Graphical Designs

Donald L. Kreher

Department of Mathematical Sciences
Michigan Technological University

24^{th} Midwestern Conference on
Combinatorics, Cryptography, and Computing
Normal Illinois
September 2010

t-wise balanced designs

Let $X = \{x_1, x_2, \ldots, x_v\}$ be a set of v-points and let $0 < t < v$ be a positive integer.

Can we find a collection

$$\mathcal{B} = \{B_1, B_2, \ldots, B_b\}$$

of subsets of X so that every t-element subset of X is contained in exactly one of them?

t-wise balanced designs

Let $X = \{x_1, x_2, \ldots, x_v\}$ be a set of v-points and let $0 < t < v$ be a positive integer.

Can we find a collection

$$\mathcal{B} = \{B_1, B_2, \ldots, B_b\}$$

of subsets of X so that every t-element subset of X is contained in exactly one of them?

Example:

$$\begin{aligned} t &= 2 \\ X &= \{1,2,3,4,5,6,7\} \\ \mathcal{B} &= \left\{ \begin{array}{c} \{1,2,3\},\{1,4,5\},\{1,6,7\},\{2,4,6\}, \\ \{2,5,7\},\{3,4,7\},\{3,5,6\} \end{array} \right\} \end{aligned}$$

Every pair is in exactly one of the chosen subsets.

123	**145**	**167**	**246**	**257**	**347**	**356**

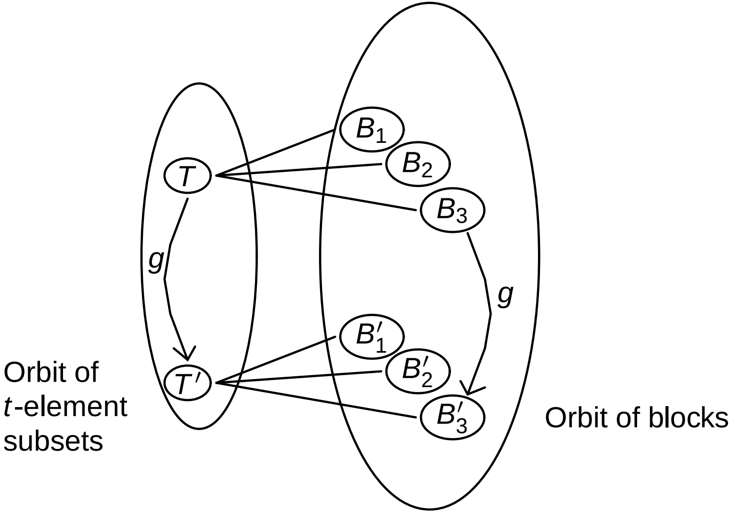
	123	145	167	246	257	347	356
12							
13							
14							
15							
16							
17							
23							
24							
25							
26							
27							
34							
35							
36							
37							
45							
46							
47							
56							
57							
67							

	123	145	167	246	257	347	356
12	√						
13	√						
14							
15							
16							
17							
23	√						
24							
25							
26							
27							
34							
35							
36							
37							
45							
46							
47							
56							
57							
67							

	123	145	167	246	257	347	356
12	√						
13	√						
14		√					
15		√					
16			√				
17			√				
23	√						
24				√			
25					√		
26				√			
27					√		
34						√	
35							√
36							√
37						√	
45		√					
46				√			
47						√	
56							√
57					√		
67			√				

Orbits

Let G be a possible automorphism group.



$$\begin{aligned} \text{Let } G &= \langle(1,3,5)(2,6,4)(7),(1,3)(6,4)(7)\rangle \\ &= \left\{ \begin{array}{c} I,(1,3,5)(2,6,4),(1,5,3)(2,4,6) \\ (1,3)(6,4),(1,5)(2,6),(2,4)(3,5) \end{array} \right\} \end{aligned}$$

$$\text{Let } G = \langle (1,3,5)(2,6,4)(7), (1,3)(6,4)(7) \rangle$$

$$= \left\{ \begin{array}{c} I, (1,3,5)(2,6,4), (1,5,3)(2,4,6) \\ (1,3)(6,4), (1,5)(2,6), (2,4)(3,5) \end{array} \right\}$$

$\{26, 24, 46\}$

$\left\{ \begin{array}{c} 12, 14, 23 \\ 36, 56, 45 \end{array} \right\}$

$\{16, 34, 25\}$

$\{13, 15, 35\}$

$\{27, 67, 47\}$

$\{17, 37, 57\}$

$$\text{Let } G = \langle (1,3,5)(2,6,4)(7), (1,3)(6,4)(7) \rangle$$

$$= \left\{ \begin{array}{c} I, (1,3,5)(2,6,4), (1,5,3)(2,4,6) \\ (1,3)(6,4), (1,5)(2,6), (2,4)(3,5) \end{array} \right\}$$

		126	136				127		
		146	156				367		
		234	345				457		
	124	346	134	123		267	147	167	137
	236	256	125	356		247	237	347	357
246	456	245	235	145	135	467	567	257	157

$\{26, 24, 46\}$

$\left\{ \begin{array}{c} 12, 14, 23 \\ 36, 56, 45 \end{array} \right\}$

$\{16, 34, 25\}$

$\{13, 15, 35\}$

$\{27, 67, 47\}$

$\{17, 37, 57\}$

$$\text{Let } G = \langle (1,3,5)(2,6,4)(7), (1,3)(6,4)(7) \rangle$$

$$= \left\{ \begin{array}{c} I, (1,3,5)(2,6,4), (1,5,3)(2,4,6) \\ (1,3)(6,4), (1,5)(2,6), (2,4)(3,5) \end{array} \right\}$$

	246	124 236 456	126 146 234 346 256 245	136 156 345 134 125 235	123 356 145	135	267 247 467	127 367 457 147 237 567	167 347 257	137 357 157
{26, 24, 46}										
{12, 14, 23 36, 56, 45}										
{16, 34, 25}										
{13, 15, 35}										
{27, 67, 47}										
{17, 37, 57}										

$$\text{Let } G = \langle (1,3,5)(2,6,4)(7), (1,3)(6,4)(7) \rangle$$

$$= \left\{ \begin{array}{c} I, (1,3,5)(2,6,4), (1,5,3)(2,4,6) \\ (1,3)(6,4), (1,5)(2,6), (2,4)(3,5) \end{array} \right\}$$

	246	124 236 456	126 146 234 346 256 245	136 156 345 134 125 235	123 356 145	135	267 247 467	127 367 457 147 237 567	167 347 257	137 357 157
{26, 24, 46}	1	1	2	0	0	0	1	0	0	0
{12, 14, 23 36, 56, 45}	0	1	1	1	1	0	0	1	0	0
{16, 34, 25}	0	0	2	2	0	0	0	0	1	0
{13, 15, 35}	0	0	0	2	1	1	0	0	0	0
{27, 67, 47}	0	0	0	0	0	0	2	2	1	0
{17, 37, 57}	0	0	0	0	0	0	0	2	1	2

$$\text{Let } G = \langle (1,3,5)(2,6,4)(7), (1,3)(6,4)(7) \rangle$$

$$= \left\{ \begin{array}{c} I, (1,3,5)(2,6,4), (1,5,3)(2,4,6) \\ (1,3)(6,4), (1,5)(2,6), (2,4)(3,5) \end{array} \right\}$$

	246	124, 236, 456	126, 146, 234, 346, 256, 245	136, 156, 345, 134, 125, 235	123, 356, 145	135	267, 247, 467	127, 367, 457, 147, 237, 567	167, 347, 257	137, 357, 157
{26, 24, 46}	1	1	2	0	0	0	1	0	0	0
{12, 14, 23, 36, 56, 45}	0	1	1	1	1	0	0	1	0	0
{16, 34, 25}	0	0	2	2	0	0	0	0	1	0
{13, 15, 35}	0	0	0	2	1	1	0	0	0	0
{27, 67, 47}	0	0	0	0	0	0	2	2	1	0
{17, 37, 57}	0	0	0	0	0	0	0	2	1	2

$$\text{Let } G = \langle (1,3,5)(2,6,4)(7), (1,3)(6,4)(7) \rangle$$

$$= \left\{ \begin{array}{c} I, (1,3,5)(2,6,4), (1,5,3)(2,4,6) \\ (1,3)(6,4), (1,5)(2,6), (2,4)(3,5) \end{array} \right\}$$

	246	124 236 456	126 146 234 346 256 245	136 156 345 134 125 235	123 356 145	135	267 247 467	127 367 457 147 237 567	167 347 257	137 357 157
{26, 24, 46}	1	1	2	0	0	0	1	0	0	0
{12, 14, 23 36, 56, 45}	0	1	1	1	1	0	0	1	0	0
{16, 34, 25}	0	0	2	2	0	0	0	0	1	0
{13, 15, 35}	0	0	0	2	1	1	0	0	0	0
{27, 67, 47}	0	0	0	0	0	0	2	2	1	0
{17, 37, 57}	0	0	0	0	0	0	0	2	1	2

Formal definition ...

A t-wise balanced design with parameters t-$(v, \mathcal{B}, \lambda)$ is a pair $(X, \mathcal{B})$ where X is a set of v *points* and $\mathcal{B}$ is a collection of subsets of X called *blocks* such that

- if $B \in \mathcal{B}$, then $|B| \in \mathscr{K}$;
- $t, v \notin \mathcal{B}$; and
- if $T \subset X$, with $|T| = t$, then there exactly λ blocks $B \in \mathcal{B}$ that contain T.

For example:

$$\begin{aligned} X &= \{1,2,3,4,5,6\} \\ \mathcal{B} &= \left\{ \begin{array}{l} \{1,2,3\},\{1,4,5\},\{2,4,6\},\{3,5,6\}, \\ \{1,3,4,6\},\{1,2,5,6\},\{2,3,4,5\} \end{array} \right\} \end{aligned}$$

is a 2-$(6, \{3,4\}, 2)$ design.

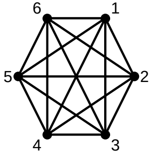
Find or classify all t-designs.

Well ... at least the interesting ones.

Graphical designs

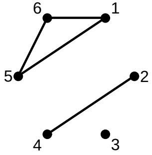
Points: X = edges of K_n the complete graph.

$K_6 \quad =$



$$X = \left\{ \begin{matrix} 12, 13, 14, 15, 16, 23, 24, \\ 25, 26, 34, 35, 36, 45, 46, 56 \end{matrix} \right\}$$

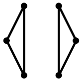
Blocks are subgraphs!



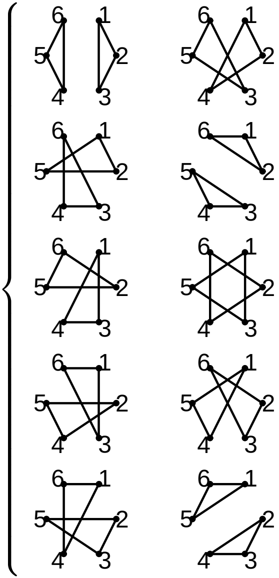
$= \quad \{15, 16, 56, 24\}$

Graphical designs

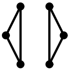
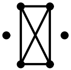
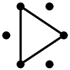
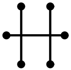
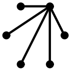
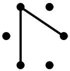
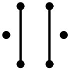
Group: S_n the automorphism group of K_n



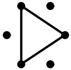
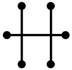
$=$

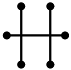
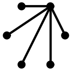


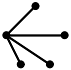
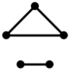
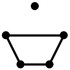
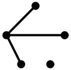
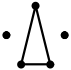
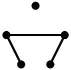
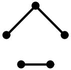
This is an orbit!

					
	1	3	1	0	1
	2	1	0	1	0

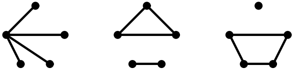
So,

  gives a 2-$(15, 3, 1)$ design.

  gives a 2-$(15, \{3, 5\}, 1)$ design.

			
	1	0	0
	0	1	0
	0	0	1
	0	1	0

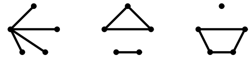
So,



gives a 3-$(10, 4, 1)$ design.

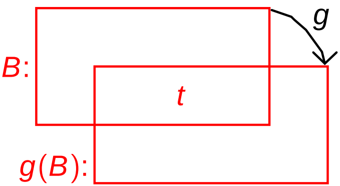
Theorem (Chouinard,Kramer,Kreher 1983)

A complete list of graphical t-$(v, \mathcal{K}, 1)$ designs.

1-(6,2,1)	$n = 4$	
$2\text{-}(15, 3, 1)$	$n = 6$	
$2\text{-}(15, \{3, 5\}, 1)$	$n = 6$	
$3\text{-}(10, 4, 1)$	$n = 5$	
$4\text{-}(15, \{5, 7\}, 1)$	$n = 6$	

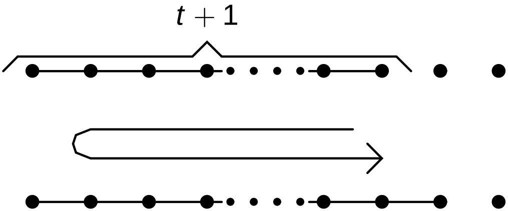
Two key ideas for graphical t-$(\binom{n}{2}, \mathcal{B}, 1)$ designs

A. Let B be a block and
let g be an automorphism.
Then $|B \cap g(B)| \geq t \Rightarrow B = g(B)$.

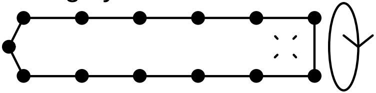


Example

If $n \geq t + 3$ a block B cannot have a path of length $t + 1$.



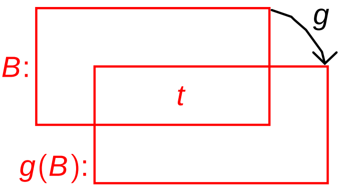
Get spanning cycle..



Repeat... and get K_n — a contradiction.

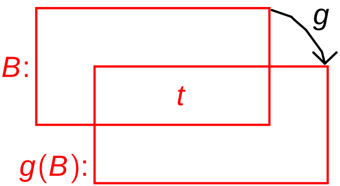
Two key ideas for graphical t-$(\binom{n}{2}, \mathcal{B}, 1)$ designs

A. Let B be a block and
let g be an automorphism.
Then $|B \cap g(B)| \geq t \Rightarrow B = g(B)$.

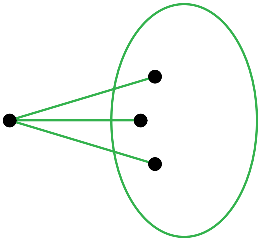


Two key ideas for graphical t-$(\binom{n}{2}, \mathcal{B}, 1)$ designs

A. Let B be a block and
let g be an automorphism.
Then $|B \cap g(B)| \geq t \Rightarrow B = g(B)$.



B. If $t \geq n - 1$, then the derived design
of a t-$(\binom{n}{2}, \mathcal{B}, 1)$ on K_n
with respect to a star $K_{1,n-1}$ is a
$(t - n + 1)$-$(\binom{n-1}{2}, \mathcal{B}', 1)$ on K_{n-1}.



The proof

The proof

1. Get a list of *small* graphical designs.

The proof

1. Get a list of *small* graphical designs.
2. Use A to show that if $t < (n-1)$, then it appears on the list.

The proof

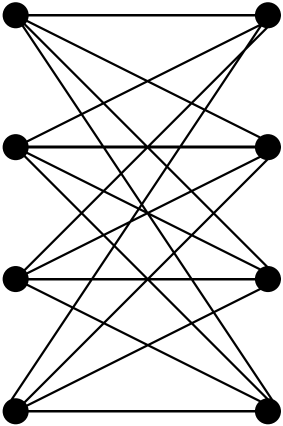
1. Get a list of *small* graphical designs.
2. Use A to show that if $t < (n-1)$, then it appears on the list.
3. Use B to show that if $t \geq (n-1)$, then it is the extension of a design on the list.

The proof

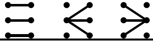
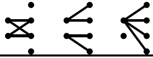
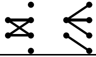
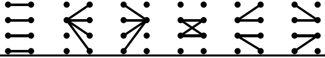
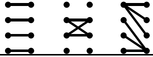
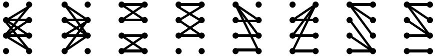
1. Get a list of *small* graphical designs.
2. Use A to show that if $t < (n-1)$, then it appears on the list.
3. Use B to show that if $t \geq (n-1)$, then it is the extension of a design on the list.
4. Find all extensions.

Bigraphical designs

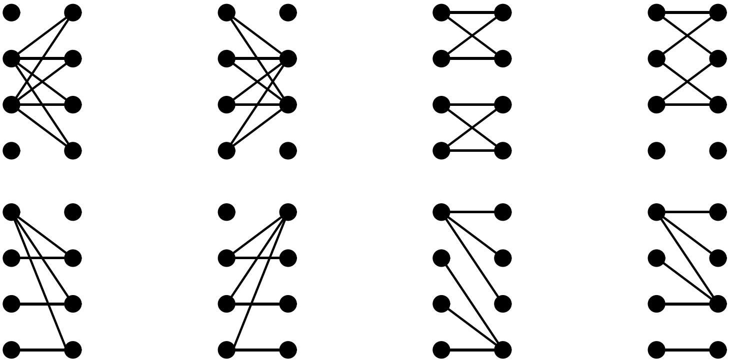
A t–wise balanced design $(X, \mathcal{B})$ of type t–$(m \cdot n, \mathcal{B}, \lambda)$ is *bigraphical* if X is the set of edges of the complete bipartite graph $K_{m,n}$ and whenever B is a block and α is an automorphism of $K_{m,n}$ (that fixes the independent sets), then $\alpha(B)$ is also a block.



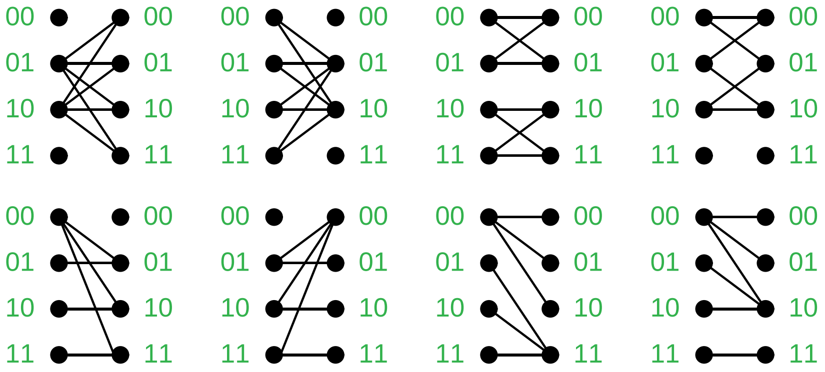
Theorem. (Hoffman and Kreher 1994) The bigraphical t-designs of index 1.

D_1:	$2 \leq m \leq n$ $1-(mn, n, 1)$	$K_{1,n}$
D_2:	$2 \leq m \leq n$ $1-(mn, m, 1)$	$K_{m,1}$
D_3:	$m = n = 2$ $1-(4, 2, 1)$	
D_4:	$m = n = 3$ $2-(9, 3, 1)$	
D_5:	$m = 2,\ n = 4$ $3-(8, 4, 1)$	
D_6:	$m = 2,\ n = 4$ $3-(8, 4, 1)$	
D_7:	$m = n = 4$ $3-(16, 4, 1)$	
D_8:	$m = n = 4$ $3-(16, \{4, 6\}, 1)$	
D_9:	$m = n = 4$ $5-(16, \{6, 8\}, 1)$	

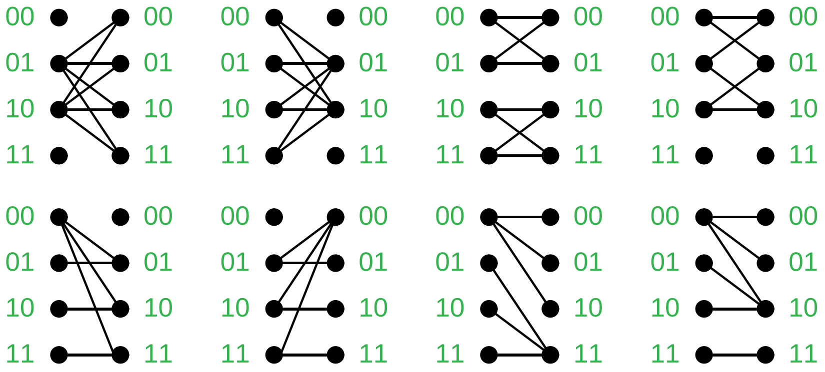
The 5–$(16, \{6, 8\}, 1)$ design



The 5–$(16, \{6, 8\}, 1)$ design



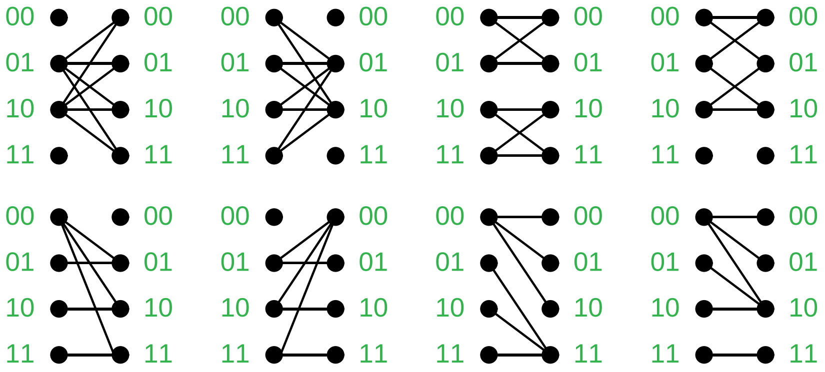
The 5–$(16, \{6, 8\}, 1)$ design



The 8-element blocks are the 3–dimensional affine subspaces.

A 6-element set $\{\vec{x_1}, \ldots, \vec{x_6}\}$ is a block $\iff$ $\vec{x_1} + \cdots + \vec{x_6} = \vec{0}$.

The 5–$(16, \{6, 8\}, 1)$ design



The 8-element blocks are the 3–dimensional affine subspaces.

A 6-element set $\{\vec{x_1}, \ldots, \vec{x_6}\}$ is a block $\iff$ $\vec{x_1} + \cdots + \vec{x_6} = \vec{0}$.

This vector space construction is due to R.M. Wilson.

Mulitgraphical designs

$$K_n^r = K_{\underbrace{n,n,n,\ldots,n}_{r}}$$

When $n = 1$ theses are the graphical designs.
There are 4 of them, with $t \geq 2$.

(Chouinard, Kramer and Kreher 1983)

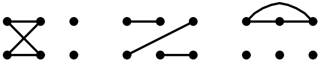
When $r = 2$ theses are bigraphical designs.
There are 7 of them, with $t \geq 2$.

(Hoffman and kreher 1994)

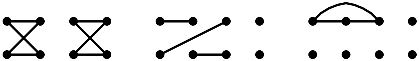
When $n > 1$ and $r > 2$ there are 2 more.

(Olsen and Kreher 1998)

A 2-$(12, \{3,4\}, 1)$ design:



A 2-$(24, \{3,4\}, 1)$ design:



Open questions

- $t = 6$
- Other graphs or incidence structures.
- A general result such as:
  Let $(X_n, \mathcal{E}_n)$ be a family of incidence structures with with automorphism groups G_n.
  Show that if $|G_n|$ is sufficiently large with respect to $|E_n|$, then there are only finitely many designs w.r.t. this group action.

- Slides: www.math.mtu.edu/∼kreher/ABOUTME/talk.html
- L.G. Chouinard, D.L. Kreher and E.S. Kramer, Graphical t-wise Balanced Designs, *Discrete Mathematics* **46** (1983) 227–240.
- D. de Caen and D.L. Kreher, The 3-hypergraphical Steiner quadruple systems of order twenty, in *Graphs, Matrices and Designs* Ed. Rolf Rees, *Lecture Notes in Pure and Applied Mathematics* **139** (1992) 85–92.
- D.G. Hoffman and D.L. Kreher, The Bigraphical t-Wise Balanced Designs of Index One, *The Journal of Combinatorial Designs* **2** (1994) 41–48.
- L.M. Weiss and D.L. Kreher, The Bigraphical t-Wise Balanced Designs of Index Two, *The Journal of Combinatorial Designs* **3** (1995) 233–255.
- C.L. Olsen and D.L. Kreher, Steiner graphical t-wise balanced designs of type n^r , *Statistical Planning and Inference* **86** (2000) 535–566.
- Y.M. Chee, D.L. Kreher, Graphical Designs *The CRC handbook of combinatorial designs* C.J. Colbourn and J.H. Dinitz (Editors) CRC Press, Boca Raton, 2007.