood graph. It too can be properly labelled.

When p is odd K_p does not have a one-factor, but it nearly does. A near one-factor of K_p, p odd, is a collection of $\frac{p-1}{2}$ disjoint edges. Observe that $\binom{p}{2}$ is precisely the number of edges in a graph of order $2p$ that is regular of degree $\frac{p-1}{2}$. A labelling of the edges of such a graph by the edges of K_p is *proper*, when the edges incident with any vertex of the graph form a near one-factor of K_p. For $p = 7$ a proper labelling of the Heawood graph is given below. However, it is not the *only* cubic graph on 14 vertices that can be properly labelled. We leave the reader to find other graphs that can be properly labelled!

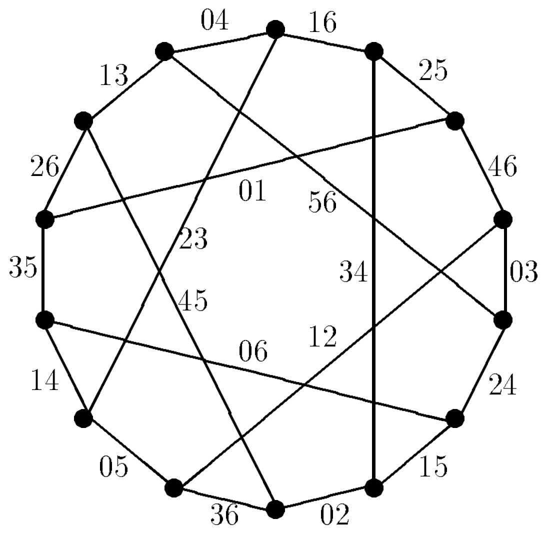

Figure 3.

References

[1] J. H. Dinitz, The existence of certain Room graphs, *Ars Combinatoria* **13** (1982) 115–128.

[2] W. Imrich, *Zehnpunktige kubische Graphen*, Aequationes Mathematicae **6** (1971) 6–10.

[3] E. S. Kramer, D. L. Kreher, S. S. Magliveras and D. M. Mesner, An assortment of Room-type designs, *Ars Combinatoria* **11** (1981) 9–29.

[4] W. D. Wallis, *Combinatorial designs*, Marcel Dekker Inc., 1988.