Covering arrays of strength 3

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- C.J. Colbourn and D.L. Kreher, Concerning difference matrices, *Designs, Codes and Cryptography*, **9**, 67-70 (1996).

- D.L. Kreher, Orthogonal arrays of strength 3, *the Journal of Combinatorial Designs*, **4** (1995).

⇒ M.A. Chateauneuf, C.J. Colbourn and D.L.Kreher, Covering arrays of strength 3, *Designs Codes and Cryptography*, **16**, 235-242 (1999).

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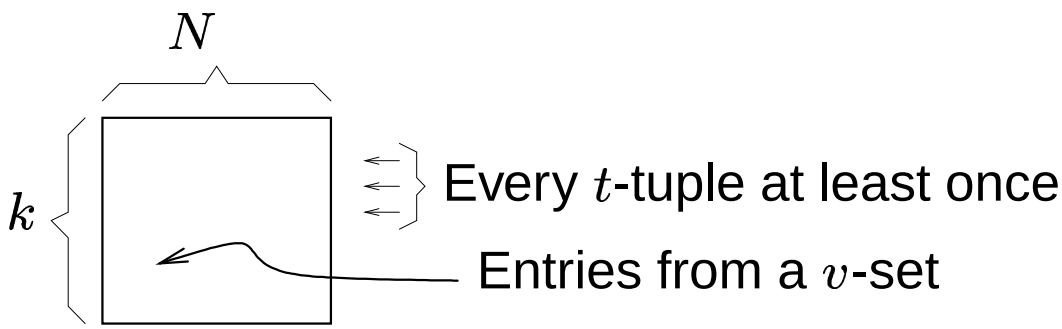
Y2K
Compliant

A Covering array $CA(N; t, k, v)$ of

size N, strength t, degree k and order v is



Example: A $CA(11, 2, 5, 3)$

$$\boxed{\begin{array}{l} 10212221010 \\ 22110120120 \\ 21201102210 \\ 12021212100 \\ 01122011220 \end{array}}$$

$CAN(t, k, v) =$ min$\{N$: there is a $CA(N, t, k, v)\}$.

So, $CAN(2, 5, 3) \leq 11$.

What's known for $t = 2$, $v = 2$

Katona 1973, Klietman & Spencer 1973:

$$k = \binom{N-1}{\lceil \frac{N}{2} \rceil} \left\{ \begin{array}{c|c} & \overbrace{}^{N-1} \\ \hline 0 & 00111 \\ 0 & 01011 \\ 0 & 10011 \\ 0 & 01101 \\ 0 & 10101 \\ 0 & 11001 \\ \vdots & 01110 \\ \vdots & 10110 \\ \vdots & 11010 \\ 0 & 11100 \end{array} \right. \leftarrow \left\{ \begin{array}{l} \lceil \frac{N}{2} \rceil\text{-subsets} \\ \text{of an } N-1\text{-set} \end{array} \right.$$

This is optimal and for k large it implies

$$N = \log k + \frac{1}{2} \log \log k \cdots$$

What's known for $t = 2$, $v > 2$

Gargano, Körner, Vaccaro 1990:

$$N = \frac{v}{2} \log k(1 + o(1))$$

Probabilistic — No explicit construction !

Only a few exact values are known.

For $t = 2$, $v = 3$:

k :	2	3	4	5	6	$7-12$	$13-16$	$\cdots$
N :	9	9	9	11	≤ 12	≤ 15	≤ 18	$\cdots$

Östegård

What's known for $t = 3$, $v = 2$

$$\underbrace{3.2\ldots\log k(1+o(1))}_{\text{Klietman \& Spencer 1973}} < CAN(3,k,2) < \underbrace{7.5\ldots\log k(1+o(1))}_{\text{Roux 1987}}$$

Probabilistic — No explicit construction !

Only a few exact values are known.

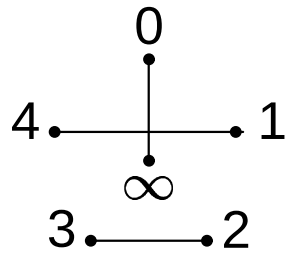
k :	3	4	5	$6-11$	$12-14$	16	$\cdots$
N :	8	8	10	12	≤ 16	≤ 17	$\cdots$

Kreher & Tonchev

A table of bounds for $k < 1,771,561$ is given in the survey:

N. Sloane, Covering Arrays and Intersecting Codes, *JCD* **1, 51-63 (1993).**

Two Small Constructions



Constructing a $CA(33; 3, 6, 3)$.

Step 1

$0 \rightarrow$	$0 \bullet\!\!-\!\!\bullet \infty$	$1 \bullet\!\!-\!\!\bullet \infty$	$2 \bullet\!\!-\!\!\bullet \infty$	$3 \bullet\!\!-\!\!\bullet \infty$	$4 \bullet\!\!-\!\!\bullet \infty$
$1 \rightarrow$	$1 \bullet\!\!-\!\!\bullet 4$	$2 \bullet\!\!-\!\!\bullet 0$	$3 \bullet\!\!-\!\!\bullet 1$	$4 \bullet\!\!-\!\!\bullet 2$	$0 \bullet\!\!-\!\!\bullet 3$
$2 \rightarrow$	$2 \bullet\!\!-\!\!\bullet 3$	$3 \bullet\!\!-\!\!\bullet 4$	$4 \bullet\!\!-\!\!\bullet 0$	$0 \bullet\!\!-\!\!\bullet 1$	$1 \bullet\!\!-\!\!\bullet 2$

0	0	1	2	2	1
1	1	0	1	2	2
2	2	1	0	1	2
3	2	2	1	0	1
4	1	2	2	1	0
∞	0	0	0	0	0

We get a starter array M

Step 2

Let $G = \text{Sym}\{0, 1, 2\}$ and apply G to M

$$[M^G, C] = [M^{g_1}, M^{g_2}, M^{g_3}, M^{g_4}, M^{g_5}, M^{g_6}, C]$$

.

$(0)(1)(2)$	$(0,1,2)$	$(0,2,1)$	$(1,2)$	$(0,1)$	$(0,2)$	C
0 1 2 2 1	1 2 0 0 2	2 0 1 1 0	0 2 1 1 2	1 0 2 2 0	2 1 0 0 1	0 1 2
1 2 2 1 0	2 0 0 2 1	0 1 1 0 2	2 1 1 2 0	0 2 2 0 1	1 0 0 1 2	0 1 2
2 2 1 0 1	0 0 2 1 2	1 1 0 2 0	1 1 2 0 2	2 2 0 1 0	0 0 1 2 1	0 1 2
2 1 0 1 2	0 2 1 2 0	1 0 2 0 1	1 2 0 2 1	2 0 1 0 2	0 1 2 1 0	0 1 2
1 0 1 2 2	2 1 2 0 0	0 2 0 1 1	2 0 2 1 1	0 1 0 2 2	1 2 1 0 0	0 1 2
0 0 0 0 0	1 1 1 1 1	2 2 2 2 2	0 0 0 0 0	1 1 1 1 1	2 2 2 2 2	0 1 2

On 3-tuples G has 5 orbits:

1. $\{(x,x,y) : x \neq y\}$
2. $\{(x,y,x) : x \neq y\}$
3. $\{(y,x,x) : x \neq y\}$
4. $\{(x,y,z) : x \neq y \neq z \neq x\}$
5. $\{(x,x,x) : x \neq y \neq z \neq x\}$

Consider any 3 rows $i, j, k \in \{0, 1, 2, 3, 4, \infty\}$.

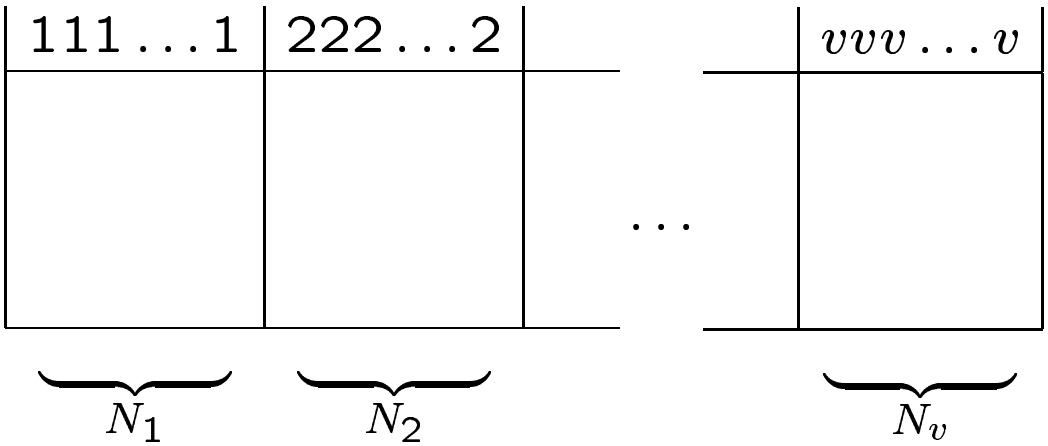
	$i \bullet\!\!-\!\!\bullet j$	$i \bullet\!\!-\!\!\bullet k$	$j \bullet\!\!-\!\!\bullet k$	$i \bullet\!\!-\!\!\bullet$	$i \bullet\!\!-\!\!\bullet$
	$k \bullet\!\!-\!\!\bullet$	$j \bullet\!\!-\!\!\bullet$	$i \bullet\!\!-\!\!\bullet$	$j \bullet\!\!-\!\!\bullet$	$j \bullet\!\!-\!\!\bullet$
	$\bullet\!\!-\!\!\bullet$	$\bullet\!\!-\!\!\bullet$	$\bullet\!\!-\!\!\bullet$	$k \bullet\!\!-\!\!\bullet$	$k \bullet\!\!-\!\!\bullet$
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$i \rightarrow$	x	x	y	x	x
$j \rightarrow$	x	y	x	y	y
$k \rightarrow$	y	x	x	z	z

Therefore $[M^G, C]$ is a $CA(33; 3, 6, 3)$.

Theorem.

$$CAN(t,k,v) \geq v \cdot CAN(t-1,k-1,v)$$

proof.



Each are $CA(N_i; t-1, k-1, v)$. []

So,

$$\underbrace{33 \geq CAN(3,6,3)}_{\text{construction}} \underbrace{\geq 3 \cdot CAN(2,5,3)}_{\text{Theorem}} = 3 \cdot \underbrace{11}_{\text{Östegård}} = 33$$

Theorem. $CAN(3,6,3) = 33$

Constructing a $CA(84; 3, 8, 4)$.

<u>Step 1</u>

Get one-factorization of K_8 with vertex set

$$\begin{aligned} \mathrm{GF}(8) &= \mathbb{Z}_2[X]/(x^3 + x + 1) \\ &= \{0, 1, x, x^2, x^3, \ldots, x^7\}. \end{aligned}$$

$0 \rightarrow$ $0 \bullet\!\!-\!\!\bullet x^0$ $0 \bullet\!\!-\!\!\bullet x^1$ $0 \bullet\!\!-\!\!\bullet x^2$ $0 \bullet\!\!-\!\!\bullet x^3$ $0 \bullet\!\!-\!\!\bullet x^4$ $0 \bullet\!\!-\!\!\bullet x^5$ $0 \bullet\!\!-\!\!\bullet x^6$

$1 \rightarrow$ $x^1 \bullet\!\!-\!\!\bullet x^3$ $x^2 \bullet\!\!-\!\!\bullet x^4$ $x^3 \bullet\!\!-\!\!\bullet x^5$ $x^4 \bullet\!\!-\!\!\bullet x^6$ $x^5 \bullet\!\!-\!\!\bullet x^0$ $x^6 \bullet\!\!-\!\!\bullet x^1$ $x^0 \bullet\!\!-\!\!\bullet x^2$

$2 \rightarrow$ $x^2 \bullet\!\!-\!\!\bullet x^6$ $x^3 \bullet\!\!-\!\!\bullet x^0$ $x^4 \bullet\!\!-\!\!\bullet x^1$ $x^5 \bullet\!\!-\!\!\bullet x^2$ $x^6 \bullet\!\!-\!\!\bullet x^3$ $x^0 \bullet\!\!-\!\!\bullet x^4$ $x^1 \bullet\!\!-\!\!\bullet x^5$

$3 \rightarrow$ $x^4 \bullet\!\!-\!\!\bullet x^5$ $x^5 \bullet\!\!-\!\!\bullet x^6$ $x^6 \bullet\!\!-\!\!\bullet x^0$ $x^0 \bullet\!\!-\!\!\bullet x^1$ $x^1 \bullet\!\!-\!\!\bullet x^2$ $x^2 \bullet\!\!-\!\!\bullet x^3$ $x^3 \bullet\!\!-\!\!\bullet x^4$

0	0	0	0	0	0	0	0
x^0	0	2	3	3	1	2	1
x^1	1	0	2	3	3	1	2
x^2	2	1	0	2	3	3	1
x^3	1	2	1	0	2	3	3
x^4	3	1	2	1	0	2	3
x^5	3	3	1	2	1	0	2
x^6	2	3	3	1	2	1	0

We get a starter array M

Step 2

Let $G = \text{Alt}\{0, 1, 2, 3\}$ and apply G to M

$$[M^G, C] = [M^{g_1}, M^{g_2}, M^{g_3}, \ldots, M^{g_{12}}, C]$$

Here $C = \left.\boxed{\begin{matrix} 0\ 1\ 2\ 3 \\ 0\ 1\ 2\ 3 \\ \vdots \\ 0\ 1\ 2\ 3 \end{matrix}}\right\}$ 8 rows

On 3-tuples G has 6 orbits:

1. $\{(x, x, y) : x \neq y\}$
2. $\{(x, y, x) : x \neq y\}$
3. $\{(y, x, x) : x \neq y\}$
4. $\{(x, y, z) : x \neq y \neq z \neq x\}$ Two orbits
5. $\{(x, x, x) : x \neq y \neq z \neq x\}$

Consider any 3 rows $i, j, k \in \{0, 1, x^1, x^2, \ldots, x^6\}$.

	i j	i k	j k	i	i	i	i
	k	j	i	j	j	j	j
				k	k	k	k
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$i \rightarrow$	x	x	y	x	x	x	x
$j \rightarrow$	x	y	x	y	y	y	y
$k \rightarrow$	y	x	x	z	z	z	z

$\underbrace{\qquad\qquad\qquad\qquad}_{\text{4 chances}}$

So this works.

Theorem.

$$CAN(3, 8, 4) \leq 7 * 12 + 4 = 88$$

Corollary.

$$CAN(2, 7, 4) \leq 22$$

Sample

On rows x^2, x^4, x^5 we get:

even=(0,3,2)

Bummer!

2	1	0	2	3	3	1
3	1	2	1	0	2	3
3	3	1	2	1	0	2

odd=(0,3,1,2)

Yea!

Using the affine group

$$AF(q) = \{x \mapsto \alpha x + \beta, \alpha, \beta \in GF(q), \alpha \neq \beta\}$$

could lead to $CAN(3, 2q, q) \leq (2q-1)q^2$,
but ..... it still eludes us.

The next case $q = 5$: The are 396 one-factorizations of K_{10}.

One-factorization of K_{10} has 9 one-factors.

One-factor's labeled by $GF(5) = \mathbb{Z}_5$ in $5! = 120$ ways.

Search space has size $120^9 \approx 2^{62}$. . . . . Too big!

Staszek observes, we use $G = AF(5)$ as automorphism group.

Only $6 = 120/(5 \cdot 4)$ cosets of $AF(5)$ in $Sym(5)$.

Search space is only of size $6^9 \approx 2^{23}$. . . . . Feasible!

The unique solutions is:

$$M = \begin{array}{|ccccccccc|}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 3 & 4 & 2 & 4 & 3 & 2 \\
1 & 1 & 0 & 4 & 3 & 4 & 2 & 2 & 3 \\
3 & 4 & 2 & 0 & 1 & 2 & 3 & 4 & 3 \\
3 & 2 & 4 & 1 & 0 & 3 & 2 & 3 & 4 \\
2 & 4 & 3 & 3 & 2 & 0 & 1 & 2 & 4 \\
2 & 3 & 4 & 2 & 3 & 1 & 0 & 4 & 2 \\
4 & 2 & 3 & 2 & 4 & 4 & 3 & 0 & 1 \\
4 & 3 & 2 & 4 & 2 & 3 & 4 & 1 & 0
\end{array}$$

$[M^G, C]$ is a $CA(185; 3, 10, 5)$. . . . . Way cool!

Permuting rows and columns we get

$$M = \begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
\begin{matrix}0\\0\end{matrix} & \begin{matrix}1\\0\end{matrix} & \begin{matrix}1\\0\end{matrix} & \begin{matrix}1\\0\end{matrix} & \begin{matrix}1\\0\end{matrix} & \begin{matrix}1\\0\end{matrix} & \begin{matrix}1\\0\end{matrix} & \begin{matrix}1\\0\end{matrix} & \begin{matrix}1\\0\end{matrix} \\
\hline
\begin{matrix}1\\1\end{matrix} & \begin{matrix}1\\0\end{matrix} & \begin{matrix}0\\1\end{matrix} & \begin{matrix}4\\2\end{matrix} & \begin{matrix}2\\4\end{matrix} & \begin{matrix}2\\3\end{matrix} & \begin{matrix}3\\2\end{matrix} & \begin{matrix}4\\3\end{matrix} & \begin{matrix}3\\4\end{matrix} \\
\hline
\begin{matrix}2\\2\end{matrix} & \begin{matrix}3\\4\end{matrix} & \begin{matrix}4\\3\end{matrix} & \begin{matrix}1\\0\end{matrix} & \begin{matrix}0\\1\end{matrix} & \begin{matrix}4\\2\end{matrix} & \begin{matrix}2\\4\end{matrix} & \begin{matrix}2\\3\end{matrix} & \begin{matrix}3\\2\end{matrix} \\
\hline
\begin{matrix}4\\4\end{matrix} & \begin{matrix}3\\2\end{matrix} & \begin{matrix}2\\3\end{matrix} & \begin{matrix}3\\4\end{matrix} & \begin{matrix}4\\3\end{matrix} & \begin{matrix}1\\0\end{matrix} & \begin{matrix}0\\1\end{matrix} & \begin{matrix}4\\2\end{matrix} & \begin{matrix}2\\4\end{matrix} \\
\hline
\begin{matrix}3\\3\end{matrix} & \begin{matrix}2\\4\end{matrix} & \begin{matrix}4\\2\end{matrix} & \begin{matrix}3\\2\end{matrix} & \begin{matrix}2\\3\end{matrix} & \begin{matrix}3\\4\end{matrix} & \begin{matrix}4\\3\end{matrix} & \begin{matrix}1\\0\end{matrix} & \begin{matrix}0\\1\end{matrix} \\
\hline
\end{array}$$

This has the form

$$\begin{array}{|c|c|c|c|c|}
\hline
\begin{matrix}0\\0\end{matrix} & B & B & B & B \\
\hline
\begin{matrix}\alpha\\\alpha\end{matrix} & I & A_1 & A_2 & A_3 \\
\hline
\begin{matrix}\alpha^2\\\alpha^2\end{matrix} & \bar{A}_3 & I & A_1 & A_2 \\
\hline
\begin{matrix}\alpha^3\\\alpha^3\end{matrix} & \bar{A}_2 & \bar{A}_3 & I & A_1 \\
\hline
\begin{matrix}\alpha^4\\\alpha^4\end{matrix} & \bar{A}_1 & \bar{A}_2 & \bar{A}_3 & I \\
\hline
\end{array}$$

Choosing different representatives and adding column of 0s

$$M = \begin{array}{|cc|cc|cc|cc|cc|}
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 2 & 2 & 4 & 4 & 3 & 3 \\
\hline
0 & 1 & 1 & 0 & 3 & 4 & 3 & 2 & 2 & 4 \\
0 & 1 & 0 & 1 & 4 & 3 & 2 & 3 & 4 & 2 \\
\hline
0 & 2 & 3 & 4 & 2 & 0 & 1 & 3 & 1 & 4 \\
0 & 2 & 4 & 3 & 0 & 2 & 3 & 1 & 4 & 1 \\
\hline
0 & 4 & 3 & 2 & 1 & 3 & 4 & 0 & 2 & 1 \\
0 & 4 & 2 & 3 & 3 & 1 & 0 & 4 & 1 & 2 \\
\hline
0 & 3 & 2 & 4 & 1 & 4 & 2 & 1 & 3 & 0 \\
0 & 3 & 4 & 2 & 4 & 1 & 1 & 2 & 0 & 3 \\
\hline
\end{array}$$

Which is symmetric

Using Gargano, Körner, Vaccaro result for k we have

$$CAN(2,k,v) = \frac{v}{2}\log(k(1+o(1))$$

so the recursive upper bound suggest that

$$CAN(3,k+1,v) \leq \frac{v^2}{2}\log(k(1+o(1))$$

Using perfect hash families and orthogonal arrays instead of starter arrays and groups we can get using a result of Atici,Magliveras, Stinson and Wei:

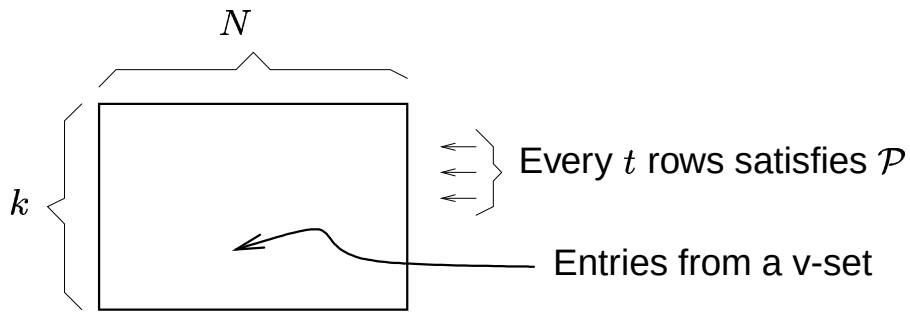
$$CAN(3,5^{2^j},v) \leq 3\cdot 4^j v^3, \text{ for all } j$$

So

$$CAN(3,k,v) \leq \frac{3v^3}{(\log 5)^2}(\log k)^2, \text{when } k = 5^{2^j}$$

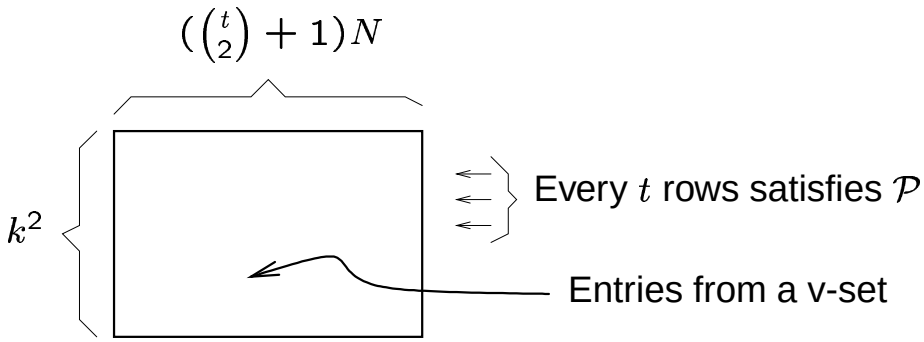
What Atici, Magilveras, Stinson and Wei really prove is:

Let $A =$



where $\mathcal{P}$ is a property that does not depend on the ordering of the t rows and $\gcd(k, \binom{t}{2}!) = 1$, then there is an array B such that

$B =$



There is a $CA(33; 3, 5, 3)$.

$$\begin{aligned} \Rightarrow CA(33 \cdot 4; 3, 5^2, 3) \quad &\Rightarrow \quad CA(33 \cdot 4^2; 3, 5^{2^2}, 3) \\ &\Rightarrow \quad CA(33 \cdot 4^3; 3, 5^{2^3}, 3) \\ &\Rightarrow \quad \cdots \Rightarrow CA(33 \cdot 4^j; 3, 5^{2^j}, 3) \end{aligned}$$

So

$$CAN(3, k, 3) \leq \frac{33}{(\log 5)^2}(\log k)^2 \approx 6.12(\log k)^2$$

when $k = 5^{2^j}$.