A hole-size bound for incomplete tBDs.

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- D.L. Kreher, R.S. Rees, A hole-size bound for incomplete t-wise balanced designs, *submitted to: the Journal of Combinatorial designs.*

- D.L. Kreher, and R.S. Rees, On the maximum size of a hole in an incomplete t-wise balanced design with specified minimum block size, *submitted to: OSU Mathematical Research Institute Monograph Series.*

  `http://www.math.mtu.edu/~kreher/ABOUTME/preprints.html`

  `http://www.math.mtu.edu/~kreher/ABOUTME/talk.html`

tBD

A *t–wise balanced design* (tBD) of type t–$(v, \mathcal{K}, \lambda)$ is a pair $(X, \mathcal{B})$

- X is a v–element set of *points*
- $\mathcal{B}$ is a collection of subsets of X called *blocks*
- $B \in \mathcal{B} \Rightarrow |B| \in \mathcal{K}$
- every t–element subset of X is in exactly λ blocks.

proper if $t < k < v$ for each $k \in \mathcal{K}$
Steiner if $\lambda = 1$

Example: A proper 2BD of type 2–$(9, \{3, 4\}, 4)$.

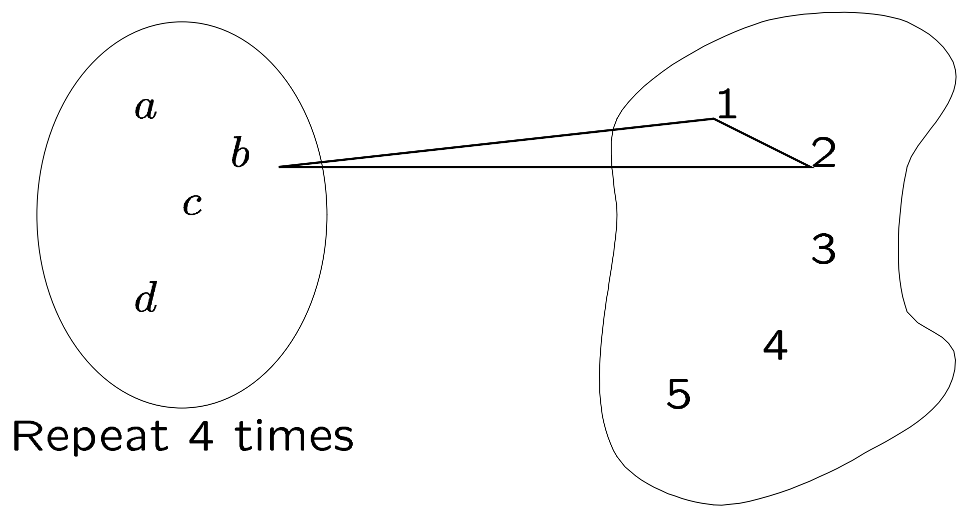
$$X=\{a, b, c, d, 1, 2, 3, 4, 5\}$$

$$\mathcal{B}=\left\{\begin{array}{l} abcd, a12, a13, a14, a15, a23, a24, a25, a34, a35, a45, \\ abcd, b12, b13, b14, b15, b23, b24, b25, b34, b35, b45, \\ abcd, c12, c13, c14, c15, c23, c24, c25, c34, c35, c45, \\ abcd, d12, d13, d14, d15, d23, d24, d25, d34, d35, d45 \end{array}\right\}$$

$$\begin{aligned}\mathcal{B}=&\{xij : x \in \{a, b, c, d\}, i, j \in \{1, 2, \ldots, 5\}\} \\ &\cup\{abcd, abcd, abcd, abcd\}\end{aligned}$$

Example

- A 2BD of type 2–$(9, \{3, 4\}, 4)$



History

1983 E.S. Kramer, Some results on t-wise balanced designs, Ars Combin. 15 (1983), 179–192.

- If B is a block in a Steiner tBD, then

$$|B| \leq (v+t-3)/2 \text{ for } t \geq 2.$$

- If B is a block in a Steiner tBD, then

$$\begin{aligned} |B| &\leq (v-1)/2 \text{ for } t = 2,4 \text{ while} \\ |B| &\leq v/2 \text{ for } t = 3,5. \end{aligned}$$

- **Conjecture:** If B is a block in a Steiner tBD, then

$$\begin{aligned} |B| &\leq (v-1)/2 \text{ for } t \text{ even while} \\ |B| &\leq v/2 \text{ for } t \text{ odd.} \end{aligned}$$

2000 M. Ira and E.S. Kramer, A block-size bound for Steiner 6-wise balanced designs, J. Combin. Designs, 8 (2000), 141-145.

- If B is a block in a Steiner 6BD, then

$$|B| \leq v/2.$$

ItBD

An *incomplete t–wise balanced design* (ItBD) of type t–$(v, h, \mathcal{K}, \lambda)$ is a triple $(X, H, \mathcal{B})$

- X is a v–element set of *points*
- H is an h–element set of points called the *hole*
- $\mathcal{B}$ is a collection of subsets of X called *blocks*
- $B \in \mathcal{B} \Rightarrow |B| \in \mathcal{K}$
- every t–element subset of X is either in the hole or in exactly λ blocks, but not both.

Example: A proper I2BD of type 2–(9, 4, {3}, 4).

$$X=\{a, b, c, d, 1, 2, 3, 4, 5\}$$

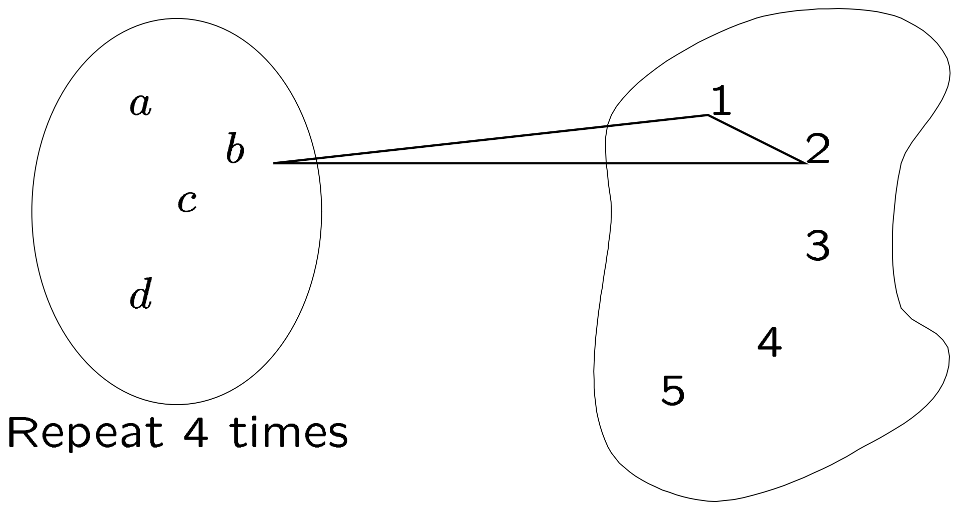
$$H=\{a, b, c, d\}$$

$$\mathcal{B}=\left\{\begin{matrix} a12,a13,a14,a15,a23,a24,a25,a34,a35,a45, \\ b12,b13,b14,b15,b23,b24,b25,b34,b35,b45, \\ c12,c13,c14,c15,c23,c24,c25,c34,c35,c45, \\ d12,d13,d14,d15,d23,d24,d25,d34,d35,d45 \end{matrix}\right\}$$

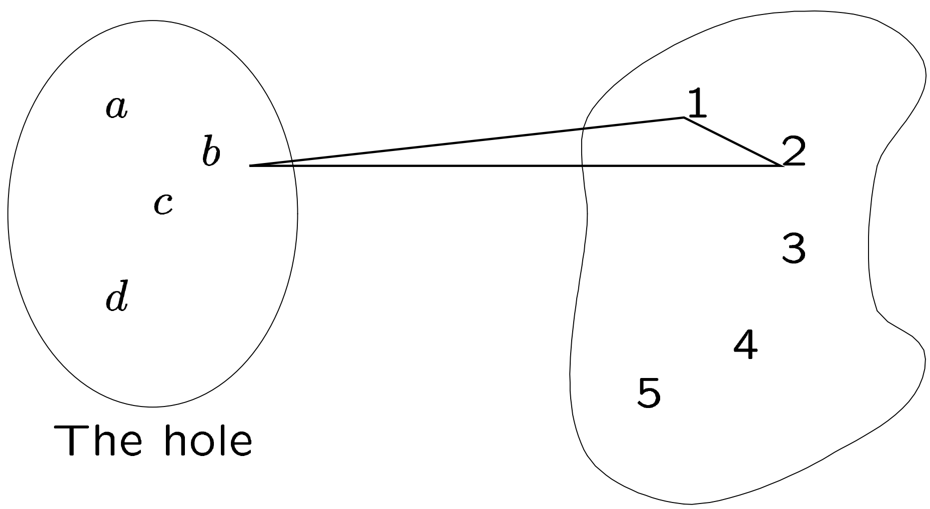
$$\mathcal{B}=\{xij : x \in \{a, b, c, d\}, i, j \in \{1, 2, \ldots, 5\}\}$$

Examples

- A 2BD of type 2–$(9, \{3, 4\}, 4)$



- A I2BD of type 2–$(9, 4, \{3\}, 4)$



Equivalence

A ItBD of type t–$(v, h, \mathcal{K}, \lambda)$ is the same as

a tBD of type t–$(v, \mathcal{K} \cup \{h\}, \lambda)$

with a block of size h repeated λ times.

A Steiner tBD is a ItBD with $\lambda = 1$ in which any block of the tBD can be considered as the hole.

Main Theorem

If H is a hole in a ItBD with any λ, then

$$\begin{aligned} |H| &\leq (v-1)/2 \text{ for } t \text{ even while} \\ |H| &\leq v/2 \text{ for } t \text{ odd.} \end{aligned}$$

Corollary

If B is a block in a Steiner tBD, then

$$\begin{aligned} |B| &\leq (v-1)/2 \text{ for } t \text{ even while} \\ |B| &\leq v/2 \text{ for } t \text{ odd.} \end{aligned}$$

Reduction to $\mathcal{K} = \{t+1\}$.

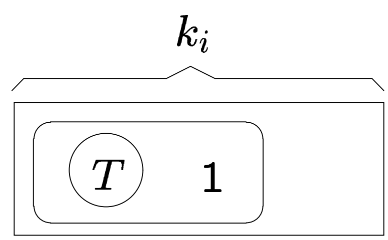
If a proper ItBD $(X, H, \mathcal{B})$ of type t–$(v, h, \mathcal{K}, \lambda)$ exists, then a proper ItBD of type t–$(v, h, \{t+1\}, \lambda')$ exists for some λ'.

Proof: $\mathcal{K} = \{k_1, k_2, \ldots, k_\ell\}$
Replace each block $B \in \mathcal{B}$ of size $|B| = k_i$ with its $\binom{k_i}{t+1}$ $(t+1)$-subsets

and repeat each

$$c_i = \prod_{j=1, j\neq i}^{\ell} (k_j - t)$$

times.



T in $k_i - t$
$(t+1)$-subsets.

Let $T \subseteq X$, $|T| = t$.

- $T \subseteq H$ no problem.

- T in r_i blocks of size k_i in $\mathcal{B}$, then $r_1 + r_2 + \cdots + r_\ell = \lambda$. In the new design T is contained in

$$\begin{aligned} & r_1 c_1 (k_1 - t) + r_2 c_2 (k_2 - t) + \cdots + r_\ell c_\ell (k_\ell - t) \\ = \ & \textstyle\sum_{i=1}^{\ell} \left\{ r_i \prod_{j=1, j\neq i}^{\ell} (k_j - t) \right\} (k_i - t) \\ = \ & (r_1 + r_2 + \cdots + r_\ell) \prod_{j=1}^{\ell} (k_j - t) \\ = \ & \lambda \prod_{k \in \mathcal{K}} (k - t) = \lambda' \text{ blocks, as required.} \quad \blacksquare \end{aligned}$$

Even suffices

If the main theorem is true when t is even, then it is also true when t is odd.

Proof:

- Suppose there is a t–$(v, h, \{t+1\}, \lambda)$ with $h > v/2$ and t odd, $t \geq 3$.
- Let x be in the hole and derive through x.
- Result: a $(t-1)$–$(v-1, h-1, \{t\}, \lambda)$
- But $t-1$ is even and
$$(v-1)-1 < (2h-1)-1 = 2(h-1).$$
A contradiction.

∎

Basis step

There is no 2–$(v, h, \{3\}, \lambda)$ with $h > (v-1)/2$.

Proof:

- $(X, H, \mathcal{B})$ a 2–$(v, h, \{3\}, \lambda)$
- derive w.r.t. $x \in H$.
- Result: A λ–regular multigraph on $X \setminus H$.
- Do this for all $x \in H$ to get a

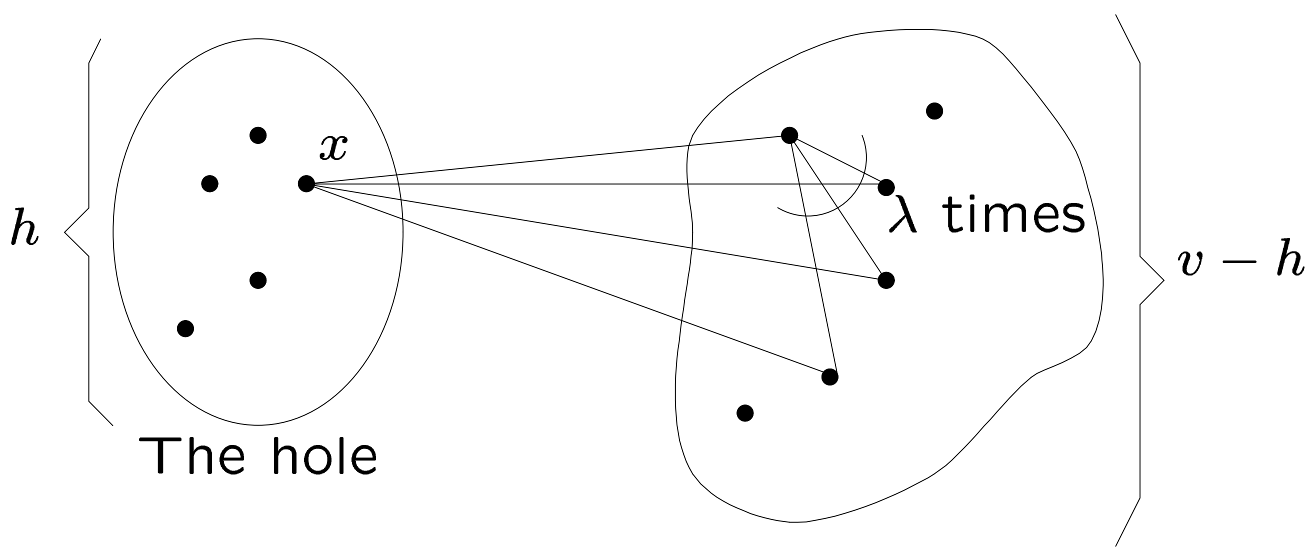
  λh–regular multigraph G on $X \setminus H$.
- Each edge in G is repeated at most λ times.
- Hence counting edges incident to a fixed vertex

  $$\lambda h \leq \lambda(v - h - 1)$$

  Thus $h \leq (v-1)/2$.

∎

A λ–regular multigraph



We need only worry about $v = 2h - 1$ and $v = 2h$.

We need to show that no ItBD exists with $h > \frac{v-1}{2}$. Thus we must rule out:

$$v = \underbrace{2h, 2h-1}_{\text{direct proof}}, \underbrace{2h-2, 2h-3 \ldots}_{\text{induction}}.$$

Suppose $t' \geq 4$ is even and we have shown

1. there are no t'–$(2h-1, h, \{t'+1\}, \lambda)$
2. there are no t'–$(2h, h, \{t'+1\}, \lambda)$
3. The Main Theorem holds for $t = t' - 2$.

Then there cannot be any

$$t'\text{–}(v, h, \{t'+1\}, \lambda) \text{ with } v \leq 2h-2$$

For if so, derive through two points in the hole to get a

$$t\text{–}(v-2, h-2, \{t+1\}, \lambda)$$

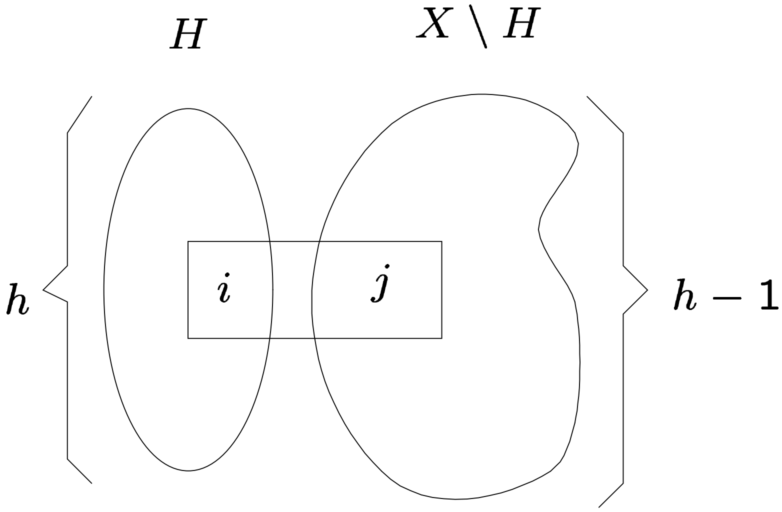
where

$$v - 2 \leq (2h-2) - 2 = 2h - 4 = 2(h-2)$$

A contradiction.

Thus the main theorem would hold for all even t' by induction.

No t–$(2h-1, h, \{t+1\}, \lambda)$ with t even.



	x_0 $(0,t+1)$	x_1 $(1,t)$	x_2 $(2,t-1)$	x_3 $(3,t-2)$	$\cdots$ $\cdots$	x_{t-1} $(t-1,2)$	$\vec{t}$
$(0,t)$	$t+1$	1					$\binom{h-1}{t}$
$(1,t-1)$		t	2				$h\binom{h-1}{t-1}$
$(2,t-2)$			$t-1$	3			$\binom{h}{2}\binom{h-1}{t-2}$
$\vdots$				$\ddots$	$\ddots$		$\vdots$
$(t-2,2)$					3	$t-1$	$\binom{h}{t-2}\binom{h-1}{2}$
$(t-1,1)$						2	$\binom{h}{t-1}(h-1)$

Solve $A\vec{x} = \vec{t}$

A is invertible, whence

$$\vec{x} = A^{-1}(\lambda\vec{t}) = \lambda A^{-1}\vec{t}.$$

Indexing the rows and columns of A^{-1} by $1, 2, \ldots, t$

$$A^{-1}[1,j] = (-1)^{j-1}\frac{1}{j\binom{t+1}{j}}$$

for $j = 1, 2, \ldots, t$. Therefore

$$\begin{aligned} x_0 &= \lambda\sum_{j=1}^{t}(-1)^{j-1}\frac{1}{j\binom{t+1}{j}}\binom{h}{j-1}\binom{h-1}{t-j+1} \\ &= -\frac{\lambda}{t+1}\sum_{j=1}^{t}(-1)^{j}\frac{1}{\binom{t}{j-1}}\binom{h}{j-1}\binom{h-1}{t-j+1} \\ &= \frac{-\lambda}{t+1}\binom{h-1}{t-1} < 0. \end{aligned}$$

This is a contradiction because,
x_0 counts the number of blocks disjoint from H
and consequently cannot be negative.
Therefore there is no t–$(2h-1, h, \{t+1\}, \lambda)$ with $h \geq t$.

No t–$(2h, h, \{t+1\}, \lambda)$ with t even.
Same matrix equation as before

$$A\vec{x} = \lambda\vec{t}$$

except this time

$$t_i = \binom{h}{i}\binom{h}{t-i}$$

for $i = 0, 1, \ldots, t-1$. So

$$\begin{aligned} x_0 &= \lambda \sum_{j=1}^{t} (-1)^{j-1} \frac{1}{j\binom{t+1}{j}} \binom{h}{j-1}\binom{h}{t-j+1} \\ &= \frac{\lambda}{t+1} \sum_{j=1}^{t} (-1)^{j-1} \frac{1}{\binom{t}{j-1}} \binom{h}{j-1}\binom{h}{t-j+1} \\ &= \frac{-\lambda t}{(t+1)(2h-t)}\binom{h}{t} < 0, \end{aligned}$$

again a contradiction. Thus no proper ItBD of type t–$(2h, h, \{t+1\}, \lambda)$ with $h \geq t$ can exist. We have therefore established the following.

Theorem 1 *There do not exist proper* ItBD*s of types* t–$(2h-1, h, \{t+1\}, \lambda)$ *or* t–$(2h, h, \{t+1\}, \lambda)$ *for any* λ *and any* $2 \leq t \leq h$*, when* t *is even.*

Hence we have established the main Theorem and so the validity of Kramer's conjecture.

Main Theorem

If H is a hole in an ItBD with any λ, then

$$\begin{aligned} |H| &\leq (v-1)/2 \text{ for } t \text{ even while} \\ |H| &\leq v/2 \text{ for } t \text{ odd.} \end{aligned}$$

Bounds are sharp

1. For each odd $t \geq 3$ and each $h \geq t+1$ there is a

$$t\text{–}(2h, h, \{t+1\}, (2h-t)t!\binom{h-1}{t}).$$

2. For each even $t \geq 2$ and each $h \geq t+1$ there is a

$$t\text{–}(2h+1, h, \{t+1\}, (2h-t+1)(t+1)!\binom{h}{t+1}).$$

For t odd we construct a $t-(2h, h, \{t+1\}, \lambda)$ with $\lambda = (2h-t)t!\binom{h-1}{t}$.

- Let $X = H \cup Y$, $|H| = |Y| = h$.
- Let $G = Sym(H) \times Sym(Y)$.
- Orbits of t sets are

  $$\Delta_i = \{T \subseteq X : |T| = t \text{ and } |T \cap H| = i\}$$

  For $i = 0, 1, 2, \ldots, t-1$. (The type $(j, t-j)$–sets.)

- Orbits of $t+1$ sets are

  $$\Gamma_j = \{K \subseteq X : |K| = t+1 \text{ and } |K \cap H| = j\}$$

  For $j = 0, 1, 2, \ldots, t-1$. (The type $(j, t+1-j)$–sets.)

Let A be the matrix given by

$$A[i, j] = |\{K \in \Gamma_j : K \supseteq T\}|$$

where $T \in \Delta_i$ is any fixed representative. A is square, upper-triangular and invertible. Thus there is a unique solution $\vec{x}$ to the matrix equation

$$A\vec{x} = \lambda J$$

where $\vec{x}$ is nonnegative and integral if $\lambda = (2h-t)t!\binom{h-1}{t}$.

Not all blocks have to be $t+1$-sets

The solution $\vec{x}$ to the equation $A\vec{x} = \lambda J$ was

$$x_{t-i} = t!\binom{h-1}{t}\left(1 + (-1)^{i-1}\frac{\binom{h-t+i}{i}}{\binom{h-1}{i}}\right).$$

$i = 1, 2, \ldots, t-1, t.$

Notice: the only $t+1$-sets that can cover type $(0, t)$ sets have type

$(0, t+1)$ i.e. orbit Γ_0

or

$(1, t)$ i.e. orbit Γ_1

When $i = t$ we have

$$x_0 = t!\left\{\binom{h-1}{t} + \binom{h}{t}\right\}.$$

When $i = t-1$ we have

$$x_1 = 0.$$

So the t-subsets of Y are covered by taking as blocks all of the $t+1$-subsets of Y each repeated x_0 times. Thus we can simply take Y as block and repeat it λ times. Thus we have a

$$t\text{–}(2h, h, \{t+1, h^\star\}, (2h-t)t!\binom{h-1}{t})$$

Bounds are sharp for even t

For t even we construct a t-$(2h+1, h, \{t+1\}, \lambda)$ with $\lambda = (2h-t+1)(t+1)!\binom{h}{t+1})$.

Proof:

- We already know that there is a I$(t+1)$BD $(H\dot{\cup}Y, H, \mathcal{B})$, of type

  $$(t+1)\text{–}(2(h+1), h+1, \{t+2\}, \lambda')$$

  with $\lambda' = (2(h+1)-(t+1))(t+1)!\binom{(h+1)-1}{t+1})) = \lambda$.

- Take the derived design through a point $x \in H$ to get a

  $$t\text{–}(2h+1, h, \{t+1\}, \lambda)$$

∎

Need $t + 1$-sets

There do not exist proper ItBDs of types

$$t\text{–}(2h+1, h, \mathcal{K}, \lambda) \qquad (t \text{ even})$$

or

$$t\text{–}(2h, h, \mathcal{K}, \lambda) \qquad (t \text{ odd})$$

for any λ and any $2 \le t \le h$,
with $\min\{k : k \in \mathcal{K}\} \ge t + 2$.

General bound

Theorem 2 *If* $(X, H, \mathcal{B})$ *is a proper* ItBD *of type* $t\text{–}(v, h, \mathcal{K}, \lambda)$ *with* $h \ge t \ge 2$ *and* $\min \mathcal{K} = k \ge t + 1$*, then*

$$h \le \frac{v + (k-t)(t-2) - 1}{k - t + 1}.$$

These bounds are sharp for $t = 2$ and $t = 3$.