On $(n, 5, 3)$-Turán systems

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Abstract

The minimal number of triples required to represent all quintuples on an n-element set is determined for $n \leq 13$ and all extremal constructions are found. In particular we establish that there is a unique minimal system on 13 points, namely the 52 collinear triples of the projective plane of order 3.

1 Introduction.

We say that a set T represents another set F if T is contained in F. A (n, k, r)-*Turán system* is a pair $(\mathcal{X}, \mathcal{B})$ where $\mathcal{B}$ is a collection of r-tuples of the n-element set $\mathcal{X}$ such every k-element subset of $\mathcal{X}$ is represented by some member of $\mathcal{B}$. The size of a (n, k, r)-Turán system $(\mathcal{X}, \mathcal{B})$ is the number of r-tuples in $\mathcal{B}$ and $T(n, k, r)$ is the minimum size required for a (n, k, r)-Turán system to exist.

*Partialy supported by NSF grant CCR-8920692.

The problem of determining $T(n,k,r)$ was solved by Turán [12] for the case $r=2$:

$$T(n,s+1,2) \;=\; mn - \frac{m(m+1)}{2}\cdot s \qquad \text{with } m \le \frac{n}{s} \le m+1$$

The general problem also was formulated by Turán and was first published in English in [13]. For $r>2$ the values of the Turán numbers were found only for the cases when n is small compared with k. For instance, it is well known that

$$T(n,s+1,r) \;=\; n-s \qquad \text{with } 1 \le \frac{n}{s} \le \frac{r}{r-1}\,.$$

The next zone was found in [7] (see also [9]) :

$$T(n,s+1,r) = \begin{cases} \lceil \frac{3r-2}{r} n \rceil - 3s & \text{with } r \equiv 0 \bmod 2 \;,\; \frac{r}{r-1} \le \frac{n}{s} \le \frac{3r}{3r-4} \\[2ex] 3n - \lfloor \frac{3r-1}{r-1} s \rfloor & \text{with } r \equiv 1 \bmod 2 \;,\; \frac{r}{r-1} \le \frac{n}{s} \le \frac{3r+1}{3r-3} \end{cases}$$

The problem is still open for large n. For any $k>r>2$, even the value

$$t(k,r) \;=\; \lim_{n\to\infty} \frac{T(n,k,r)}{\binom{n}{r}}$$

is not determined. Moreover, there is no reasonable conjecture about values $t(k,r)$ with $r \ge 4$. It was conjectured by Turán [14] that

$$T(qm, q\cdot(r-1), r) = q\binom{m}{r} \tag{1}$$

but it was disproved for $r \ge 4$ (see [6]). However, it is quite possible that (1) holds for $r=3$, and

$$t(s+1,3) = \frac{4}{s^2}\,.$$

Sidorenko in [7, 8] has proved that

$$T(n,s+1,3) = \begin{cases} n-s & \text{with } 1 \le \frac{n}{s} \le \frac{3}{2} \\[2ex] 3n-4s & \text{with } \frac{3}{2} \le \frac{n}{s} \le 2 \\[2ex] 4n-6s & \text{with } 2 \le \frac{n}{s} \le \frac{9}{4}\,,\; 4n-9s \ne -1 \\[2ex] 4n-6s+2 & \text{with } 4n-9s = -1,1,2 \end{cases} \tag{2}$$

The aim of this paper is to determine $T(n,5,3)$ for small n. The case $n \le 9$ was solved by Surányi [11], and $n=10$ by Stanton and Bate [10]. We will find $T(n,5,3)$ for $n=11,12,13$ as well as all extremal constructions, for $n \le 13$

If $\mathcal{C}$ is any collection of subsets of $\mathcal{X}$ and $S \subseteq \mathcal{X}$, we denote by $\deg_{\mathcal{C}}(S)$ the number of sets in $\mathcal{C}$ that contain S. We will make frequent use of the following identity for a $(n,5,3)$-Turán system $(\mathcal{X},\mathcal{B})$:

$$3|\mathcal{B}| = \sum_{x\in\mathcal{X}} \deg_{\mathcal{B}}(x) = \sum_{x,y\in\mathcal{X}} \deg_{\mathcal{B}}(x,y) \tag{3}$$

The *residual system with respect to* $S \subseteq \mathcal{X}$ of a $(n,5,3)$-Turán system $(\mathcal{X},\mathcal{B})$ is $(\mathcal{X}-S,\mathcal{B}^S)$ where $\mathcal{B}^S = \{T \in \mathcal{B} : S \cap T = \emptyset\}$. It is a $(n-|S|,5,3)$-Turán system and thus

$$|\mathcal{B}^S| \geq T(n-|S|,5,3)\ . \tag{4}$$

The Schönheim bound (see [5] or [3]) follows from (3) and (4) :

$$T(n,5,3) \geq \lceil \frac{n}{n-3} \cdot T(n-1,5,3) \rceil\ .$$

2 General Constructions of $(n,5,3)$-Turán systems.

Turán observed in 1961 that if you divide n elements into two almost equal sized groups and take all triples contained in either group, then every quintuple must contain at least one of these triples. Thus,

$$T(n,5,3) \leq \binom{\lfloor n/2 \rfloor}{3} + \binom{\lceil n/2 \rceil}{3} = f(n).$$

In this section we describe some constructions that produce better upper bounds on $T(n,5,3)$ for odd n. These constructions were first established by A. Sidorenko in [8] published in Russian.

Construction 1.

The finite projective plane of order 3 has 13 points and 13 lines, Each line contains 4 points. For any 5 points there is a line containing at least 3 of them. Hence the system of $13 \cdot 4 = 52$ collinear triples is a (13,5,3)-Turán system.

Construction 2.

Divide the n elements into 9 disjoint sets $A_1, A_2, \ldots, A_9$ which correspond to the points $a_1, a_2, \ldots, a_9$ of the finite affine plane of order 3. This plane has 4 families of parallel lines. Color two of the families by red and the other two by blue. Take triples $x \in A_i$, $y \in A_j$ and $z \in A_k$ satisfying

1. $i = j = k$, or
2. i, j, k are pairwise distinct and a_i, a_j, a_k are collinear, or
3. $i = j,\ \ k \neq i$ and a_i, a_k defines a red line.

Then these triples form a $(n,5,3)$-Turán system.

Construction 3.

Denote the 12 lines of the finite affine plane of order 3 as follows:

$$\begin{array}{lll}
L_1 = \{a_1, a_6, a_8\}, & L_9 = \{a_2, a_4, a_9\}, & M_1 = M_9 = \{a_1, a_5, a_9\}, \\
L_2 = \{a_1, a_2, a_3\}, & L_8 = \{a_7, a_8, a_9\}, & M_2 = M_8 = \{a_2, a_5, a_8\}, \\
L_3 = \{a_3, a_4, a_8\}, & L_7 = \{a_2, a_6, a_7\}, & M_3 = M_7 = \{a_3, a_5, a_7\}, \\
L_4 = \{a_1, a_4, a_7\}, & L_6 = \{a_3, a_6, a_9\}, & M_4 = M_6 = \{a_4, a_5, a_6\}.
\end{array}$$

Note that $a_i \in L_i$ and $a_i, a_5 \in M_i$ with $i = 1,2,3,4,6,7,8,9$. Take the same triples as in Construction 2 but instead of condition 3 use

$3'$. $i = j$, $\; k \neq i$ and $a_k \in L_i \cup M_i$.

If we also require that $|A_5| = 1$, then the resulting triples form a $(n,5,3)$-Turán system.

Construction 1 gives

$$T(13,5,3) \leq 52 = f(13) - 3 .$$

Construction 2 is good for some small values of n. Let, for definiteness, a_7, a_8, a_9 form a blue line. Choosing

$$|A_1| = |A_2| = |A_3| = |A_4| = |A_5| = |A_6| = |A_7| = 1 \quad \text{and} \quad |A_8| = |A_9| = 2,$$

a (11,5,3)-Turán system of size 29 is obtained. For $n = 12$ when

$$|A_1| = |A_2| = |A_3| = |A_4| = |A_5| = |A_6| = 1 \quad \text{and} \quad |A_7| = |A_8| = |A_9| = 2,$$

is chosen, then non-regular (12,5,3)-system of size 40 is produced. The six elements in $A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6$ are of degree 9, and the six elements in $A_7 \cup A_8 \cup A_9$ have degree 11. Observe that the same systems for $n = 11, 12$ can also be obtained from Construction 3. The only situation when Construction 2 is better than Construction 3 is for $n = 15$. Choosing

$$|A_1| = |A_2| = |A_3| = |A_4| = |A_5| = |A_6| = 2, \quad \text{and} \quad |A_7| = |A_8| = |A_9| = 1,$$

a (15,5,3)-Turán system of size 89 is obtained. Thus

$$T(15,5,3) \leq 89 = f(15) - 2.$$

Construction 3 gives the best known result for any odd $n \neq 13, 15$. Namely, choosing

$$|A_1| \geq |A_3| \geq |A_9| \geq |A_7| \geq |A_2| \geq |A_6| \geq |A_8| \geq |A_4| \geq |A_1| - 1,$$

$$|A_1| + |A_2| + |A_3| + |A_4| + |A_6| + |A_7| + |A_8| + |A_9| = n - 1,$$

we get

$$T(n,5,3) \leq f(n) - \frac{n-1}{4} \quad \text{with } n \equiv 1 \bmod 8,$$

$$T(n,5,3) \leq f(n) - \frac{n-3}{8} \quad \text{with } n \equiv 3 \bmod 8,$$

$$T(n,5,3) \leq f(n) - \frac{n-9}{4} \quad \text{with } n \equiv 5 \bmod 8,$$

$$T(n,5,3) \leq f(n) - \frac{n-7}{8} \quad \text{with } n \equiv 7 \bmod 8.$$

3 $T(n,5,3)$ with $n \leq 9$.

Proposition 1 *$T(6,5,3) = 2$, $T(7,5,3) = 5$ and $T(8,5,3) = 8$.*

Proof: The unique minimal $(6,5,3)$-Turán system is given by two disjoint triples. The analysis for 7 and 8 points is tedious but straight forward. In Tables I and II a complete list of all minimal $(7,5,3)$ and $(8,5,3)$-Turán systems is given. ∎

Table I: The $(7,5,3)$-Turán systems of size 5

No.	Turán system				
1	145	467	567	236	237
2	456	457	467	567	123
3	127	347	567	234	156
4	127	347	567	135	246

Table II: The $(8,5,3)$-Turán systems of size 8

No.	Turán system							
1	123	147	168	258	357	348	456	267
2	123	124	134	234	567	568	678	578
3	123	124	345	346	567	568	178	278

Proposition 2 *Let $(\mathcal{X},\mathcal{B})$ be a $(n,5,3)$-Turán system. Then*

(i) $\deg_{\mathcal{B}}(x) \le |\mathcal{B}| - T(n-1,5,3)$ *and*

(ii) $\deg_{\mathcal{B}}(x) + \deg_{\mathcal{B}}(y) - \deg_{\mathcal{B}}(x,y) \le |\mathcal{B}| - T(n-2,5,3)$.

Proof: This follows from inclusion exclusion, $|\mathcal{B}^x| = |\mathcal{B}| - \deg_{\mathcal{B}}(x)$ and $|\mathcal{B}^{\{x,y\}}| = |\mathcal{B}| - (\deg_{\mathcal{B}}(x) + \deg_{\mathcal{B}}(y) - \deg_{\mathcal{B}}(x,y))$. ∎

Proposition 3 *The only $(9,5,3)$-Turán system of size 12 is the unique Steiner system $S(2,3,9)$*

Proof: Let $(\mathcal{X},\mathcal{B})$ be a $(n,5,3)$-Turán system of size 12. Then applying Propositions 1 and 2(i) we have $\deg_{\mathcal{B}}(x) \le 4$ for all $x \in \mathcal{X}$. Also, $\sum_{x\in\mathcal{X}} \deg_{\mathcal{B}}(x) = 12\binom{3}{1} = 4\cdot 9$ and therefore $\deg_{\mathcal{B}}(x) = 4$ for all $x \in \mathcal{X}$. Now for $x,y \in \mathcal{X}, x \ne y$ we have by Propositions 1 and 2(ii) that $\deg_{\mathcal{B}}(x,y) \ge 1$. But, $\sum_{x,y\in\mathcal{X}} \deg_{\mathcal{B}}(x,y) = 12\binom{3}{2} = \binom{9}{2}$. Consequently $\deg_{\mathcal{B}}(x,y) = 1$ for all $x,y \in \mathcal{X}, x \ne y$ and thus $(\mathcal{X},\mathcal{B})$ is an $S(2,3,9)$ system as claimed. It is given in Table III. ∎

Table III: The unique $(9,5,3)$-Turán system of size 12

123	147	168	258	357	348	456	267	159	789	369	249

Theorem 1 *A $(9,5,3)$-Turán system of size 13 contains a superfluous triple.*

Proof: Let $(\mathcal{X},\mathcal{B})$ be a $(9,5,3)$-Turán system of size 13. Then Propositions 1 and 2(i) imply that $\deg_{\mathcal{B}}(x) \leq 5$ for all $x \in \mathcal{X}$. There are at least 3 points of degree 5 in $\mathcal{B}$, since $39 = \sum_{x\in\mathcal{X}} \deg_{\mathcal{B}}(x)$. If $a \in \mathcal{X}$ is any point of degree 5, then $\deg_{\mathcal{B}}(ax) = \deg_{\mathcal{B}}(x) - 3$ for all $x \in \mathcal{X} - \{a\}$, since every $(8,5,3)$-Turán system of size 8 is regular of degree 3. In particular we have $\deg_{\mathcal{B}}(x) \geq 3$ for all $x \in \mathcal{B}$.

Suppose $x \in \mathcal{X}$ has $\deg_{\mathcal{B}}(x) = 3$. Then $39 = \sum_{x\in\mathcal{X}} \deg_{\mathcal{B}}(x)$ implies there are at least 4 points of degree 5 in $\mathcal{B}$. Fix any point a of degree 5 in $\mathcal{B}$. Then since there are at least 3 other points of degree 5 the derived system $\mathcal{B}^{\{a,x\}}$ is a $(7,5,3)$-Turán system of size 5 that contains at least 3 points of degree 3. The only such $(7,5,3)$-Turán system is No. 2 in Table I. It has three points u, v, w of degree 1 and four points b, c, d, e of degree 3. Further uvw, bcd, bce, bde and cde are the triples in this system. Thus b, c, d, and e are also points of degree 5 in $\mathcal{B}$. This argument was independent of the initial choice of a point of degree 5. Consequently $\mathcal{B}$ contains all of the *ten* 3-element subsets of $\{a,b,c,d,e,\}$ as well as uvw and the *three* triples containing x. This accounts for 14 triples in $\mathcal{B}$ contrary to $|\mathcal{B}| = 13$. Therefore $\mathcal{B}$ contains no point of degree 3.

Now that we know that there are no points of degree 3 the degree sequence of $\mathcal{B}$ is completely determined. It has 3 points a, b and c of degree 5 and exactly 6 points x_1, x_2, x_3, x_4, x_5 and x_6 of degree 4. Again, since the derived system with respect to a point of degree 5 is an $(8,5,3)$-Turán system of size 8 and they are all regular of degree 3 , we have:

$$\deg_{\mathcal{B}}(a,b) = \deg_{\mathcal{B}}(a,c) = \deg_{\mathcal{B}}(b,c) = 2$$

and

$$\deg_{\mathcal{B}}(a,x_i) = \deg_{\mathcal{B}}(b,x_i) = \deg_{\mathcal{B}}(c,x_i) = 1, \quad i = 1,2,\ldots 6.$$

We will show that $abc \in \mathcal{B}$. If $abc \notin \mathcal{B}$, then $\mathcal{B} = \mathcal{F} \cup \mathcal{M}$ where

$$\mathcal{F} = \{abx_1, abx_2, acx_3, acx_4, bcx_5, bcx_6, ax_5x_6, bx_3x_4, cx_1x_2\}$$

and $\mathcal{M}$ is some set of four more triples from $\{x_1, x_2, \ldots, x_6\}$.

Note that $H = \langle (x_1,x_2),(x_3,x_4),(x_5,x_6)\rangle$ is an automorphism group of $\mathcal{F}$ and that $\deg_{\mathcal{M}}(x_i) = 2$ for each i. Thus without loss we may assume that $\mathcal{M}$ is one of the following 5 possibilities:

$$\{x_1x_2x_3, x_1x_5x_6, x_3x_4x_5, x_2x_4x_6\}$$

$$\{x_1x_2x_3, x_1x_2x_5, x_3x_4x_6, x_4x_5x_6\}$$

$$\{x_1x_2x_3, x_1x_5x_6, x_3x_4x_6, x_2x_4x_5\}$$

$$\{x_1x_2x_3, x_1x_2x_5, x_4x_5x_6, x_3x_4x_6\}$$

$$\{x_1x_2x_3, x_1x_5x_6, x_4x_5x_6, x_2x_3x_4\}$$

However in each case either $ax_1x_3x_4x_6$ or $ax_1x_3x_4x_5$ has not been represented. Consequently, abc is indeed a triple in $\mathcal{B}$.

If $\deg_{\mathcal{B}}(x_i, x_j) = 0$, for some $i \neq j$, then the residual system $\mathcal{B}^{\{x_i,x_j\}}$ is a $(7, 5, 3)$-Turán system of size 5. Furthermore

$$\deg_{\mathcal{B}^{\{x_i,x_j\}}}(a) = \deg_{\mathcal{B}^{\{x_i,x_j\}}}(b) = \deg_{\mathcal{B}^{\{x_i,x_j\}}}(c) = 3$$

Thus, $\mathcal{B}^{\{x_i,x_j\}}$ is Turán system No. 2 in Table I. Consequently it contains a fourth point d of degree 3 and $\deg_{\mathcal{B}^{\{x_i,x_j\}}}(a, d) = 2$. Therefore $\deg_{\mathcal{B}}(d) = 5$ and this contradicts $\mathcal{B}$ having exactly three points of degree 5. Thus, $\deg_{\mathcal{B}}(x_i, x_j) \geq 1$ for all $i \neq j$.

Let $\mathcal{B}' = \mathcal{B} - \{abc\}$. Then $\deg_{\mathcal{B}'}(p, q) \geq 1$ for all $p, q \in \mathcal{X}$. it follows that $\mathcal{B}'$ is an S(2,3,9). Whence abc is a superfluous triple. ∎

The reader should note that $G = \langle \alpha, \beta \rangle$ where $\alpha = (1, 2, 3)(4, 5, 6)(7, 8, 9)$ and $\beta = (1)(2, 8, 4, 5, 3, 6, 7, 9)$, is the full automorphism group of the $S(2, 3, 9)$ given in Table III. There are precisely two orbits of triples under this group and they are represented by 123 and 124. Thus by proposition 3 it follows that there is a, unique up to isomorphism, $(9, 5, 3)$-Turán system of size 13. It is given in table IV.

Table IV: The unique $(9, 5, 3)$-Turán system of size 13

123	147	168	258	357	348	456	267	159	789	369	249	124

4 $T(10, 5, 3) = 20$

A $(n, 5, 3)$-Turán system $(\mathcal{X}, \mathcal{B})$ is an *extension* of the $(n - 1, 5, 3)$-Turán system $(\mathcal{Y}, \mathcal{A})$ if there is a point $x \in \mathcal{X}$ such that $\mathcal{Y} = \mathcal{X} - \{x\}$ and $\mathcal{A} = \mathcal{B}^x$. Throughout the remainder of this section $\mathcal{X}$ will be the point set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, x\}$ and $\mathcal{S}$ will be the $S(2, 3, 9)$ given in Table III. Also we let $\mathcal{K}$ be the the set of all 5-element subsets of $\mathcal{X}$ containing 1 and x that are not represented by some triple in $\mathcal{S}$. It is elementary to show that $|\mathcal{K}| = 24$ and a triple xab where $a, b \neq 1$ and $1ab \notin \mathcal{S}$ represents exactly 3 of the 5-element subsets in $\mathcal{K}$.

Theorem 2 *There is a unique extension of* $S(2, 3, 9)$ *to a* $(10, 5, 3)$-*Turán system of size 20.*

Proof: Let $(\mathcal{X}, \mathcal{B})$ be a $(10, 5, 3)$-Turán system of size 20 and suppose $x \in \mathcal{X}$ has degree 8. Then without loss $\mathcal{B}^x = \mathcal{S}$. Let Γ be the graph on $\{1, 2, \ldots, 9\}$ with edges $\mathcal{E} = \{T - x : x \in T \in \mathcal{B}\}$. Then by Proposition 2(i) we have $0 \leq \deg_{\mathcal{E}}(i) \leq 4$.

Case 1: $\deg_{\mathcal{E}}(i) = 4$ or 3 for some i.

Without loss we may assume $i = 1$. Then by Proposition 3 or Theorem 1 $\mathcal{E}$ is forced to contain the edges $\{23, 47, 59, 68\}$. Observe that $\beta = \langle (2, 8, 4, 5, 3, 6, 7, 9) \rangle$ is an automorphism of $\mathcal{B}$ and the forced edges. Thus up to permutation by β the neighborhood of 1 in Γ is

{9,2,3,5}, {9,2,3,4}, {9,2,5,6}, {9,2,4,5}, {9,3,4,6},
{9,2,5,7}, {9,2,3,7}, {9,2,5,8}, {9,2,4,8} or {9,5,6,8}, if $\deg_{\mathcal{E}}(1) = 4$

and it is

{0,2,5}, {0,2,3}, {0,3,4}, {0,2,4}, {0,2,6}, {0,2,7} or {0,5,6}, if $\deg_{\mathcal{E}}(1) = 3$.

When considering representing 5-element sets the only possibility that survives for the neighborhood of 1 is $\{9, 5, 6, 8\}$ and the resulting $(10, 5, 3)$-Turán system of size 20 is given in table V.

Case 2: $\deg_{\mathcal{E}}(i) \leq 2$ for all i.

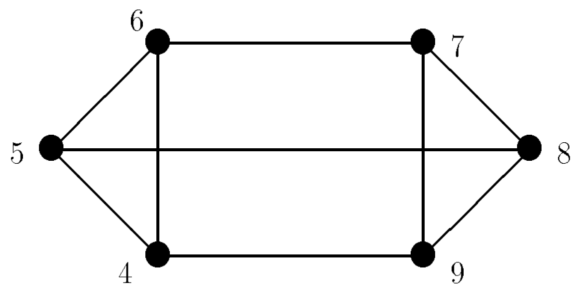
If N_d is the number of vertices of Γ of degree d, then $N_0 + N_1 + N_2 = 9$ and $N_1 + 2N_2 = 16$. Thus either $N_0 = 1$ and $N_2 = 8$ or $N_1 = 2$ and $N_2 = 7$.

If $N_0 = 1$, then without loss we may assume that $\deg_{\mathcal{E}}(1) = 0$. Thus the 24 subsets in $\mathcal{K}$ must be represented by exactly 8 triples, $\mathcal{T}$, of the form xuv, $u, v \in \{2, 3, \ldots, 9\}$. It is easy to determine that if $\deg_{\mathcal{K}}(x, u, v) \neq 0$, then $\deg_{\mathcal{K}}(x, u, v) = 3$. Thus $\deg_{\mathcal{K}}(x, u, v) = 3$ for every triple $xuv \in \mathcal{T}$ and in particular no two triples in $\mathcal{T}$ can represent the same triple in $\mathcal{K}$. This implies that if xuv and xuw are in $\mathcal{T}$, then $x1uvw \notin \mathcal{K}$. Hence some 3-element subset of $1uvw$ is a triple of $\mathcal{S}$. Consider any edge $\{a, b\}$ of Γ. There are triples $xau, xab, xbv, \in \mathcal{T}$ since, $N_2 = 8$. If $u \neq v$, then the 3-element subsets in $1abu$ and $1abv$ that are triples $\mathcal{S}$ must both contain 1 for otherwise they would cover a pair twice. Neither $1au$, $1bv$ nor $1ab$ can be a triples in $\mathcal{S}$, since $\deg_{\mathcal{K}}(x, a, u) = \deg_{K}(x, b, v) = \deg_{\mathcal{K}}(x, a, b) = 3$. Consequently, $1bu$ and $1av$ must be triples in $\mathcal{S}$. The above argument implies Γ is the union of two 4-cycles $a_1b_1a_2b_2$ and $c_1d_1c_2d_2$ for otherwise we force 1 to appear in more than 4 triples of $\mathcal{S}$. An obvious contradiction. Furthermore $1a_1a_2$, $1b_1b_2$, $1c_1c_2$ and $1d_1d_2$ are the triples in $\mathcal{S}$ containing 1. Unfortunately the five element set $xa_1a_2c_1c_1$ has not been represented and so $N_0 \neq 1$.

If instead $N_1 = 2$ and $N_2 = 7$, then we may assume $\deg_{\mathcal{E}}(1) = 1$ and that $T_0 = x12$ is a chosen triple. Thus $\{1, 2\}$ is an edge of Γ and there a 6 more edges in Γ left to choose. Note that T_0 represents exactly 9 of the 24 5-sets in $\mathcal{K}$. The remaining 15 are:

$$\hat{\mathcal{K}} = \left\{ \begin{array}{ccccc} x1345, & x1346, & x1349, & x1356, & x1358, \\ x1367, & x1378, & x1379, & x1389, & x1458, \\ x1469, & x1489, & x1567, & x1578, & x1679 \end{array} \right\}$$

Let Δ be the graph whose edges are $\{B \setminus \{x, 1, 3\} : B \in \hat{\mathcal{K}}\}$. Then Δ is



The edges of Δ must be represented by $\{E \setminus \{3\} : E$ is an edge of $\Gamma\}$. This set contains at most two singletons since $\deg_{\mathcal{K}}(3)$ is 1 or 2. When these singleton vertices are deleted form Δ the edges of the resulting graph must be edges of Γ. Thus the resulting graph can have almost $6-2=4$ edges. Consequently, there are exactly two vertices u and v that are adjacent to vertex 3 in Γ, and furthermore they are not adjacent in Γ. Without loss, $u \in \{4,5,6\}$ and $v \in \{7,8,9\}$. Then either $1uv$, $2uv$, or $3uv$ is a triple in $\mathcal{S}$.

Let 123, $1uv$, $1a_1a_2$ and $1b_1b_2$ be the triples in $\mathcal{S}$ that contain i, $i=1,2$ or 2.

> If $i=1$ or 3, then the above argument shows that Γ contains the six edges 12, $3u$, $3v$, a_1b_1, b_1a_2 and $a2b_2$. The two remaining edges must in particular represent $x2ja_1a_2$, $x2jb_1b_2$, $xiua_1b_2$ and $xiva_1b_2$: where $j=1$, if $i=3$ and $j=3$ if, $i=1$. This is clearly impossible without violating $N_1=2$ and $N_2=9$.
>
> If $i=2$, then the above argument shows that Γ contains the seven edges 12, $3u$, $3v$, a_1a_2, a_2b_1, b_1b_2 and b_2a_1. The remaining edge must in particular represent $x23cd$, where $1cd \in \mathcal{S}$ and is disjoint from uv. This is clearly impossible without violating $N_1=2$ and $N_2=9$.

∎

Table V

The unique extension of $S(2,3,9)$ to a $(10,5,3)$-Turán system of size 20.

123	456	789	147	258	369	159	267	348	168
249	357	x23	x47	x59	x68	x19	x15	x16	x18

Theorem 3 $T(10,5,3)=20$ *(Note that Stanton and Bate proved this by computer in 1980, see [10])*

Proof: Let $(\mathcal{X},\mathcal{B})$ be a (10,5,3)-Turán system. Then the Schönheim bound gives us $|\mathcal{B}| \geq 18$.

Suppose $|\mathcal{B}|=18$ or 19. Then by Proposition 2(i) and equation (3) there are points $x,y \in \mathcal{X}$, $x \neq y$ such that $|\mathcal{B}|-12=\deg_{\mathcal{B}}(x) \geq \mathcal{B}_(y) \geq 6$. Thus without loss $\mathcal{B}^x$ is the (9,5,3)-Turán system $\mathcal{S}$ and $y=1$. Furthermore $\mathcal{B}^y$ is a (9,5,3)-Turán system of size 12 or 13. The uniqueness of such systems forces $\mathcal{B}$ to contain $x23$, $x68$, $x59$ and $x47$. This accounts for all but two or three of the triples in $\mathcal{B}$. These remaining triples two or three triples must represent all of $\mathcal{K}$, since no quintuple in $\mathcal{K}$ has been represented so far. An easy inclusion exclusion argument shows that it is impossible to represent $\mathcal{K}$ with fewer than four triples. Consequently $|\mathcal{B}| \geq 20$.

The (10,5,3)-Turán system of the Table V is of size 20. ∎

Lemma 1 *There is no extension of a (9,5,3)-Turán system of size 13 to a (10,5,3)-Turán system of size 20.*

Proof: Suppose that $(\mathcal{X}, \mathcal{B})$ is a (10,5,3)-Turán system of size 20 extending a (9,5,3)-Turán system of size 13 by the point x. According to Theorem 1, $\mathcal{B}^x$ contains a superfluous triple yzu. Any 5-element subset of $\mathcal{X}$, represented only by this triple, must contain x. Thus, replacing yzu by xzu, we obtain an (10,5,3)-Turán system $(\mathcal{X}, \mathcal{B}')$ ($\mathcal{B}$ cannot already contain xzu since $|\mathcal{B}'| \geq 20$ by Theorem 3). The degree of x in $\mathcal{B}'$ is equal to 8, and $(\mathcal{X}, \mathcal{B}')$ is an extension of the unique (9,5,3)-Turán system of size 12. Such extension is also unique, and one was described in Theorem 2. It is easy to check that we could not replace x by another vertex y in one triple of $\mathcal{B}'$ without destroying the Turán condition. ∎

5 $T(11, 5, 3) = 29$

Through out this section $\mathcal{X}$ will be the point set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, x, y\}$ and $(\mathcal{X}, \mathcal{B})$ will be a (11,5,3)-Turán system. The Schönheim bound gives $T(11, 5, 3) \geq 28$, since by Theorem 3, $T(10, 5, 3) = 20$.

Corollary 1 *There is no extension of the Turán system in Table V to a* $(11, 5, 3)$*-Turán system of size 28.*

Proof: Suppose there is such an extension. Then there is a point $y \in \mathcal{X}$ such that $\mathcal{B}^y$ is the $(10, 5, 3)$-Turán system of size 20 given in Table V. Note that Proposition 2(i) implies that $\deg(a) \leq 8$ for all $a \in \mathcal{X}$. Thus in particular $\deg(y, 1) = 0$. But $\mathcal{B}^x$ is also a $(10, 5, 3)$-Turán system of size 20 that extends an $S(2, 3, 9)$. It is therefore, by Theorem 2, isomorphic to the Turán system in Table V and this contradicts $\deg(y, 1) = 0$. ∎

Proposition 4 *If x is a point of degree 8 in a (11,5,3)-Turán system $(\mathcal{X}, \mathcal{B})$ of size 28, then $\deg_{\mathcal{B}}(xy) = \deg_{\mathcal{B}}(y) - 6$ for all other points $y \in \mathcal{X}$. In particular the degree of every point in $(\mathcal{X}, \mathcal{B})$ is at least* 6.

Proof: Let $(\mathcal{X}, \mathcal{B})$ be a (11,5,3)-Turán system of size 28. Then the average degree of a point is

$$\frac{1}{11} \sum_{x \in \mathcal{X}} \deg_{\mathcal{B}}(x) = \frac{3 \cdot 28}{11} = 7\frac{7}{11} .$$

Thus there is at least one point x of degree 8 in $(\mathcal{X}, \mathcal{B})$. Observe that $(\mathcal{X} - \{x\}, \mathcal{B}^x)$ is a (10,5,3)-Turán system of size 20. Theorem 1 and Lemma 1 give us that the only (10,5,3)-Turán system of size 20 with a point of degree greater than or equal to 7 is the unique system given in Table 2. Corollary 1 insures us that this system can not be extended to a (11,5,3)-Turán system of size 28. Moreover in a (10,5,3)-Turán system of size 20, if there are no points with degree ≥ 7, then all points y have degree 6. The truth of the proposition now follows. ∎

Lemma 2 *If there is (11,5,3)-Turán system of size 28, it has 7 points with degree 8 and 4 points with degree 7.*

Proof: Let $(\mathcal{X}, \mathcal{B})$ be a (11,5,3)-Turán system of size 28. By Proposition 2(i), for any point x, $\deg_{\mathcal{B}}(x) \leq 8$.

Let N_d be the number of points of degree d in $\mathcal{B}$, then the equations

$$\sum_{d=1}^{8} N_d = 11 \quad \text{and} \quad \sum_{d=1}^{8} dN_d = 3 \cdot 28$$

hold. Observe that $N_d = 0$ for $d < 6$ by Proposition 4. Moreover if there is a point y of degree 6 in $\mathcal{B}$, then, also by Proposition 4, $\deg_{\mathcal{B}}(xy) = 0$ for all points x with $\deg_{\mathcal{B}}(x) = 8$. Thus in order to construct the 6 distinct triples containing y, there must be at least 4 points of degree less than 8.

Hence the only possible solution to the equations is $N_8 = 7$ and $N_7 = 4$ ∎

Lemma 3 $T(11, 5, 3) > 28$

Proof: Suppose that $(\mathcal{X}, \mathcal{B})$ be a (11,5,3)-Turán system of size 28. Then by Lemma 2, there is a set $\mathcal{P} = \{1, 2, \ldots, 7\} \subseteq \mathcal{X}$ of 7 points all of degree 8 in $\mathcal{B}$ and the remaining 4 points $\mathcal{Q} = \{a, b, c, d\} = \mathcal{X} - \mathcal{P}$ all have degree 7. It is useful to note that by proposition 4, $\deg_{\mathcal{B}}(pp') = 2$ and $\deg_{\mathcal{B}}(pq) = 1$ for all $p, p' \in \mathcal{P}$ and $q \in \mathcal{Q}$.

A subset $A \subseteq \mathcal{X}$ will be said to be of type i if $|A \cap \mathcal{P}| = i$. Let N_i be the number of triples in $\mathcal{B}$ of type i. Then counting the number of triples in two ways gives:

$$N_0 + N_1 + N_2 + N_3 = 28;$$

counting pairs of type 2 in two ways we get:

$$N_2 + 3N_3 = 2\binom{7}{2};$$

and counting pairs of type 1 in two ways we get:

$$2N_1 + 2N_2 = 4 \cdot 7.$$

There are exactly 5 solutions to these equations and they are given in Table VI.

Table VI

	N_0	N_1	N_2	N_3
A	0	14	0	14
B	1	11	3	13
C	2	8	6	12
D	3	5	9	11
E	4	2	12	10

We eliminate each of solutions A, B, C, D, and E in turn.

Elimination of solution A.

Suppose $(\mathcal{X}, \mathcal{B})$ satisfies the distribution given by solution A and let $\mathcal{D}$ be the 14 triples of type 3. All of these triples come from the point set $\mathcal{P}$ and $\deg_{\mathcal{D}}(ij) = 2$, for all $i, j \in \mathcal{P}$, since $N_2 = 0$. Thus $\mathcal{D}$ is a 2-(7,3,2) design and the complement $\bar{\mathcal{D}} = \{uvw \subset \mathcal{P} : uvw \notin \mathcal{D}\}$ is a 2-(7,3,3) design. For each pair $x, y \in \mathcal{Q}$ the 5-element subsets $\{xyuvw : uvw \in \bar{\mathcal{D}}\}$ must be represented by triples of the form xyi, where $i \in \mathcal{P}$. Thus $\deg_{\mathcal{B}}(xy) \geq 3$, since $\deg_{\bar{\mathcal{D}}}(i) = 9$ and $\deg_{\bar{\mathcal{D}}}(i,j) = 3$. This accounts for at least $3 \cdot 6 = 18$ type 1 triples in $\mathcal{B}$, which contradicts $N_1 = 14$.

Elimination of solution B.

Let $(\mathcal{X}, \mathcal{B})$ satisfies solution B. Then we may assume that $abc \in \mathcal{B}$.

If $\deg_{\mathcal{B}}(xd) = 0$ for some $x \in \{a, b, c\}$, then there are 7 type 1 triples in $\mathcal{B}$ containing d and 5 other type 1 triples in $\mathcal{B}$ containing x. This accounts for 12 type 1 triples in $\mathcal{B}$ contrary to $N_1 = 11$.

If $\deg_{\mathcal{B}}(xd) = 1$ for some $x \in \{a, b, c\}$, then since $\deg_{\mathcal{B}}(x) = 7$ and $\deg_{\mathcal{B}}(xp) = 1$ for all $p \in \mathcal{P}$ there is exactly one type 2 triple in $\mathcal{B}$ containing x. Thus we may assume without loss that $xd1$ and $x23$ are triples in $\mathcal{B}$. The only possibility to represent the quintuples $xd245$, $xd246$ and $xd247$ is to include the triples 245, 246 and 247, but this contradicts $\deg_{\mathcal{B}}(24) = 2$.

Therefore $\deg_{\mathcal{B}}(xd) \geq 2$ for all $x \in \{a, b, c\}$ and consequently we may assume without loss that $\deg_{\mathcal{B}}(ad) = 3$, $\deg_{\mathcal{B}}(bd) = 2$. and $\deg_{\mathcal{B}}(cd) = 2$, since $\deg_{\mathcal{B}}(d) = 7$ Also, since $\deg_{\mathcal{B}}(a) = \deg_{\mathcal{B}}(b) = \deg_{\mathcal{B}}(c) = 7$ and $\deg_{\mathcal{B}}(xp) = 1$ for all $p \in \mathcal{P}$ and $x \in \{a, b, c\}$ it follows that we may assume that the 7 triples containing a are abc, $ad1$, $ad2$, $ad3$, $ab4$, $ac5$ and $a67$. Furthermore it is easy to see that there is a unique type 2 triple containing b. It is now impossible to represent the 5-element subsets $ab123$, $ab125$, $ab126$, $ab127$, $ab235$, $ab236$, $ab237$, $ab135$, $ab136$ and $ab137$ without forcing the degree of some pair in $\mathcal{P}$ to exceed 2.

Elimination of solution C.

Suppose that $(\mathcal{X}, \mathcal{B})$ has the distribution given by solution C. The without loss we may assume that the 2 type 0 triples in $\mathcal{B}$ are abc and abd. Since $\deg_{\mathcal{B}}(pq) = 1$ for all $p \in \mathcal{P}$ and $q \in \mathcal{Q}$ it follows that of the 6 type 2 triples two contain a, two contain b, one has c and one has d. Without loss we may assume that $a12, a34 \in \mathcal{B}$.

Claim: *$\mathcal{B}$ contains no triple of the form xij, $x \in \{c, d\}$, $i, j \in \{5, 6, 7\}$*

Suppose for example that $\mathcal{B}$ contained the triple $c56$, then in order to represent the 5-element set $acijk$ where $i \in \{1, 2\}$, $j \in \{3, 4\}$ and $k \in \{5, 6\}$. The triple ijk is required. Thus $\mathcal{B}$ contains 135, 136, 145, 146, 235, 236, 245, 246. Note that among the triples chosen so far the degree of 13 and 24 is 2. Consequently in order to represent $c1234$ either $c12$ or $c34$ must be chosen. It is now impossible to represent $adc15$, $adc35$, $adc16$ and $adc36$ and have $\deg_{\mathcal{B}}(di) = 1$ for all $i \in \{1, 2, \ldots, 3\}$. Therefore the claim holds.

It is now impossible to represent the 5-element sets of the form $acijk$ and $adijk$ where $i, j \in \{5, 6, 7\}$ and $k \in \{1, 2, 3, 4\}$ without contradicting the claim.

Elimination of solution D.

Let $(\mathcal{X}, \mathcal{B})$ have the distribution given by solution D. Then without loss the type 0 triples in $\mathcal{B}$ are

$$abc,\ acd, \text{ and } abd.$$

Also since $\deg_{\mathcal{B}}(ap) = 1$ for all $p \in \mathcal{P}$ and since (b, c, d) is an automorphism of the type 0 triples we may assume without loss that

$$ab1,\ a23,\ a45 \text{ and } a67$$

are triples in $\mathcal{B}$. Thus all triples in $\mathcal{B}$ containing a are accounted for. Now since c must appear with each $p \in \mathcal{P}$ exactly once and $\deg_{\mathcal{B}}(c) = 7$, there are exactly 2 type 2 triples in $\mathcal{B}$ that contain c. Say these triples are chi, cjk where h, i, j and k are distinct elements of $\mathcal{P}$. Without loss of generality $2 \in \mathcal{P} - \{h, i, j, k\}$. Consider the four 5-element subsets $ac124$, $ac125$, $ac126$ and $ac127$. At most one of them is represented by chi and cjk and none are represented by a triple containing a. This implies that at least three of the triples 124, 125, 126 and 127 are in $\mathcal{B}$ contrary to $\deg_{\mathcal{B}}(12) = 2$.

Elimination of solution E.

If $(\mathcal{X}, \mathcal{B})$ satisfies solution E then the 4 triples of type 0 in $\mathcal{B}$ are

$$abc,\ \ abd,\ \ acd, \text{ and } bcd.$$

Also, since $\deg_{\mathcal{B}}(pq) = 1$ for all $p \in \mathcal{P}$ and $q \in \mathcal{Q}$, we may assume without loss of generality that

$$ab1 \text{ and } cd2$$

are the two triples of type 1 in $\mathcal{B}$. Since there is a unique triple containing a and 2 and a unique triple containing d and 1 we also have in $\mathcal{B}$ without loss of generality

$$a23 \text{ and } d14 \quad \text{or} \quad a23 \text{ and } d13.$$

Now the only way to represent $ad125$, $ad126$ and $ad127$ is to include 125, 126 and 127 in $\mathcal{B}$. This contradicts $\deg_{\mathcal{B}}(12) = 1$. ∎

Lemma 4 *There is a unique way to doubly extend $S(2, 3, 9)$ to a (11,5,3)-Turán system of size 29.*

Proof: Let $\mathcal{X} = \{1, 2, 3, \ldots, 9, x, y\}$ and suppose $(\mathcal{X}, \mathcal{B})$ is a (11,5,3)-Turán system of size 29 such that $\mathcal{B}^{\{x,y\}} = \mathcal{S}$ the $S(2, 3, 9)$ in Table III. Then without loss we may assume that $\deg_{\mathcal{B}}(y) = 9$, $\deg(x,y) \leq 1$ and $\deg_{\mathcal{B}}(x) = 8$ if $\deg_{\mathcal{B}}(x,y) = 0$; or $\deg_{\mathcal{B}}(x) = 9$ if $\deg_{\mathcal{B}}(x,y) = 1$; since the maximum degree is 9 and $12 + \deg_{\mathcal{B}}(x) + \deg_{\mathcal{B}}(y) - \deg_{\mathcal{B}}(x,y) = 29$. Thus by Theorem 2 we may assume that $\mathcal{B}^y$ is the (10,5,3)-Turán system of size 20 in Table V.

Suppose $\deg_{\mathcal{B}}(yx) = 0$ and $\deg_{\mathcal{B}}(y1) = 0$. Then in order to represent the 5-element sets $\{\ yx1ij \ : \ i,j \in \{2,3,4,7\}\ \}$ the triples $y24$, $y27$, $y34$ and $y37$ must be included in $\mathcal{B}$. There are only 5 more triples that are as yet unspecified. But, then to represent with fewer than 5 triples the 5-element sets $\{\ yxijk \ : \ i \in \{2,3,4,7\}$ and $j,k \in \{5,6,8,9\}\ \}$ the triples $y56$, $y58$, $y69$ and $y89$ must be included, leaving only one more triple in $\mathcal{B}$ to be chosen. Unfortunately $y2359$ and $y4768$ have not been represented and contain no common triple. Consequently since the permutation $(1,x)$ is an automorphism of the the system in Table V we may assume that $\deg_{\mathcal{B}}(x,y) = 1$.

Thus $\mathcal{B}^x$ is a (10,5,3)-Turán system of size 20 on the point set $\{1,2,3,\ldots,9,y\}$ that is an extension of $\mathcal{S}$. Applying Theorem 2 again it is not to difficult to show that up to isomorphism there is a unique way to specify the triples containing y in $\mathcal{B}$ so that $\mathcal{B}$ is a (11,5,3)-Turán system as was required. It is given in Table VII. ∎

Theorem 4 $T(11,5,3) = 29$

Proof: The above two Lemmas establish that $T(11,5,3) = 29$. ∎

Table VII: A $(11,5,3)$-Turán system of size 29

123	147	168	258	357	348	456	267	159	789	369	249
x23	x47	x68	x59	y57	y48	12y	y69	x2y			
x16	x18	x15	x19	y34	y38	y35	y37				

The (11,5,3)-Turán system of size 29 in Table VII can also be obtained using construction 3 in section 2.

6 Computational Results.

Independently from the results presented in the previous sections, we have performed extensive machine computations with $(n,5,3)$-Turán systems. This section sketches most important algorithms used and presents the results of our computations. The software used consisted of:

(1) *Turexp.* An algorithm for extending (n,k,l) systems to $(n+1,k,l)$ systems.

(2) *Nauty.* A very efficient set of procedures written by B.D. McKay [1] for determining the automorphism group of a graph, and optionally for canonically labeling it. Two graphs are isomorphic iff they have identical canonical labelings, thus *nauty* can be used as a powerful tool to detect isomorphs in large

families of graphs, and indirectly via graphs in families of any reasonable finite objects, in particular Turán systems.

(3) A variety of set system manipulation programs for basic operations, changes of representation, checking the Turán condition, and others.

Most of the algorithms mentioned above are modifications of the algorithms described in [2]. In this paper the first classical Ramsey number for hypergraphs $R(4,4;3) = 13$ was computed and a strong relationship between Turán systems $(n,5,4)$ and the Ramsey number $R(4,4;3)$ was exploited.

A simple counting of how many times each k-element set is represented by a chosen l-element sets permits a fairly efficient implementation of *turexp*. For any (n,k,l)-Turán system $(\mathcal{X},\mathcal{B})$ of size b let q_i denote the number of k-sets containing exactly i members of $\mathcal{B}$, for $1 \le i \le \binom{k}{l}$. Define $Q = Q(\mathcal{X},\mathcal{B})$ by

$$Q = \sum_{i=2}^{\binom{k}{l}} (i-1) q_i,$$

the number of multiple representations of k-element subsets of $\mathcal{X}$ there are in S. It is easy to see that

$$Q = \binom{n-l+1}{l-k} b - \binom{n}{l}.$$

Hence Q depends only on the size and the parameters of $(\mathcal{X},\mathcal{B})$. and so we can write $Q(\mathcal{X},\mathcal{B}) = Q(n,k,l,b)$. When extending a (n,k,l)-Turán system $(\mathcal{X},\mathcal{B})$ to a $(n+1,k,l)$-Turán system we use a variable *qsum* which is initialized to the $Q(n,k,l,b)$. For each new l-element set, if it appears in a k-element set already represented by previously chosen l-element sets, we increment *qsum* by one for each such k-set. When for a partial extension of $(\mathcal{X},\mathcal{B})$ the variable *qsum* exceeds $Q(n+1,k,l,p)$ then we know that this extension cannot be completed to any $(n+1,k,l)$ system with at most p blocks.

Turexp is a recursive backtracking algorithm organized as follows. Let $(\mathcal{X},\mathcal{B})$ be a (n,k,l)-Turán system of size b and let $y \notin \mathcal{X}$ be a new point. Consider the $\binom{n}{l-1}$ (0,1)-*variables* corresponding to possible new blocks passing through point y, and the $m \le \binom{n}{k-1} - b$ *conditions* corresponding to k-sets passing through point y, which have to be represented by new blocks. *Turexp* assigns recursively 0 or 1 to these variables, and at each level of recursion maintains: the current list of not yet satisfied conditions, the current number of blocks, the current degrees of all points, and *qsum*. This enables the algorithm to return to a higher level of recursion before a full assignment to the variables has been done in the following situations:

(a) The number of blocks exceeds the desired value p.

(b) The degree of some point exceeds the maximal degree permitted.

(c) The value of *qsum* exceeds the maximal possible value $Q(n+1,k,l,p)$.

Similar, but weaker situations, enforce new assignments to variables, solutions or contradiction as follows:

(d) The number of blocks reaches p. We have a solution or contradiction.

(e) The degree of some point reaches the maximal possible degree. Set to 0 all variables containing a point which reached maximal degree.

(g) If some condition contains exactly one variable with a not yet assigned value, then either we can force this variable to be 1 or we have a contradiction.

The bottleneck of computations was the time consumed by the algorithm *turexp*. The results of the above algorithms are summarized in the table VIII showing the number of nonisomorphic $(n, 5, 3)$ systems on b blocks for different values of n and b.

Table VIII: Number of nonisomorphic $(n, 5, 3)$ systems of size b

n	b	number of systems
9	12	1
9	13	1
9	14	29
10	19	0
10	20	5
10	21	95
11	28	0
11	29	1
11	30	166
12	39	0
12	40	16
13	52	1
14	67	0

The unique $(13, 5, 3)$-Turán system on 52 blocks has 5616 automorphisms, and it can be obtained by taking all triplets of collinear points in the well known projective plane of order 3. It seems that further progress in the study of $(n, 5, 3)$-Turán systems could be achieved by a careful analysis of the 16 $(12, 5, 3)$ systems on 40 blocks. 15 of them are regular of degree 10, and one has 6 points of degree 9 and 6 points of degree 11. These systems have automorphism groups of surprisingly large size, namely: 48, 64 (2 systems), 96, 128, 144, 256 (3 systems), 288, 384, 432, 768, 4608, 5184, and 1036800.

Acknowledgments. Portions of this paper was written while D. L. Kreher was visiting the Department of Mathematics at the University of Wyoming. He would like to thank the department for its warm hospitality. We would also like to thank John Van Rees for his very useful comments.

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