Isofactorization of Circulant Graphs

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ADK Alspach, Dyer, and Kreher,
On Isomorphic Factorizations of Circulant Graphs
Journal of Combinatorial Designs 14, (2006), to appear.

KW Kreher and Westlund,
On n-Isofactorization of Circulant Graphs,
in preparation.

Slides: `www.math.mtu.edu/~kreher/ABOUTME/talk.html`

Goal: *Decompose the edges of the circulant graph* $G = \textsc{Circ}(n;\ S)$ *into pairwise isomorphic subgraphs.*

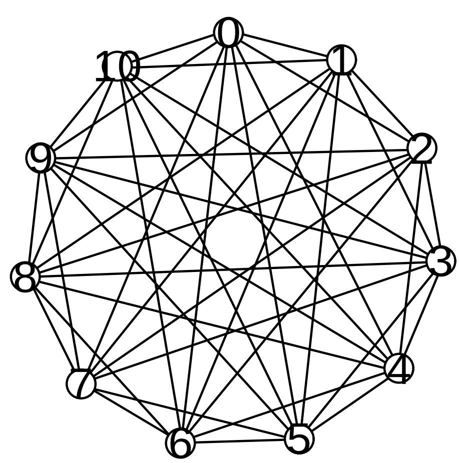
- *vertices* are elements from $\mathbb{Z}_n$.
- $S \subseteq \mathbb{Z} \setminus \{0\}$ is the *connection set*.
- Require $\ell \in S \Leftrightarrow -\ell \in S$.
- $\{x, y\}$ is an *edge* just when $x - y \in S$.
- G is connected $\Leftrightarrow$ S generates $\mathbb{Z}_n$.

$$\gcd(n, \ell_1, \ell_2, \ldots, \ell_t) = 1$$

  where $S = \{\pm\ell_1, \pm\ell_2, \ldots, \pm\ell_t\}$

- There are $\frac{n|S|}{2}$ edges.

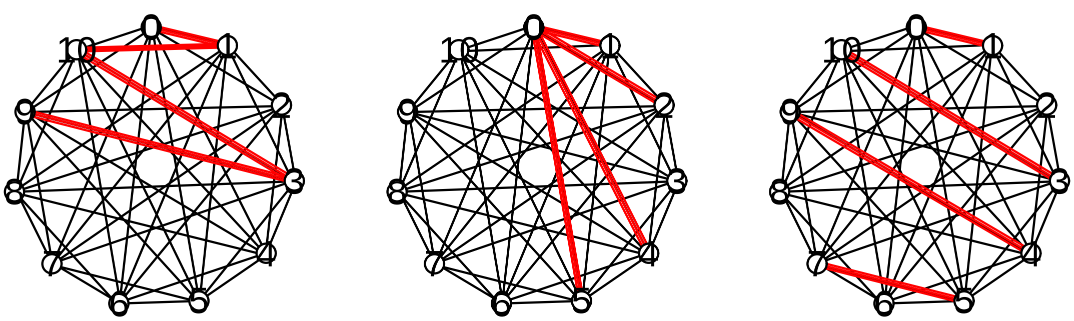
Example.



$$G = \textsc{Circ}(11; \{\pm1, \pm2, \pm4, \pm5\})$$

A k-isofactorization is a partition of the edges into isomorphic subgraphs, each of size k. So k must divide $|E(G)| = n|S|/2$.

Alspach Conjecture (1982): If k divides $|E(G)|$ for a circulant graph G, then G has a k-isofactorization.



Zig-zag path   Star   4-Matching

Some 4-isofactorizations of $G = \textsc{Circ}(11; \{\pm 1, \pm 2, \pm 4, \pm 5\})$

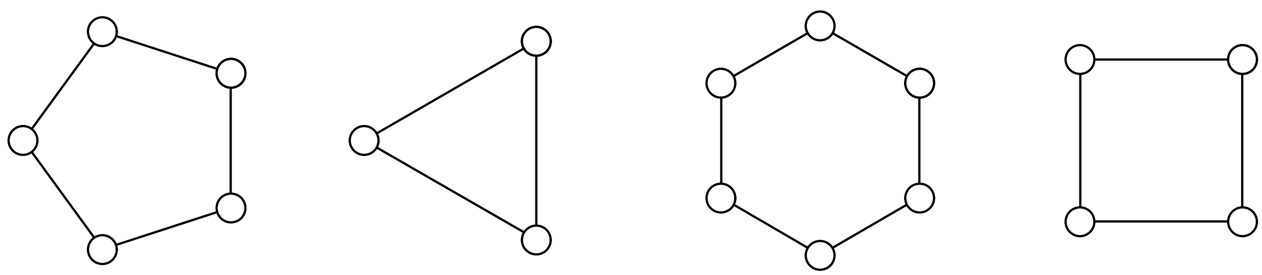
k-matchings are k independent edges.

Lemma 1 *(ADK) Let G be a regular graph of order n and valency 1 or 2. If k is a proper divisor of $|E(G)|$, then G can be decomposed into k-matchings except when $n = 2k$ and at least one component of G has odd order.*

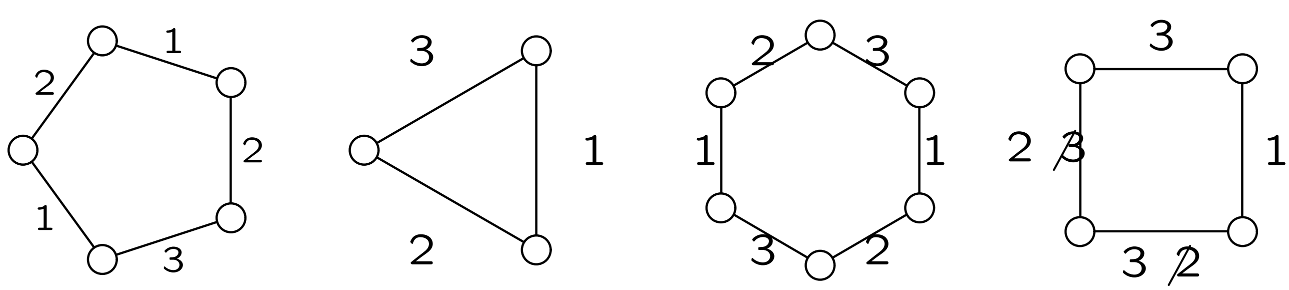
PROOF.

Valency 1: G is itself a $\frac{n}{2}$-matching.

Valency 2:



Find a proper edge coloring with $d = \dfrac{n}{k}$ colors so that each color class has k edges. ($k = 6$, $d = 3$)



Always possible unless $d = 2$ and there is an odd cycle.

■

Theorem 2 *(ADK) If $G = \text{Circ}(n; S)$ is connected, k is a proper divisor of n, and k divides $|E(G)|$, then there is a decomposition of G into k-matchings.*

PROOF.

For $k = n/2$ use Stong's result on the 1-factorization of Caley graphs.

Otherwise each length $\ell \in S$ generates a 1 or 2-regular graph and we use Lemma 1 to independently decompose each into k-matchings.

■

Theorem 3 *(ADK) Let $X = \text{Circ}(n; S)$ be a connected circulant graph of order n. If k divides $|S|/2$, then there is a k-isofactorization of X into stars, i.e. $K_{1,k}$s.*

PROOF. Partition the $\frac{|S|}{2}$ "positive" lengths into blocks of size k, draw the stars and rotate. ■

In general we prove:

Theorem 4 *(ADK) Let $X = \textsc{Circ}(n; S)$ be a connected circulant graph of order n. If k divides $|E(X)|$ and either k properly divides n or k divides $|S|$, then there is a k-isofactorization of X.*

Notice the omission of the case $k = n$.

n-Isofactorization of connected $G = \textsc{Circ}(n;\, S)$

Here $\dfrac{|S|}{2} = \dfrac{n|S|/2}{n}$ is an integer.

So if n is even, then $n/2 \notin S$ because $n/2 \equiv -n/2 \mod n$ and $|S|$ would be odd.

Theorem 5 *(ADK) If $|S|/2$ divides n, then G has an n-isofactorization.*

A Hamilton decomposition is one type of n-isofactorization.

Alspach Conjecture (1985): Every connected circulant graph of valency $2t$ has a decomposition into t edge-disjoint Hamilton cycles.

The conjecture has been shown for the following circulant graphs:

- $t = 1$: the entire graph is one Hamilton cycle.

- Bermond, Favaron, and Maheon (1989): For connected graphs when $t = 2$, i.e. valency 4.

- Dean (2006): For connected $G = \textsc{Circ}(n;\, S)$ with $t = 3$ when n is odd, or n is even and there exists some element $l \in S$ such that $\gcd(n, l) = 1$.

Theorem 6 *(ADK) Partition S into 4-subsets, so that*

$$S = \{\pm l_1, \pm l_2\} \cup \{\pm l_3, \pm l_4\} \cup \cdots \cup \{\pm l_{t-1}, \pm l_t\}$$

If, for each pair, the $\gcd(n, l_i, l_{i+1}) = 1$*, then G has an n-isofactorization into Hamilton cycles.*

Theorem 7 *(ADK) If $S = \{\pm(l+i) : i = 0, 1, 2, \ldots, t-1\}$ where t is even, then there exists a Hamilton decomposition of G.*

(Here $\gcd(l, l+1) = \gcd(n, l, l+1) = 1$) for all $l \in S$)

Valency 8: n-isofactorization for small lengths

Forward Edges: $T \subseteq E(G)$ with $S = \{\pm l_1, \pm l_2, ..., \pm l_j\}$.

- $0 < |l_i| < n/2$, when we assume w.l.g. $S = S^+ \dot\cup S^-$ where
- $S^+ = \{l_i : \ i = 1, 2, \ldots, j\}$ and
- $S^- = \{-l_i : \ i = 1, 2, \ldots, j\}$.

T_V is the set of *forward edges* on the vertices in V:

$$T_V = \{\{v, \ v + l\} : \ l \in S^+, \ v \in V\}.$$

Note: $T_V = \bigcup_{x \in V} T_{\{x\}}$.

$$\begin{aligned} \text{Example:} S &= \{\pm 1, \pm 2, \pm 4, \pm 5\}, \quad n = 11 \\ T_{\{1,7\}} &= \{\{1,2\},\{1,3\},\{1,5\},\{1,6\}, \\ &\qquad \{7,8\},\{7,9\},\{7,0\},\{7,1\}\} \end{aligned}$$

Theorem 8 *(KW) The circulant graph* $G = \text{Circ}(n; S)$ *where,*

- $S = \pm\{l_1, l_2, l_3, l_4\}$
- $n = 4x+p$ for $p = 4, 5, 6, 7$

has an n-isofactorization when one of the following is true:

- $n \equiv 0 \mod 4$
- $1 < l_i \leq x$ for $i = 1, 2, 3, 4$, $x \geq 5$
- $x = 1, 2, 3, 4$.

If $n \equiv 1 \mod 4$ or $x = 1, 2, 3, 4$, then we may also include $1 \in S$.

If $n \equiv 0 \mod 4$, then $4 = \frac{|S|}{2}$ divides n. This has an n-isofactorization by Alspach, Dyer, and Kreher. (Theorem 5)

Sample construction for $n \equiv 1 \mod 4$ $(n = 4x + 5)$

Let $G = \textsc{Circ}(25; \pm\{2,3,4,6\})$. Here $x = 5$.

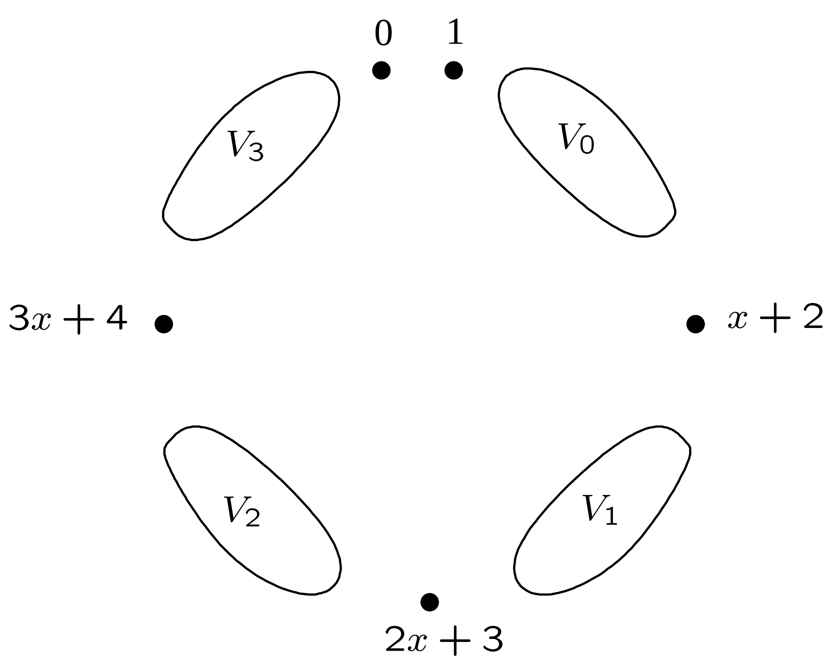
Partition the vertices:

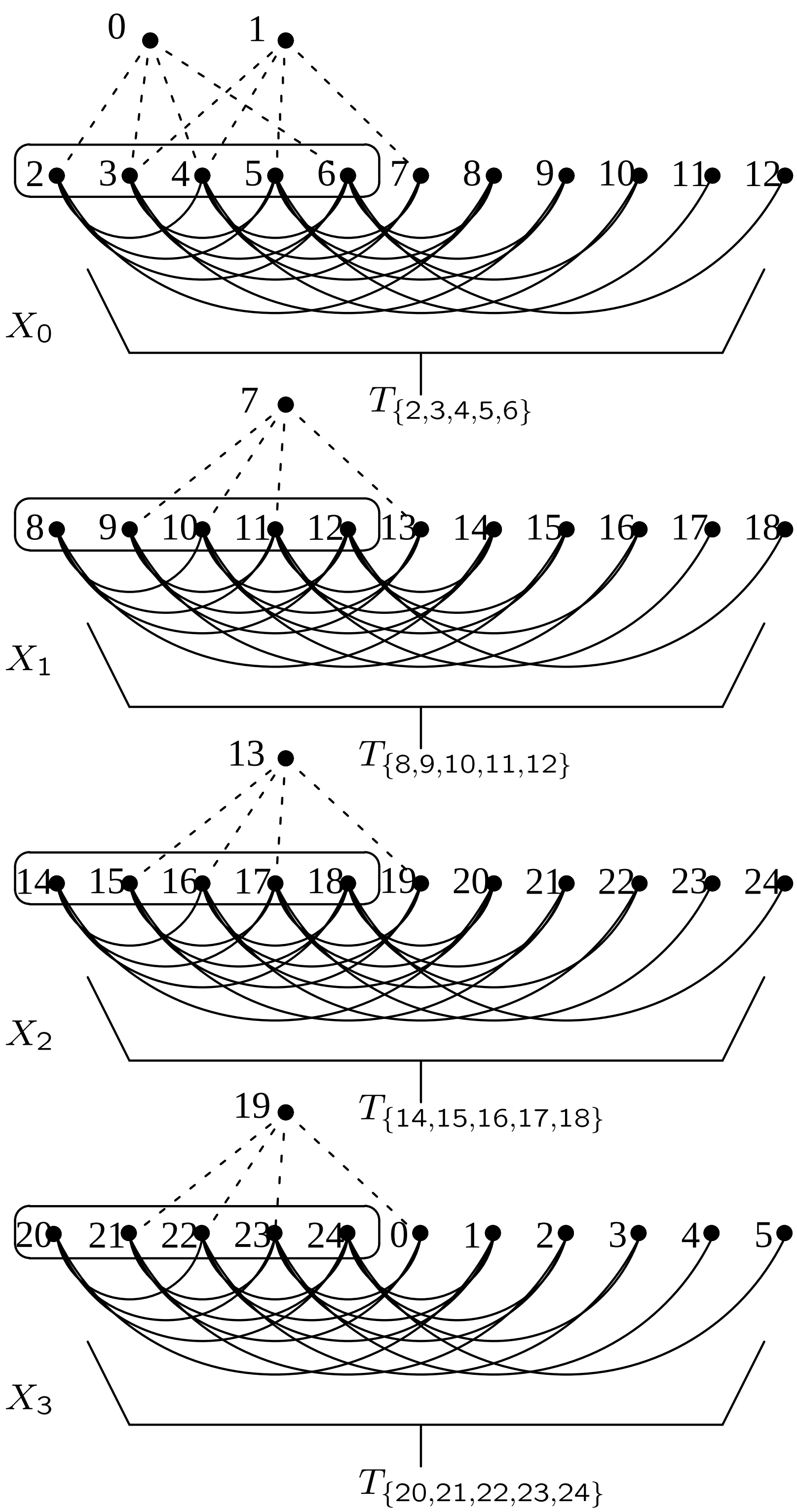
$$\mathbb{Z}_{25} = U \,\dot\cup\, V_0 \,\dot\cup\, V_1 \,\dot\cup\, V_2 \,\dot\cup\, V_3,$$

where $U = \{0, 1, x{+}2, 2x{+}3, 3x{+}4\} = \{0, 1, 7, 13, 19\}$ and

$$\begin{aligned}
V_0 &= \{2, 3, \ldots, x{+}1\} &&= \{2,3,4,5,6\},\\
V_1 &= \{x{+}3, x{+}4, \ldots, 2x{+}2\} &&= \{8,9,10,11,12\},\\
V_2 &= \{2x{+}4, 2x{+}5, \ldots, 3x{+}3\} &&= \{14,15,16,17,18\},\\
V_3 &= \{3x{+}5, 3x{+}6, \ldots, 4x{+}4\} &&= \{20,21,22,23,24\}.
\end{aligned}$$

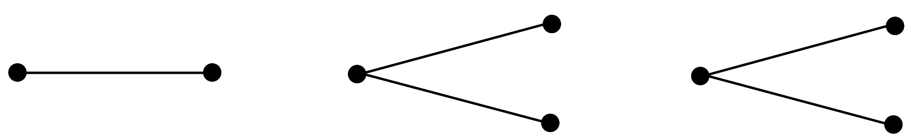
$X_i = \langle T_{V_i} \rangle$, the subgraph induced by T_{V_i}.
As $|V_i| = x$ and each V_i consists of consecutive vertices, X_0, X_1, X_2, and X_3 are pairwise isomorphic, each having 20 edges.




X_0
$T_{\{2,3,4,5,6\}}$
X_1
$T_{\{8,9,10,11,12\}}$
X_2
$T_{\{14,15,16,17,18\}}$
X_3
$T_{\{20,21,22,23,24\}}$

We now distribute the 20 forward edges $(T_U = T_{\{0,1,7,13,19\}})$ preserving isomorphism.

Adjoin a single edge and a pair of 2-paths to each subgraph.



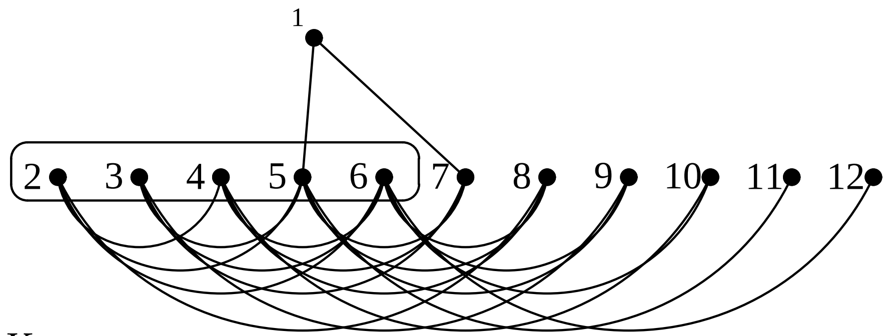
Without loss:
adjoin $\{0, l_1\} = \{0, 2\}$ and $\{0, l_2\} = \{0, 3\}$ to X_1,
adjoin $\{0, l_3\} = \{0, 4\}$ and $\{0, l_4\} = \{0, 6\}$ to X_2.

$\{1, l_1\} \notin E(G)$ (otherwise $l_1 - 1 \in S$). If $\{1, l_2\} \in E(G)$ adjoin it to X_2. If $\{1, l_2\} \notin E(G) \Rightarrow \exists$ at least one edge, call it $\{1, k\}$ where $k \notin \{x + 2, l_3, l_4\} = \{7, 4, 6\}$. Adjoin $\{1, k\}$ to X_2.

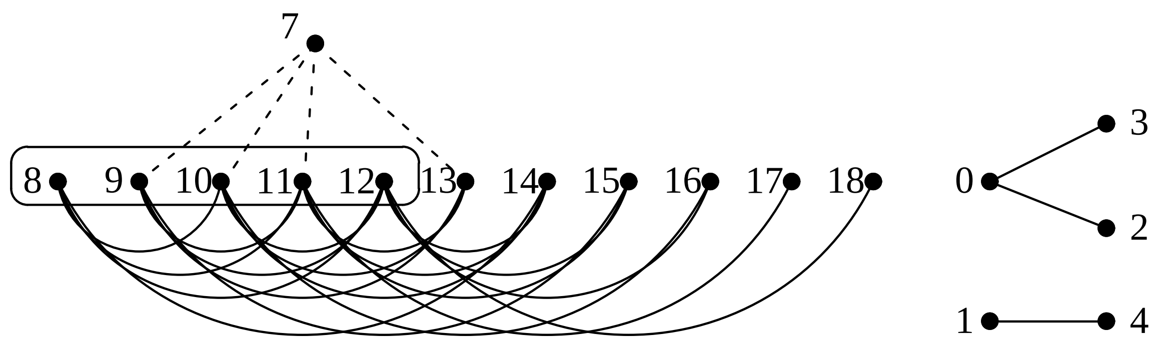
As $\{1, l_2\} = \{1, 3\} \in E(G)$, adjoin it X_2.

Thus $\exists$ $\{1, s\}, \{1, t\} \in E(G)$ where $s, t \neq \{k, x + 2, l_1, l_2\}$. Without loss, adjoin $\{1, s\} = \{1, 4\}$ to X_1.

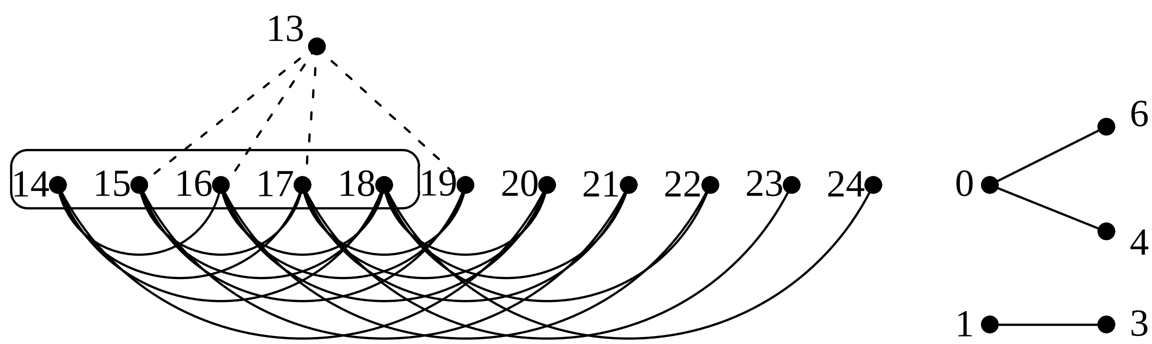
Finally, adjoin $\{1, t\} = \{1, 5\}$ and $\{1, x + 2\} = \{1, 7\}$ to X_0.


1
2 3 4 5 6 7 8 9 10 11 12

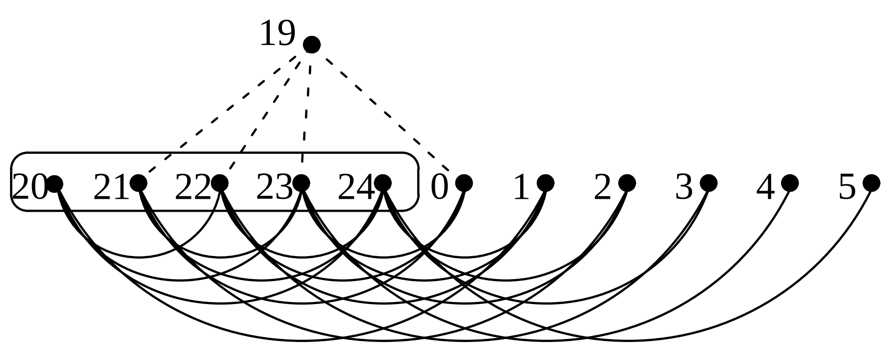
X_0


7
8 9 10 11 12 13 14 15 16 17 18
0 3 2
1 4

X_1


13
14 15 16 17 18 19 20 21 22 23 24
0 6 4
1 3

X_2


19
20 21 22 23 24 0 1 2 3 4 5

X_3

To preserve isomorphism:

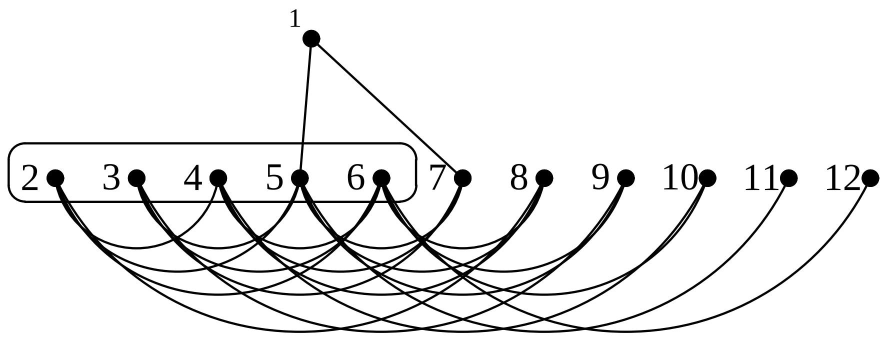
Adjoin to X_1:
$\{x+2, 2t+1\} = \{7, 11\}$
$\{x+2, (x+2)+(x+1)\} = \{x+2, 2x+3\} = \{7, 13\}$.

Adjoin to X_2:
$\{2x+3, 3t+2\} = \{13, 17\}$
$\{2x+3, (2x+3)+(x+1)\} = \{2x+3, 3x+4\} = \{13, 19\}$.

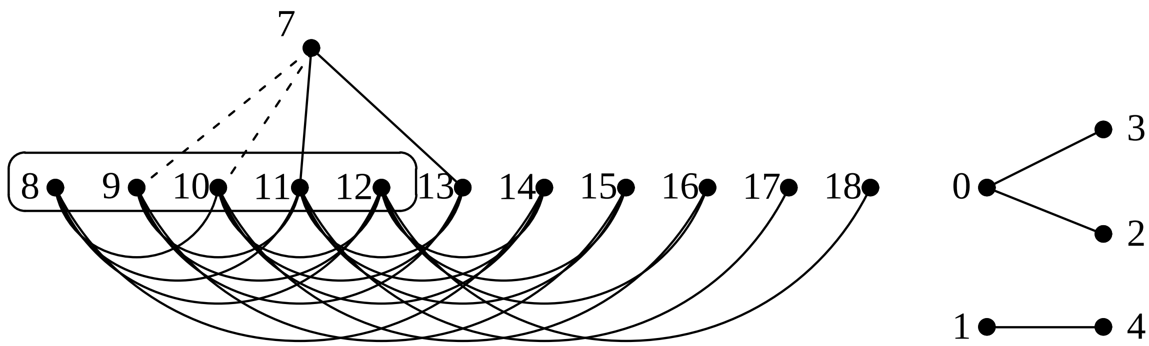
Adjoin to X_3:
$\{3x+4, 4t+3\} = \{19, 23\}$
$\{3x+4, (3x+4)+(x+1)\} = \{3x+4, 0\} = \{19, 0\}$.

There now exist only 6 forward edges left to distribute.

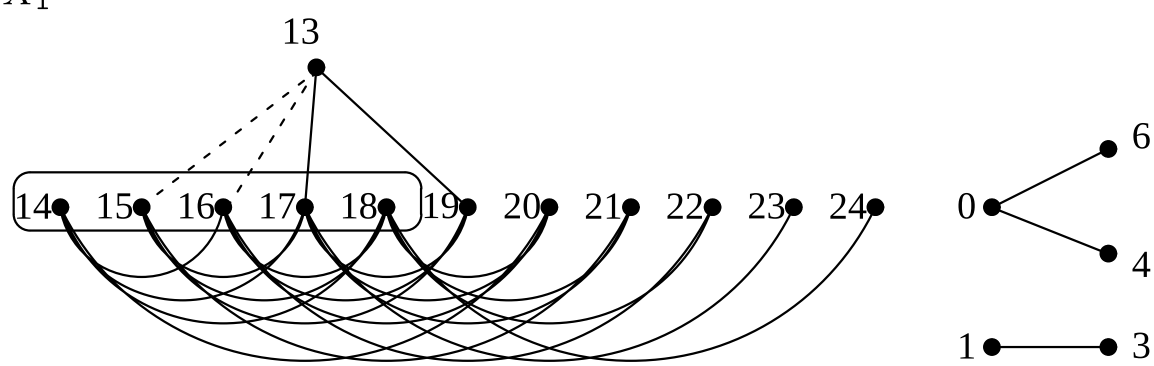
Two from each of $U \setminus \{0, 1\} = \{7, 13, 19\}$.


1
2 3 4 5 6 7 8 9 10 11 12

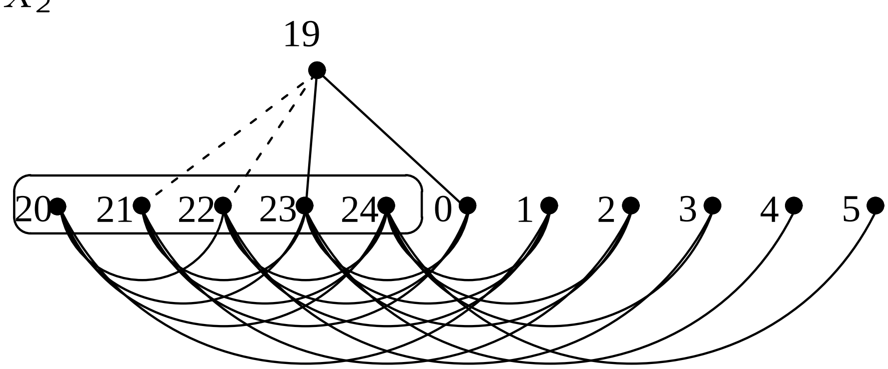
X_0


7
8 9 10 11 12 13 14 15 16 17 18
0 3 2
1 4

X_1


13
14 15 16 17 18 19 20 21 22 23 24
0 6 4
1 3

X_2


19
20 21 22 23 24 0 1 2 3 4 5

X_3

Remaining forward edges to distribute to X_0 and X_3:

$$\left.\begin{array}{l}\{x+2, y_1\} = \{7, 9\} \\ \{x+2, z_1\} = \{7, 10\}\end{array}\right\}\text{ cannot adjoin to } X_0$$

$$\left.\begin{array}{l}\{2x+3, y_2\} = \{13, 15\} \\ \{2x+3, z_2\} = \{13, 16\}\end{array}\right\}\text{ can adjoin to either } X_0 \text{ or } X_3.$$

$$\left.\begin{array}{l}\{3x+4, y_3\} = \{19, 21\} \\ \{3x+4, z_3\} = \{19, 22\}\end{array}\right\}\text{ cannot adjoin to } X_3$$

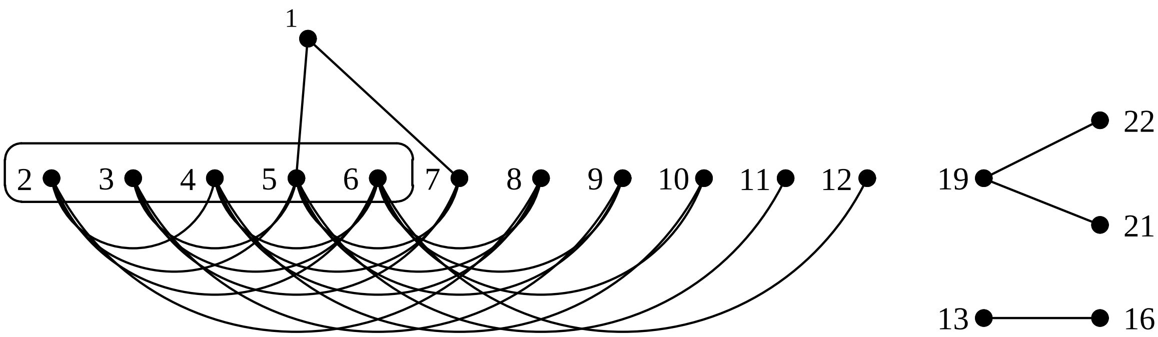
Adjoin $\{19, 21\}$ and $\{19, 22\}$ to X_0.

Adjoin $\{7, 9\}$ and $\{7, 10\}$ to X_3.

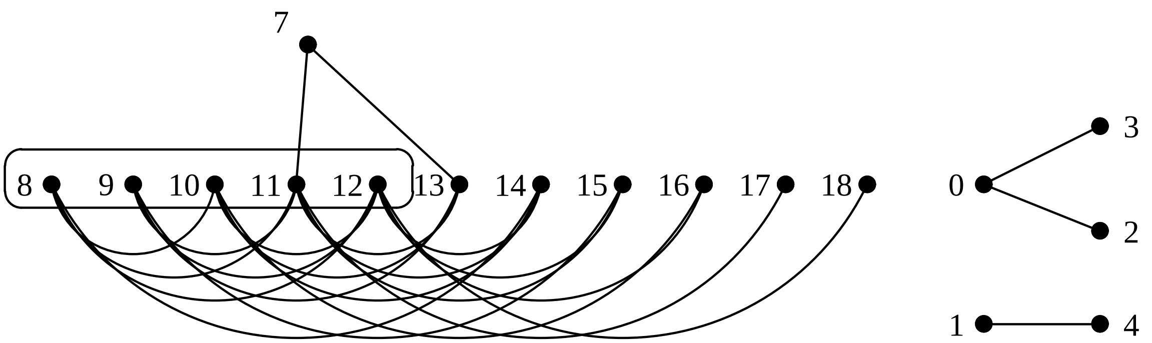
Without loss of generality,

Adjoin $\{13, 16\}$ to X_0.
Adjoin $\{13, 15\}$ to X_3.

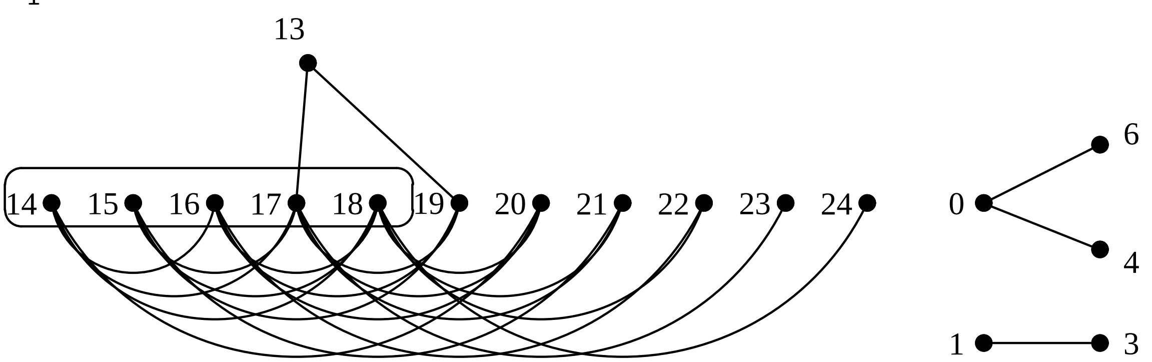
The 25-isofactorization of $G = \text{CIRC}(25; \pm\{2, 3, 4, 6\})$ is complete.


1
2 3 4 5 6 7 8 9 10 11 12
19 22 21
13 16

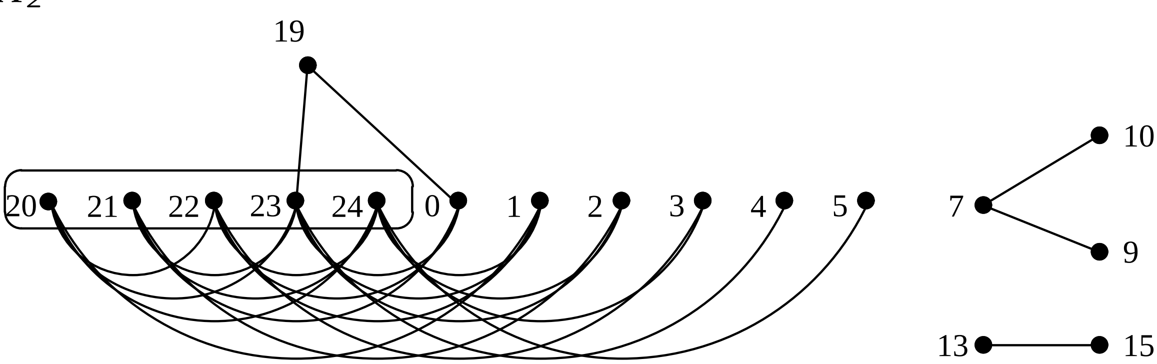
X_0


7
8 9 10 11 12 13 14 15 16 17 18
0 3 2
1 4

X_1


13
14 15 16 17 18 19 20 21 22 23 24
0 6 4
1 3

X_2


19
20 21 22 23 24 0 1 2 3 4 5
7 10 9
13 15

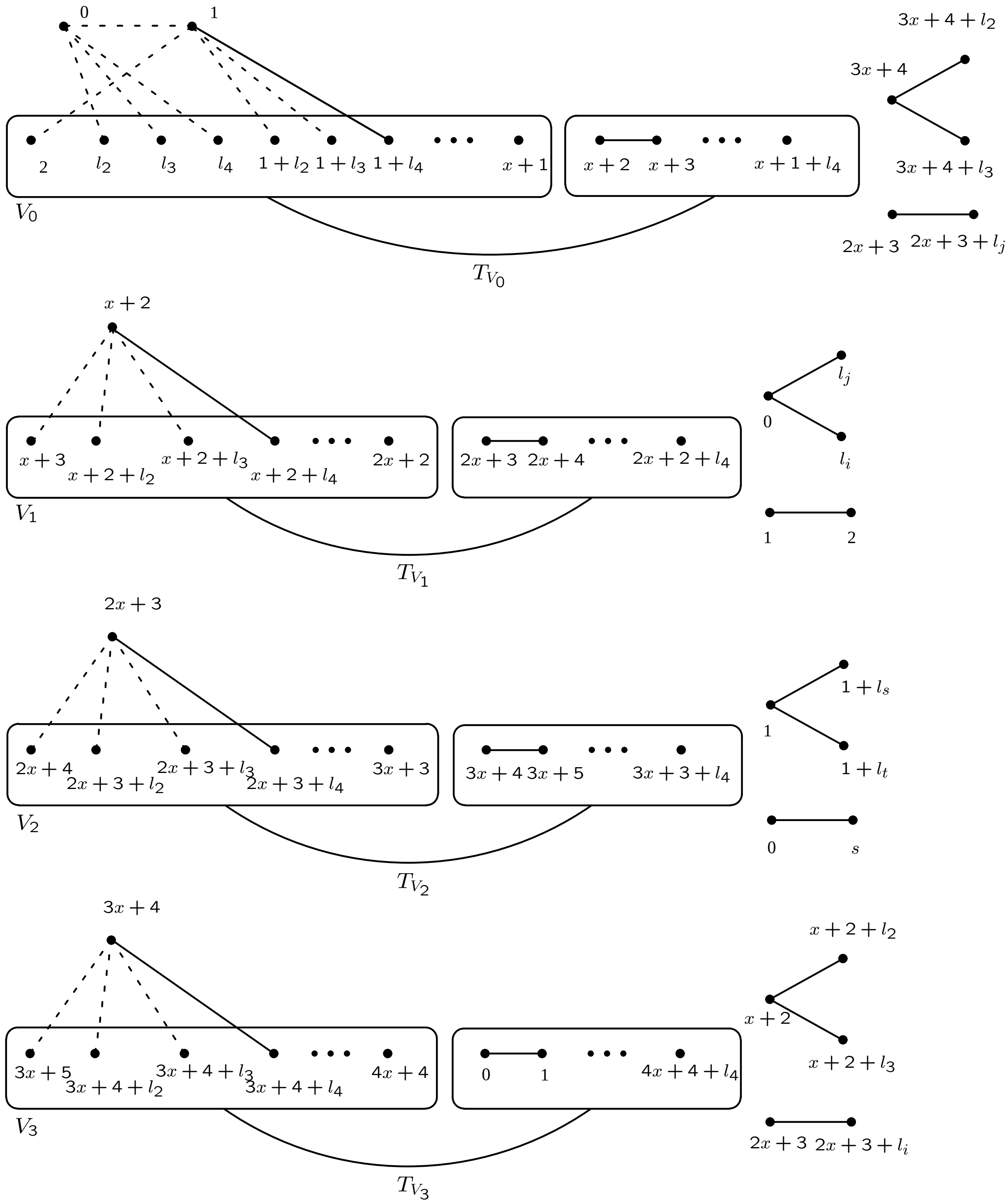
X_3

An adaption of this construction allows:

Theorem 9 *(KW) The circulant graph* $G = \textsc{Circ}(n;\, S)$ *where,*

- $n = 4x + 5$ *for* $x \geq 5$
- $S = \pm\{1, l_2, l_3, l_4\})$

has an n*-isofactorization when* $1 < l_2 < l_3 < l_4 \leq x$.



0
1
2
l_2
l_3
l_4
$1+l_2$
$1+l_3$
$1+l_4$
$x+1$
$x+2$
$x+3$
$x+1+l_4$
V_0
T_{V_0}
$3x+4+l_2$
$3x+4$
$3x+4+l_3$
$2x+3$
$2x+3+l_j$
$x+2$
$x+3$
$x+2+l_2$
$x+2+l_3$
$x+2+l_4$
$2x+2$
$2x+3$
$2x+4$
$2x+2+l_4$
V_1
T_{V_1}
l_j
0
l_i
1
2
$2x+3$
$2x+4$
$2x+3+l_2$
$2x+3+l_3$
$2x+3+l_4$
$3x+3$
$3x+4$
$3x+5$
$3x+3+l_4$
V_2
T_{V_2}
$1+l_s$
1
$1+l_t$
0
s
$3x+4$
$3x+5$
$3x+4+l_2$
$3x+4+l_3$
$3x+4+l_4$
$4x+4$
0
1
$4x+4+l_4$
V_3
T_{V_3}
$x+2+l_2$
$x+2$
$x+2+l_3$
$2x+3$
$2x+3+l_i$

For $\textsc{Circ}(n; S)$ with $n = 4x + p$ ($p = 4, 5, 6, 7$) and $x = 1, 2, 3, 4$, we use separate constructions for each case:

Example: $p = 6$

x	n	Possible connection set S
1	10	$\pm\{1, 2, 3, 4\}$
2	14	$S^+ \subset \{1, 2, 3, 4, 5, 6\}$
3	18	$S^+ \subset \{1, 2, 3, 4, 5, 6, 7, 8\}$
4	22	$S^+ \subset \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$n = 10$: has a 10-isofactorization as S^+ contains four consecutive integers.

$n = 14 = 2 \cdot 7$ and $n = 22 = 11 \cdot 2$ as $7, 11 \equiv 3 \mod 4$, we are guaranteed Hamilton decompositions by Alspach's result.

$n = 18$: if S^+ contains two or three elements co-prime with 18 $\Rightarrow$ Hamilton decomposition by Alspach, Dyer, and Kreher.

If $G = \textsc{Circ}(18; \pm\{2, 4, 6, 8\}) \Rightarrow G$ is isomorphic to two copies of $G^* = \textsc{Circ}(9; \pm\{1, 2, 3, 4\})$. As G^* is Hamilton-decomposable, pair up eight 9-isofactors to achieve an 18-isofactorization of G.

Remaining cases were found Hamilton-decomposable by random computer search or previous theorems.

Valency $2t$: The n-isofactorization for small lengths

Using a similar construction for valency 8, we can generalize to valency $2t$ where $n \equiv 0, 1, 2 \mod t$:

Theorem 10 *(KW) The circulant graph* $G = \textsc{Circ}(n; S)$ *where*

- $n = tx + t + p$ *for* $p = 0, 1, 2$
- $S = \pm\{l_1, l_2, \ldots, l_t\}$

has an n*-isofactorization when* $n \equiv 0 \mod t$ *or when* t *is even and*

- $t \geq 6$ if $p = 1$
- $t \geq 8$ if $p = 2$
- $1 < l_1 < l_2 < \cdots < l_t \leq x$ for all $i = 1, 2, \ldots, t-1$.

Example: Let $n = 6x + 7$. Here $t = 6$, $p = 1$, valency 12.

Partition

$$V(G) = \mathbb{Z}_n = V \,\cup\, V_0 \,\cup\, V_1 \,\cup\, \cdots\, V_{t-1},$$

where

$V = \{0, 1, x+2, 2x+3, 3x+4, 4x+5, 5x+6\}$.

$V_0 = \{2, 3, \ldots, x, x+1\}$.

$V_j = \{v + j(x+1) :\ v \in V_0\},\ j = 1, 2, \ldots, t-1$.

Let $X_i = \langle T_{V_i} \rangle$ for $i = 0, \ldots, t-1$. Because $|V_i| = x \ \forall \ i$ and all forward edges have been chosen $\Rightarrow X_i \cong X_j$, where $|E(X_i)| = 6x$. The remaining 42 edges of G are T_V.

Let,

$I = \{v \in V_0 : \ \{1, v\}, \{0, v\} \in E(G)\}$,

$L = \{v \in V_0 : \ \{1, v\} \in E(G), \ \{0, v\} \notin E(G)\}$.

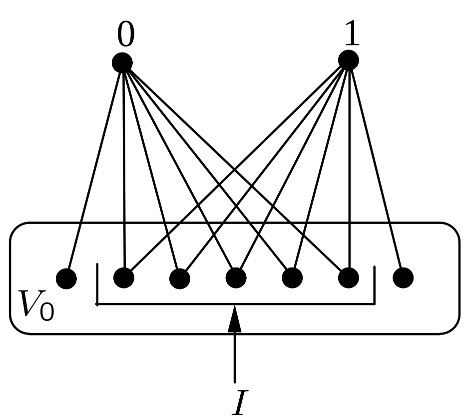
0 and 1 cannot share more than five other vertices in V_0 as common neighbors.

In particular, the 2-path $\{0, l_1, 1\}$ cannot exist, otherwise $1_1 - 1 \in S$.

Obviously the 2-path $\{0, 1 + l_6, 1\}$ cannot exist otherwise $1 + l_6 \in S$.

If $I \neq \emptyset$, and $|I| = j \Rightarrow j \in \{1, 2, 3, 4, 5\}$.

Example: $j = 5$.



If $|I| = k$ where $0 \leq k \leq 4$, (the case $k = 5$ is simpler) then let $L' \subseteq L$ be any set of $4 - k$ vertices from L.

Let $E(\{1\}) = \{\{1, v\} : \ v \in I \dot\cup L'\}$.

Clearly, $|E(\{1\})| = 4$, relabeled as:

$$E(\{1\}) = \{\{1, y_1\}, \{1, y_2\}, \{1, y_3\}, \{1, y_4\}\}.$$

For $v \in \overline{V} \setminus \{0, 1\}$, let $E(\{v\}) =$

$\{\{v, \ v + (y_i - 1)\} : y_i \in \{1, y_i\} \in E(\{1\}), \ v \in \overline{V} \setminus \{0, 1\}\}$.

Adjoin accordingly:

$E(\{1\}) \rightarrow X_0$
$E(\{x + 2\}) \rightarrow X_1$
$E(\{2x + 3\}) \rightarrow X_2$
$E(\{3x + 4\}) \rightarrow X_3$
$E(\{4x + 5\}) \rightarrow X_4$
$E(\{5x + 6\}) \rightarrow X_5$
$\{0, l_1\}, \{0, l_2\} \rightarrow X_2$
$\{0, l_3\}, \{0, l_4\} \rightarrow X_3$
$\{0, l_5\}, \{0, l_6\} \rightarrow X_4$

As $v \leq (x + 1) \ \forall \ v \in V_0$ and

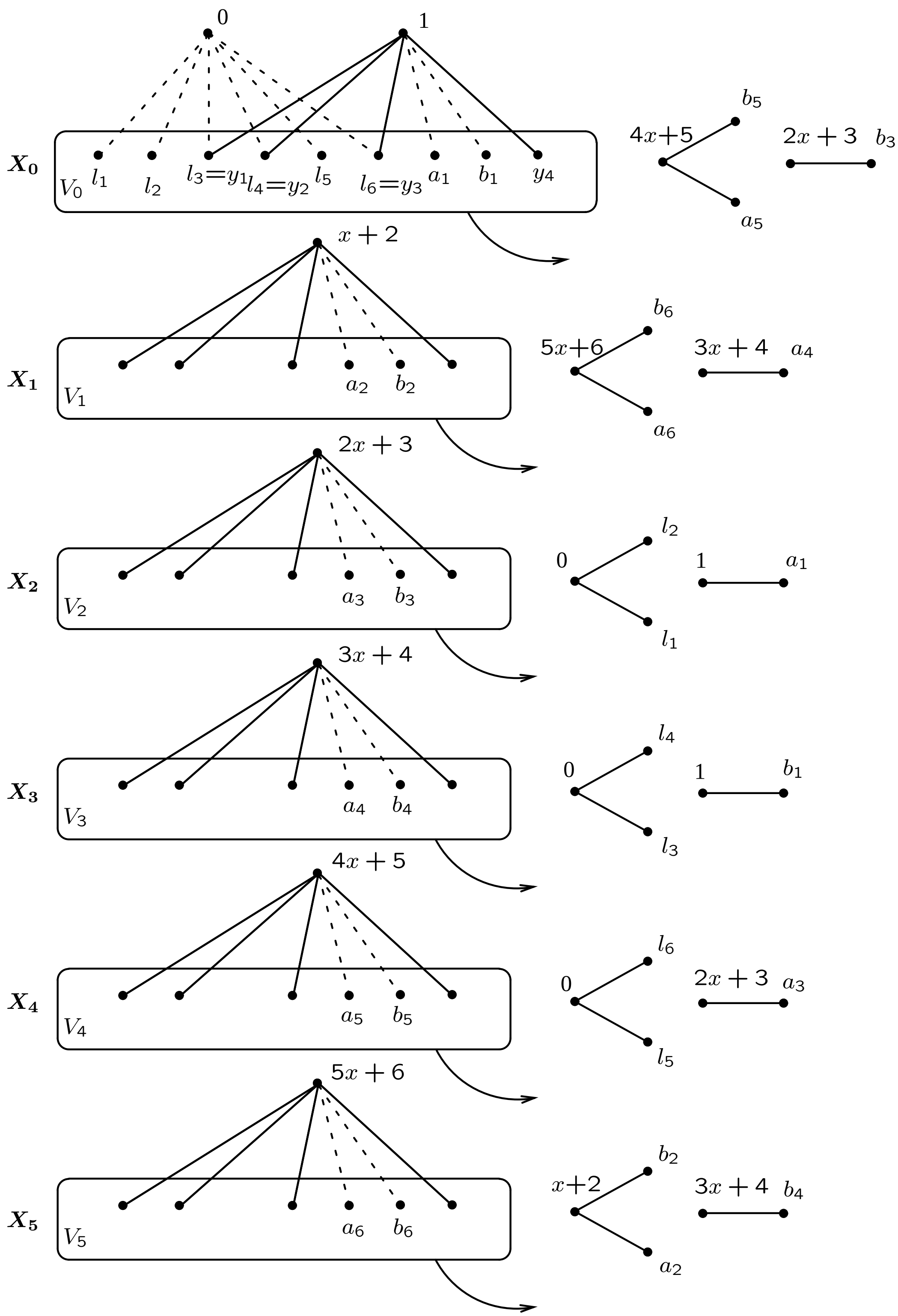
$$(x + 1) + l_6 \leq (x + 1) + x = 2x + 1 \notin V_2,$$

we have *no edges* of the form, $\{\{v, v'\} : \ v \in V_0, \ v' \in V_2\}$.

In general, there are no edges of the form,

$$\{\{v, v'\} : \ v \in V_i, \ v' \in V_{i+2}\},$$

(subscript addition $i + 2$ is modulo 6.)


0
1
X_0
V_0
l_1
l_2
$l_3{=}y_1$
$l_4{=}y_2$
l_5
$l_6{=}y_3$
a_1
b_1
y_4
$4x{+}5$
b_5
a_5
$2x+3$
b_3
$x+2$
X_1
V_1
a_2
b_2
$5x{+}6$
b_6
a_6
$3x+4$
a_4
$2x+3$
X_2
V_2
a_3
b_3
0
l_2
l_1
1
a_1
$3x+4$
X_3
V_3
a_4
b_4
0
l_4
l_3
1
b_1
$4x+5$
X_4
V_4
a_5
b_5
0
l_6
l_5
$2x+3$
a_3
$5x+6$
X_5
V_5
a_6
b_6
$x{+}2$
b_2
a_2
$3x+4$
b_4

Conclusions and Further Research Problems

M. Dean's result for valency 6 is limited to odd order or even order circulants providing there exists $l \in S$ such that $\mathtt{gcd}(l, n) = 1$.

Open Problem: Complete results for valency 6.

Valency 8: Complete results when $n = 4x + p$, and $l \leq x$ for all $l \in S^+$, but not H-decomposable when $x \geq 5$.

Open Problem: Hamilton-decompositions of the valency 8 circulant graphs where $x \geq 5$.

Open Problem: Develop appropriate constructions to allow for $1 \in S$.

Valency $2t$: Complete results when $n = tx + t + p$, where $1 < l < x$ for all $l \in S^+$, $p = 0, 1, 2$, and even $t \geq 6$ if $p = 1$, or even $t \geq 8$ if $p = 2$.

Open Problem: Hamilton-decompositions of the valency $2t$ circulants.

Open Problem: Resolve conjecture that every circulant of order $2p$ (p is prime) has a Hamilton decomposition.