Transverse t-designs

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- Kimberly A. Lauinger, *Computing transverse t–designs* M.S. report, Michigan Technological University.
- D.L. Kreher, K.A. Lauinger, R.S. Rees, and D.R. Stinson *Computing transverse Steiner Quadruple Systems*. *in preparation*
- M.S. Keranen, *some new results*.

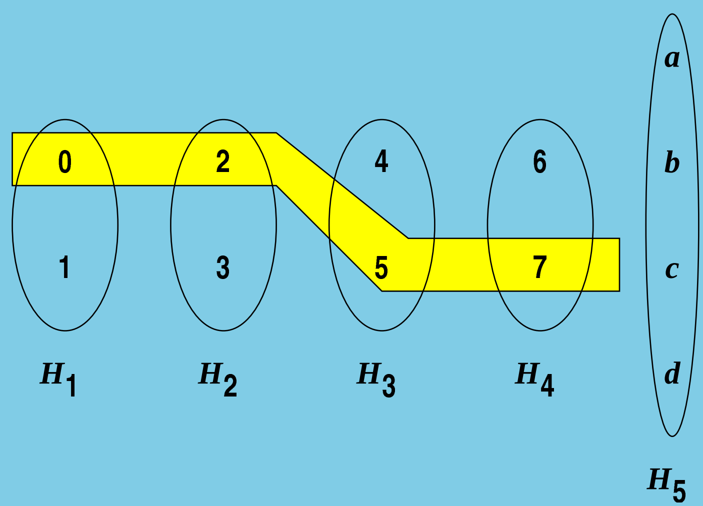
Transverse subsets $X =$ the point set.

$\mathcal{H} = [H_1, H_2, \ldots, H_r]$ is a partition of X into *holes*.

Type is $\prod j^{m[j]}$ where there are $m[j]$ holes of size j.

$T \subseteq X$ is *transverse* w.r.t $\mathcal{H}$ if $|T \cap H_i| = 0$ or 1, $\forall i$.

Example: A transverse subset.



Partition type $2^4 4^1$,

Transverse t-design A transverse t-*design* with parameters t–(v, k, λ) is a triple $(X, \mathcal{H}, \mathcal{B})$

- X is a v–element set of *points*
- $\mathcal{H} = \{H_1, H_2, \ldots, H_r\}$ is a partition of X into *holes*;
- $\mathcal{B}$ is a collection of transverse subsets of X called *blocks*;
- every transverse t–element subset is in exactly λ blocks.

Example: A Transverse 3-$(12, 4, 1)$ design of type $2^4 4^1$.

$$X = \{0, 1, 2, 3, 4, 5, 6, 7, a, b, c, d\}$$

$$\mathcal{H} = [\{0, 1\}, \{2, 3\}, \{4, 5\}, \{6, 7\}, \{a, b, c, d\}]$$

$$\mathcal{B} = \left\{ \begin{array}{cccc} \{3,5,7,a\} & \{3,5,6,b\} & \{3,4,7,d\} & \{3,4,6,c\} \\ \{2,5,7,c\} & \{2,5,6,d\} & \{2,4,7,b\} & \{2,4,6,a\} \\ \{1,5,7,b\} & \{1,5,6,c\} & \{1,4,7,a\} & \{1,4,6,d\} \\ \{1,3,7,c\} & \{1,3,6,a\} & \{1,3,5,d\} & \{1,3,4,b\} \\ \{1,2,7,d\} & \{1,2,6,b\} & \{1,2,5,a\} & \{1,2,4,c\} \\ \{0,5,7,d\} & \{0,5,6,a\} & \{0,4,7,c\} & \{0,4,6,b\} \\ \{0,3,7,b\} & \{0,3,6,d\} & \{0,3,5,c\} & \{0,3,4,a\} \\ \{0,2,7,a\} & \{0,2,6,c\} & \{0,2,5,b\} & \{0,2,4,d\} \end{array} \right\}$$

Connections

- A transverse t–(v, k, λ) design of type 1^v is an (ordinary) t–design.

- Steiner if $\lambda = 1$.

  ⋆ No nontrivial (block size $<$ number of holes) Steiner transverse t–design is
  ⋆ known to exist with $t \geq 6$.

- A transverse 2–design is a group divisible design.

- Uniform if $|H| = u$ for all $H \in \mathcal{H}$ i.e. type u^r.

- A uniform transverse t–(u^r, r, λ) design of type u^r is

  ⋆ an orthogonal array of strength t
  ⋆ a.k.a a fractional factorial design.

Automorphisms

An automorphism must preserve
The points X, the holes $\mathcal{H}$, and the blocks $\mathcal{B}$.

If $S \subset X$, and $g \in \mathrm{Sym}(X)$, then $g(S) = \{g(x) : x \in S\}$.

So g is an automorphism if
$g \in \mathrm{Sym}(X)$, $g(H) \in \mathcal{H}$ for each $H \in \mathcal{H}$, and $g(B) \in \mathcal{B}$ for each $B \in \mathcal{B}$.

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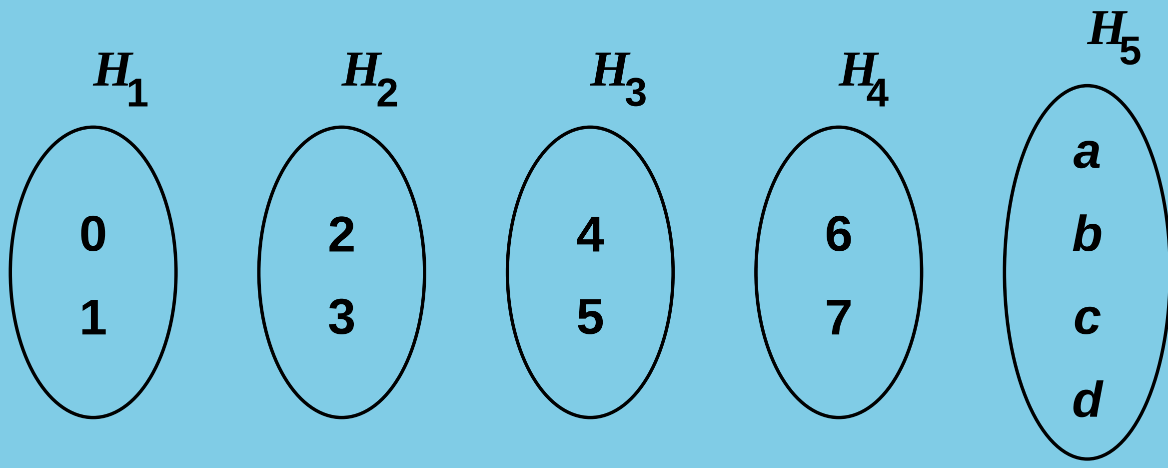
Example: $g = (0, 2)(1, 3)(4, 6)(5, 7)(a, b)(c, d)$



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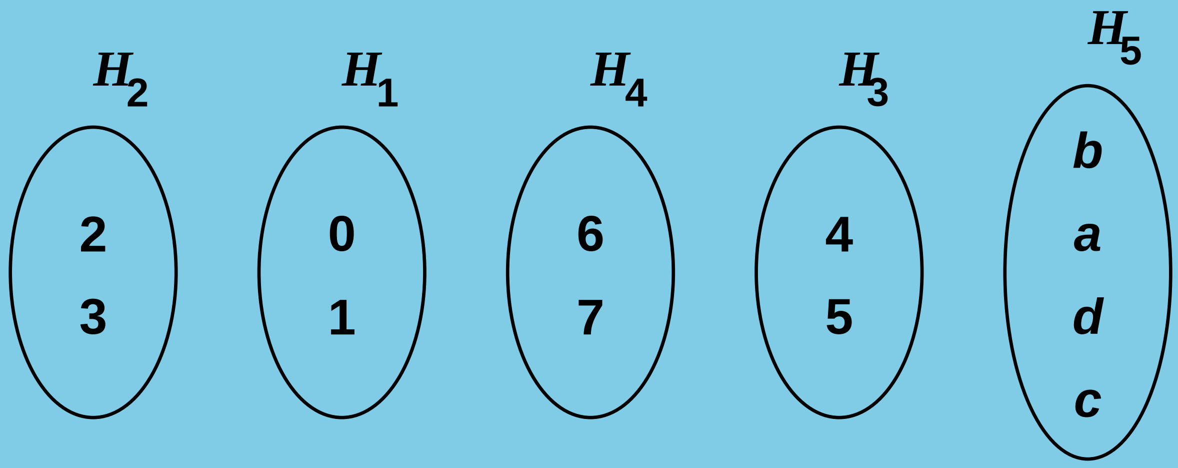

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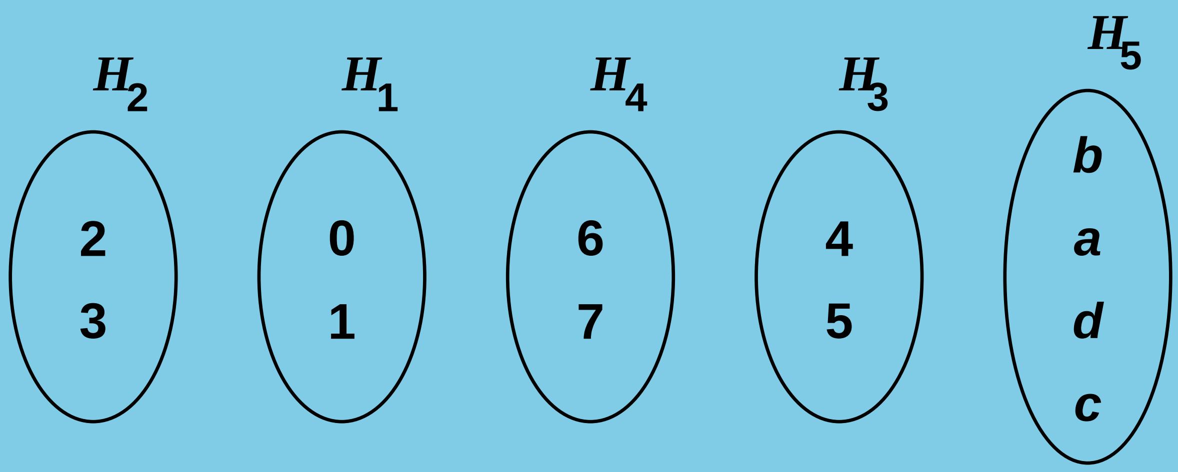

Incidence matrix

Let G be a possible automorphism group of a transverse t-(v, k, λ) design $(X, \mathcal{H}, \mathcal{B})$.

Let;

- $\Delta_1, \Delta_2, \ldots, \Delta_{N_t}$ be the orbits of transverse t-subsets;
- $\Gamma_1, \Gamma_2, \ldots, \Gamma_{N_k}$ be the orbits of transverse k-subsets;
- $A_{tk}[\Delta_i, \Gamma_j] = |\{K \in \Gamma_j : K \supseteq T\}|$, $T \in \Delta_i$ fixed.

Then there is a transverse t-(v, k, λ) design with automorphism group G if and only if

$$A_{tk} U = \lambda J$$

has a $(0, 1)$-valued solution U.

Example

Holes: $\{0\}$ $\{1\}$ $\{2\}$ $\{3\}$ $\{4,5,6\}$ $\{7,8,9\}$ (Type $1^4 3^2$.)

Group generators: $(0,1)(2,3), (0,2)(1,3), (0,1,2)(4,5,6)(7,8,9)$

Orbits of transverse triples:
Δ_0={**012**, **013**, **023**, **123**}
Δ_1={**014**, **234**, **125**, **026**, **035**, **036**, **136**}
Δ_2={**015**, **235**, **126**, **024**, **036**, **034**, **134**}
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Orbits of transverse quadruples:
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Row Δ_4 of A_{tk} is $[0,0,2,1,0,0,1,0,0,1,0,0]$.

Example continued

$A_{34} =$

	Γ_0	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7	Γ_8	Γ_9	Γ_{10}	Γ_{11}
Δ_0	1	3	3	0	0	0	0	0	0	0	0	0
Δ_1	0	2	0	1	1	1	0	0	0	0	0	0
Δ_2	0	2	0	0	0	0	1	1	1	0	0	0
Δ_3	0	2	0	0	0	0	0	0	0	1	1	1
Δ_4	0	0	2	1	0	0	1	0	0	1	0	0
Δ_5	0	0	2	0	1	0	0	1	0	0	1	0
Δ_6	0	0	2	0	0	1	0	0	1	0	0	1
Δ_7	0	0	0	1	0	0	0	1	0	0	0	1
Δ_8	0	0	0	0	1	0	0	0	1	1	0	0
Δ_9	0	0	0	0	0	1	1	0	0	0	1	0

Example continued

$A_{34} =$

	Γ_0	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7	Γ_8	Γ_9	Γ_{10}	Γ_{11}
Δ_0	1	3	3	0	0	0	0	0	0	0	0	0
Δ_1	0	2	0	1	1	1	0	0	0	0	0	0
Δ_2	0	2	0	0	0	0	1	1	1	0	0	0
Δ_3	0	2	0	0	0	0	0	0	0	1	1	1
Δ_4	0	0	2	1	0	0	1	0	0	1	0	0
Δ_5	0	0	2	0	1	0	0	1	0	0	1	0
Δ_6	0	0	2	0	0	1	0	0	1	0	0	1
Δ_7	0	0	0	1	0	0	0	1	0	0	0	1
Δ_8	0	0	0	0	1	0	0	0	1	1	0	0
Δ_9	0	0	0	0	0	1	1	0	0	0	1	0

Example continued

$A_{34} =$

	Γ_0	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7	Γ_8	Γ_9	Γ_{10}	Γ_{11}
Δ_0	1	3	3	0	0	0	0	0	0	0	0	0
Δ_1	0	2	0	1	1	1	0	0	0	0	0	0
Δ_2	0	2	0	0	0	0	1	1	1	0	0	0
Δ_3	0	2	0	0	0	0	0	0	0	1	1	1
Δ_4	0	0	2	1	0	0	1	0	0	1	0	0
Δ_5	0	0	2	0	1	0	0	1	0	0	1	0
Δ_6	0	0	2	0	0	1	0	0	1	0	0	1
Δ_7	0	0	0	1	0	0	0	1	0	0	0	1
Δ_8	0	0	0	0	1	0	0	0	1	1	0	0
Δ_9	0	0	0	0	0	1	1	0	0	0	1	0

Example continued

$A_{34} =$

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Δ_0	1	3	3	0	0	0	0	0	0	0	0	0
Δ_1	0	2	0	1	1	1	0	0	0	0	0	0
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Δ_5	0	0	2	0	1	0	0	1	0	0	1	0
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Δ_8	0	0	0	0	1	0	0	0	1	1	0	0
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$A_{34}U = J$, where $U = [1, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, 1, \mathbf{0}, 1, \mathbf{0}, 1, \mathbf{0}, \mathbf{0}]^T$.

So $\Gamma_0 \cup \Gamma_5 \cup \Gamma_7 \cup \Gamma_9$ is a transverse Steiner quadruple system of type $1^4 3^2$

$\Gamma_0 = \{\mathbf{0123}\}$

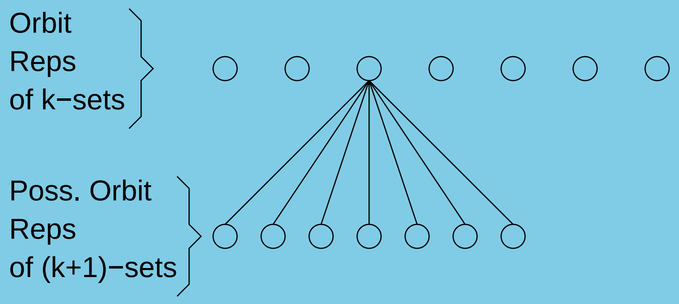
$\Gamma_5 = \{\mathbf{0149}, \mathbf{2349}, \mathbf{1257}, \mathbf{0168}, \mathbf{0357}, \mathbf{1368}\}$

$\Gamma_7 = \{\mathbf{0158}, \mathbf{2358}, \mathbf{1269}, \mathbf{0147}, \mathbf{0369}, \mathbf{1347}\}$

$\Gamma_9 = \{\mathbf{0167}, \mathbf{2367}, \mathbf{1248}, \mathbf{0159}, \mathbf{0348}, \mathbf{1359}\}$

Holes: $\{\mathbf{0}\}$ $\{\mathbf{1}\}$ $\{\mathbf{2}\}$ $\{\mathbf{3}\}$ $\{\mathbf{4}, \mathbf{5}, \mathbf{6}\}$ $\{\mathbf{7}, \mathbf{8}, \mathbf{9}\}$

Remarks on Orbit representatives



Order subsets by Lexicographical order.

If B is a minimal orbit rep. of transverse $(k+1)$-sets, then for some $x \in B$,

$$A = B \setminus \{x\},$$

is a minimal orbit rep. of transverse k-sets.

Let

$$C = C_A = \cup\{H \in \mathcal{H} : H \cap A = \emptyset\}$$

Consider all $A \cup \{x\}$ where $x \in C_A$.

Choose the minimal ones.

Computing orbit representatives

procedure $\textsc{TransReps}(G, \mathcal{H}, \mathcal{R}, \mathcal{S})$

comment:
- G: group
- $\mathcal{H}$: holes
- $\mathcal{R}$: known orb. reps. of trans. k-sets
- $\mathcal{S}$: computed orb. reps. of trans. $(k+1)$-sets

$\mathcal{S} \leftarrow$ the empty list
for each A in the list $\mathcal{R}$
do
- **comment:** Compute $C = C_A$.
- $C \leftarrow \emptyset$
- **for** $H \in \mathcal{H}$ **do if** $A \cap H = \emptyset$ **then** $C \leftarrow C \cup H$
- **for each** $x \in C$
- **do**
  - $B \leftarrow A \bigcup \{x\}$
  - **comment:** Find min. orbit rep. $B^\star$ of $G(B)$
  - $B^\star \leftarrow B$
  - **for each** $g \in G$ **do if** $g(B) < B^\star$ in lex. order **then** $B^\star \leftarrow g(B)$
  - **if** $B^\star$ is not in the list $\mathcal{S}$ **then** Insert $B^\star$ into list $\mathcal{S}$

Computing the number of orbits

The number of orbits of transverse k-sets

$$N[k] = \frac{1}{|G|} \sum_{g \in G} \textsc{Fix}(g);$$

where

$$\textsc{Fix}(g) = \text{ the number of trans. } k\text{-sets fixed by } g$$

So, we need to compute $\textsc{Fix}(g)$.

Computing Fix(g)

Write $g = C_0 C_1 C_2 \cdots C_{s-1}$ into disjoint cycles and set $\overline{C_j} = \{H : H \cap C_j \neq \emptyset\}$.

If K is a transverse k–element subset fixed by g, then

1. K is a union of cycles $C_{j[1]}, C_{j[2]}, \ldots, C_{j[\ell]}$,
2. each $C_{j[i]}$ is transverse to $\mathcal{H}$ for $i = 1, 2, \ldots, \ell$, and
3. $\overline{C_{j[h]}} \cap \overline{C_{j[i]}} = \emptyset$ for all $h \neq i$.

Define graph $\mathcal{G}_g$

- vertices $= \{C_j : C_j$ is transverse to $\mathcal{H}\}$.
- C_j is adjacent to C_i if and only if $\overline{C_j} \cap \overline{C_i} = \emptyset$

If A is a clique in $\mathcal{G}_g$, then g fixes a subset of size

$$\sum_{C_j \in A} |C_j|$$

namely the subset

$$K = \{x : x \in C_j \text{ and } C_j \in A\}.$$

Procedure for the num. of trans. orbits

procedure $\textsc{TransNorb}(G)$
for $i \leftarrow 0$ **to** n
  do $N[i] = 0$
for each $g \in G$

$$\textbf{do}\ \begin{cases} \text{Construct } \Gamma_g \\ \textbf{for each} \text{ clique } K \text{ of } \Gamma_g \\ \quad \textbf{do}\ \begin{cases} j \leftarrow 0 \\ \textbf{for each} \text{ cycle } C \in K \textbf{ do } j \leftarrow j + |C| \\ N[j] \leftarrow N[j] + 1 \end{cases} \end{cases}$$

for $i \leftarrow 0$ **to** n **do** $N[i] \leftarrow N[i]/|G|$
output (N)

Computing orbit representatives

procedure $\textsc{TransReps}(G, \mathcal{H}, \mathcal{R}, \mathcal{S})$

comment: $\begin{cases} G\text{: group} \\ \mathcal{H}\text{: holes} \\ \mathcal{R}\text{: known orb. reps. of trans. } k\text{-sets} \\ \mathcal{S}\text{: computed orb. reps. of trans. } (k+1)\text{-sets} \end{cases}$

$\mathcal{S} \leftarrow$ the empty list; $\quad n \leftarrow 0$

for each A in the list $\mathcal{R}$

do
- **comment:** Compute $C = C_A$.
- $C \leftarrow \emptyset$
- **for** $H \in \mathcal{H}$ **do if** $A \cap H = \emptyset$ **then** $C \leftarrow C \cup H$
- **for each** $x \in C$
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  - **for each** $g \in G$ **do if** $g(B) < B^\star$ in lex. order **then** $B^\star \leftarrow B$
  - **if** $B^\star$ is not in the list $\mathcal{S}$ **then**
    - Insert $B^\star$ into list $\mathcal{S}$
    - $n \leftarrow n + 1;$ **if** $n \geq N[k+1]$ **then exit**

Computing orbit representatives

procedure $\textsc{TransReps}(G, \mathcal{H}, \mathcal{R}, \mathcal{S})$

comment:
- G: group
- $\mathcal{H}$: holes
- $\mathcal{R}$: known orb. reps. of trans. k-sets
- $\mathcal{S}$: computed orb. reps. of trans. $(k+1)$-sets

$\mathcal{S} \leftarrow$ the empty list; $n \leftarrow 0$

for each A in the list $\mathcal{R}$

do
- **comment:** Compute $C = C_A$.
- $C \leftarrow \emptyset$
- **for** $H \in \mathcal{H}$ **do if** $A \cap H = \emptyset$ **then** $C \leftarrow C \cup H$
- **for each** $x \in C$
- **do**
  - $B \leftarrow A \bigcup \{x\}$
  - **comment:** Find min. orbit rep. $B^\star$ of $G(B)$
  - $B^\star \leftarrow B$
  - **for each** $g \in G$ **do if** $g(B) < B^\star$ in lex. order **then** $B^\star \leftarrow B$
  - **if** $B^\star$ is not in the list $\mathcal{S}$ **then**
    - Insert $B^\star$ into list $\mathcal{S}$
    - $n \leftarrow n + 1$; **if** $n \geq N[k+1]$ **then exit**

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$$\textbf{comment:} \begin{cases} G\text{: group} \\ \mathcal{H}\text{: holes} \\ \mathcal{R}\text{: known orb. reps. of trans. } k\text{-sets} \\ \mathcal{S}\text{: computed orb. reps. of trans. } (k+1)\text{-sets} \end{cases}$$

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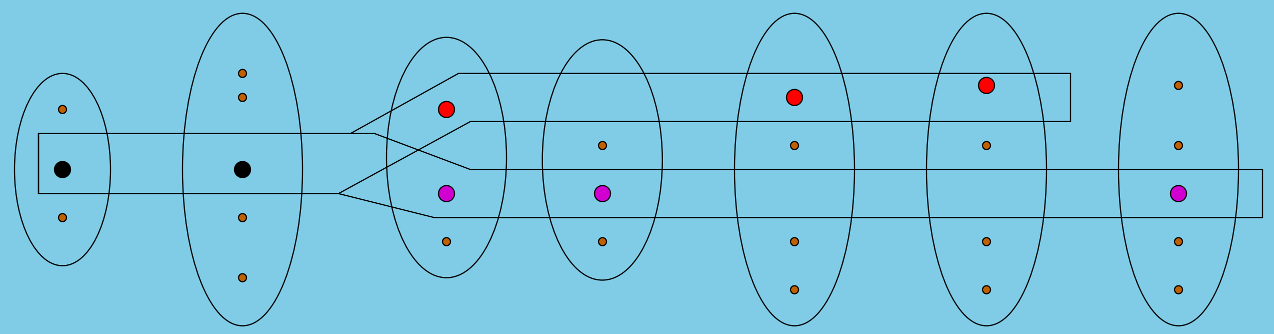
Derived design

Let $(X, \mathcal{H}, \mathcal{B})$ be a transverse t–(v, k, λ) design of type $h_1 h_2 \cdots h_r$ and let $S \subseteq X$, $s = |S| < t$.

Then the *derived design* w.r.t. S is the transverse $(t - s)$–$((v - s), (k - s), \lambda)$ design $(X', \mathcal{H}', \mathcal{B}')$; where

- $X' = X \setminus S$;
- $\mathcal{H} = \{H \in X, H \cap S = 0\}$
- $B' = \{B \setminus S : S \subseteq B \in \mathcal{B}\}$

Example with $t = 3$, $s = 2$ and $k = 5$.



Necessary Conditions for transverse SQS

Theorem :. *Suppose that a transverse* $\mathrm{SQS}(v)$ *of type* $h_1h_2\ldots h_n$ *exists and* $n \geq 4$*. Then* $v = \sum_{i=1}^{n} h_i$ *and the following hold:*

1. $h_i + h_j \equiv v \pmod 2, \forall i \neq j$;
2. *There exists a 3–*GDD *of type* $\prod_{i \neq \ell} h_i, \forall \ell$;
3. $\sum_{0<i<j<k \leq n} h_i h_j h_k \equiv 0 \pmod 4$.

Corollary :. *Suppose that a transverse* $\mathrm{SQS}(v)$ *of type* $h_1h_2\ldots h_n$ *exists. Then* $v = \sum_{i=1}^{n} h_i$ *and*

1. $h_1 \equiv h_2 \equiv \ldots \equiv h_n \pmod 2$, $n \geq 4$;
2. $v \equiv 0 \pmod 2$, $n \geq 4$.

Recursive and direct constructions

T.1. (Mills) For $u \geq 4$, $u \neq 5$:
$\exists$ trans-SQS of type $h^u \Leftrightarrow hu$ is even
and $h(u-1)(u-2) \equiv 0 \pmod 3$

. . . .

(Stanton, Mullin) $\nexists$ trans-SQSof type 2^5

. . . .

$\exists$ trans-SQS of type h^5, $\forall\, h \equiv 0, 4, 6, 8 \pmod{12}$.

. . . .

Open problem: Existence of trans-SQS of type h^5 when $h \equiv 2$ or $10 \pmod{12}$, $h > 2$.

T.2. (G.K.) For each $w > 0$ $\exists$ trans-SQS of type w^4.

T.3. $\exists$ trans-SQS of type $h_1 h_2 \ldots h_k$, $\Rightarrow$ $\exists$ trans-SQS of type $(wh_1)(wh_2)\ldots(wh_k)$

T.4. $\exists$ trans-SQS of type $m^s((s-2)m)^1 \Leftrightarrow s(s-1)m^2 \equiv 0 \pmod 6$, $(s-1)m \equiv 0 \pmod 2$, and $(m,s) \neq (1,7)$.

Recursive and direct constructions cont.

T.5. $\exists$ trans-SQS of type m^x and n^y with $m(x-1) = n(y-1)$, then $exists$ trans-SQS of type $m^x n^y$.

T.6. $\exists$ trans-SQS of type m^x and $g = m(x-1)$, $\Rightarrow \exists$ trans-SQS of type $m^x g^2$.

T.7. mn is even and $\exists$ trans-SQS of type $(mn)^r(s+t)^1$ and trans-SQS of type $m^n s^1 t^1 \Rightarrow \exists$ trans-SQS of type $m^{rn} s^1 t^1$.

T.8. $\nexists$ a trans-SQS(16) of type $1^1 3^5$.

T.2 For each $w > 0$ $\exists$ trans-SQS of type w^4.

- Holes: $H_i = \mathbb{Z}_w \times \{i\}$, $i = 1, 2, 3, 4$
- Blocks: $\{(x_1, 1), (x_2, 2), (x_3, 3), (x_4, 4)\}$ such that $x_1 + x_2 + x_3 + x_4 \equiv 0 \pmod 4$.

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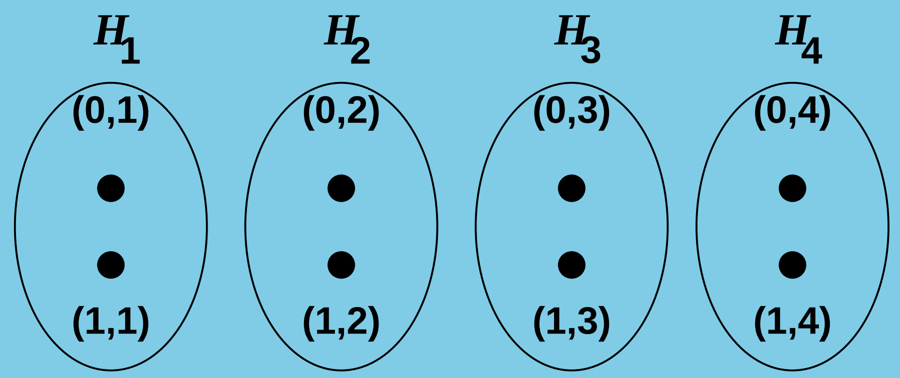

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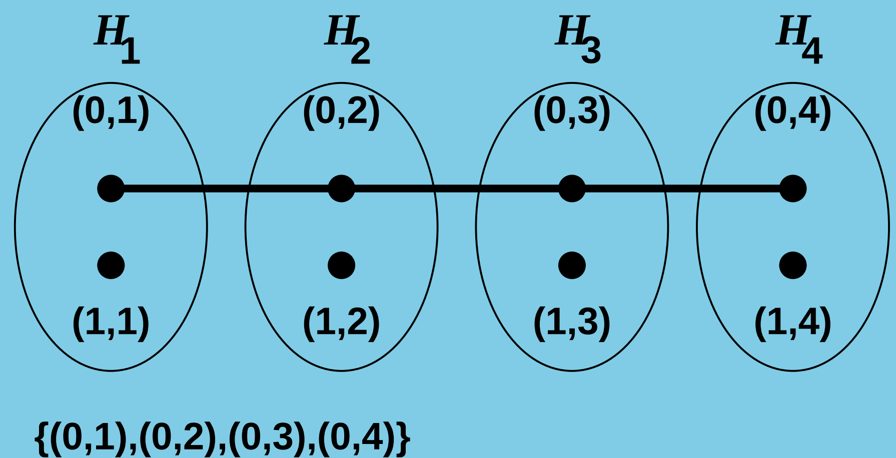

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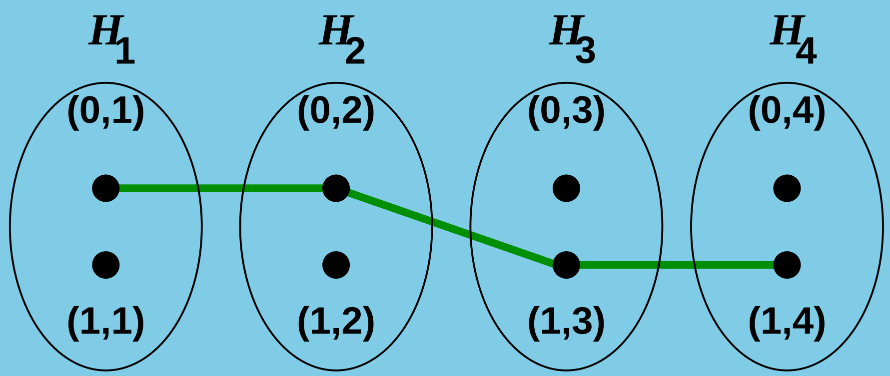

{(0,1),(0,2),(0,3),(0,4)}

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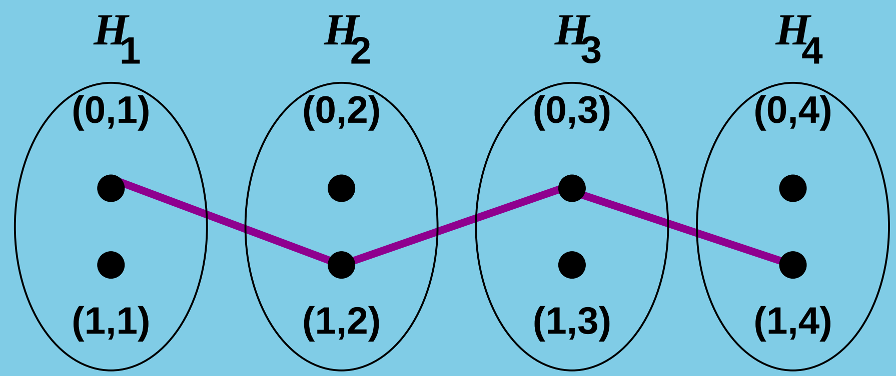

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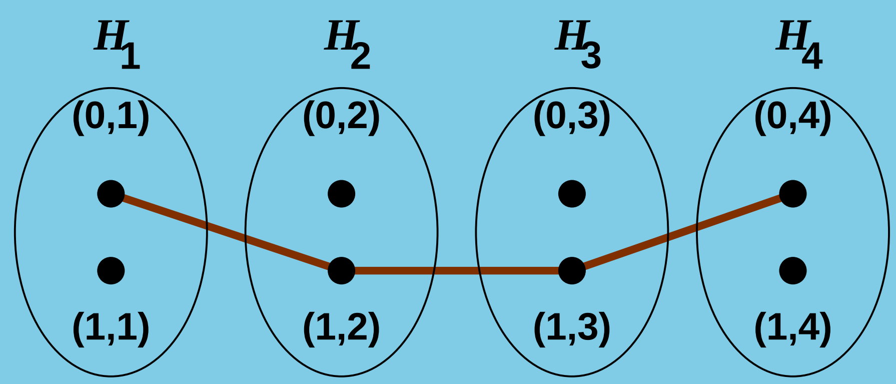

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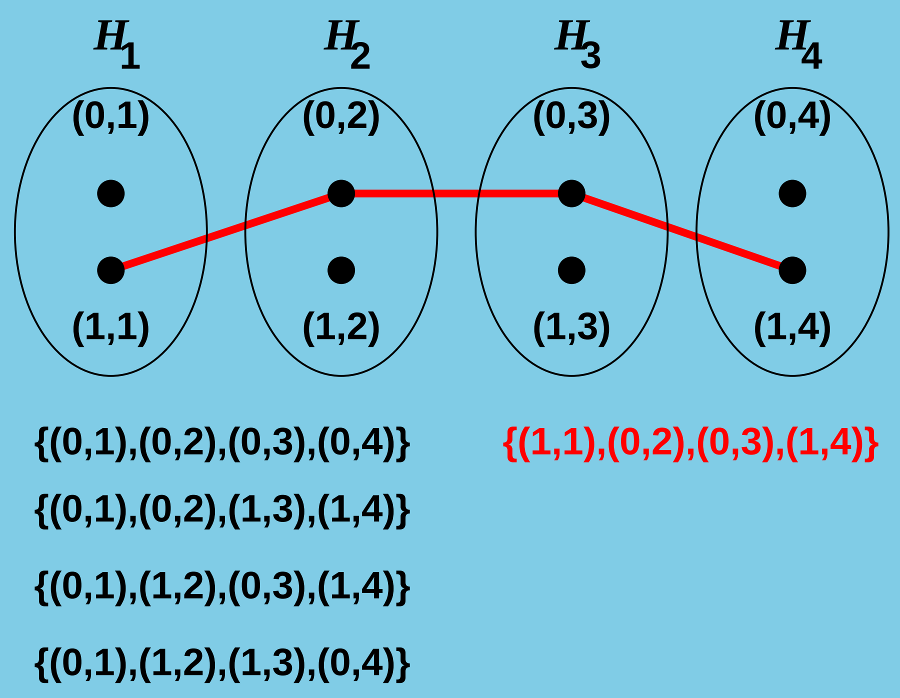

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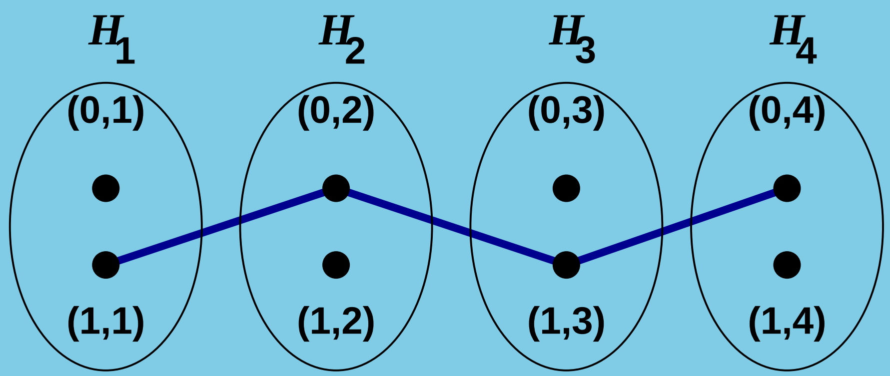

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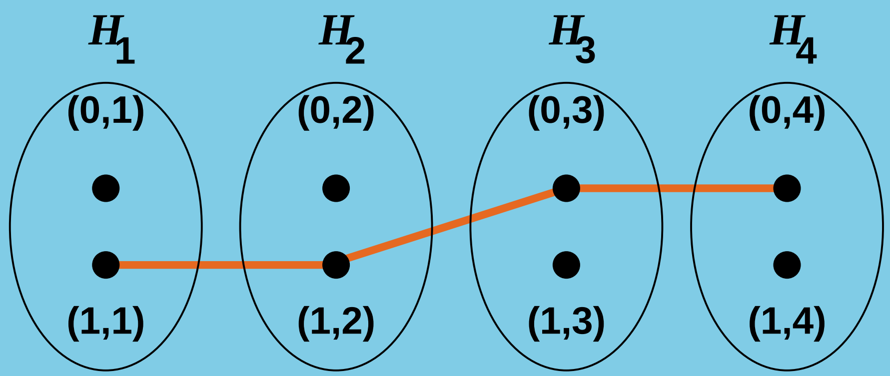

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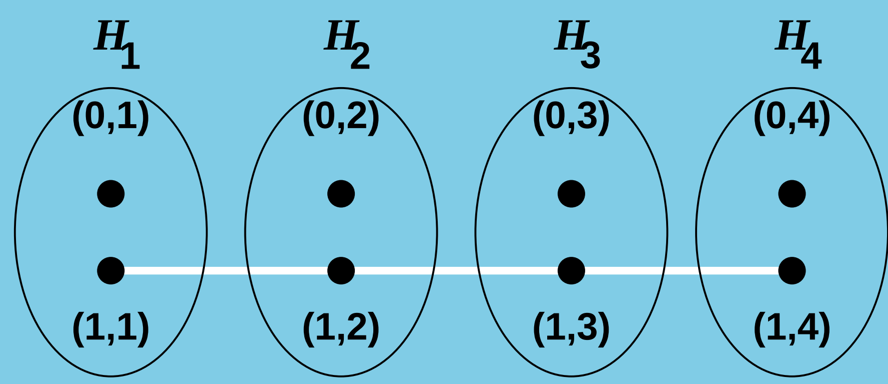

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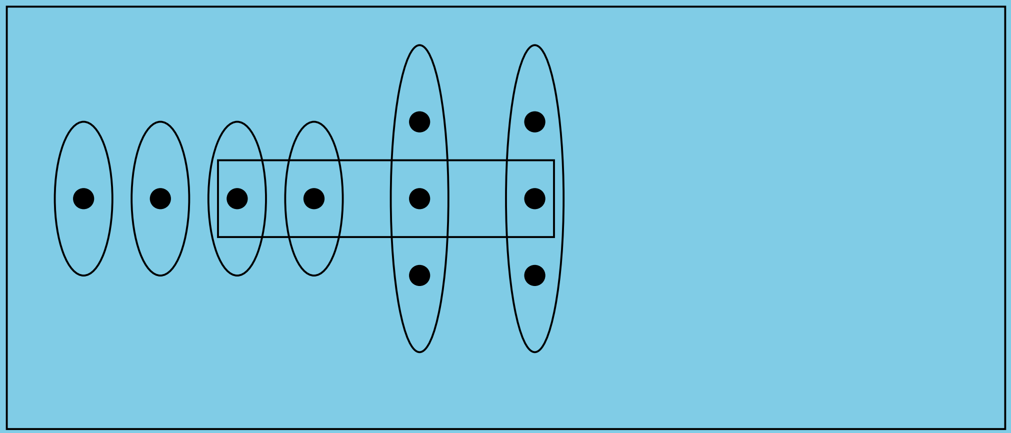

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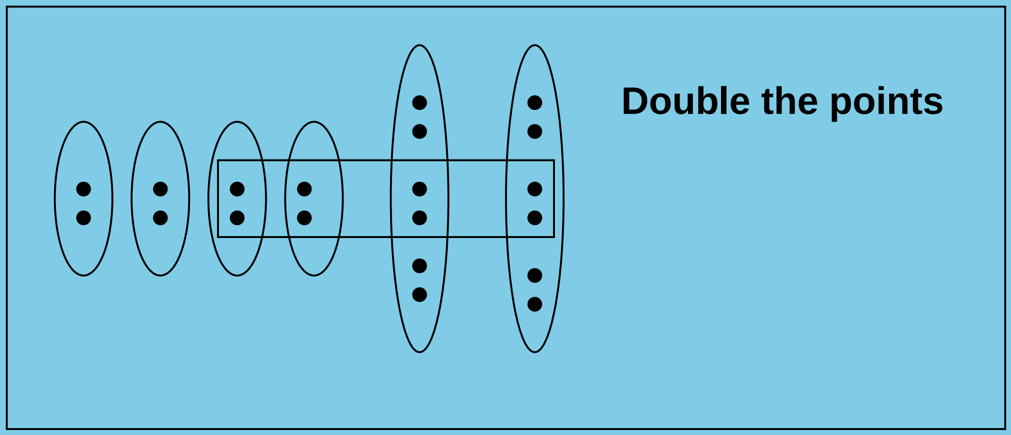

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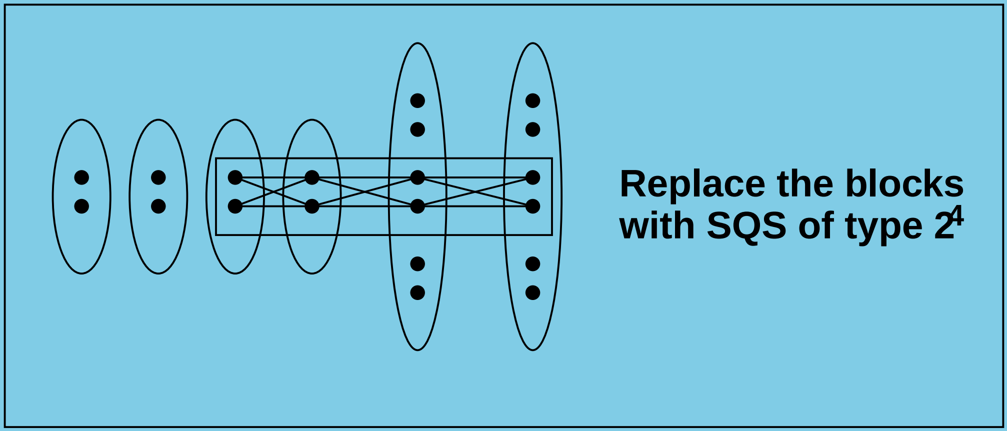


T.4 $\exists$ trans-SQS of type $m^s((s-2)m)^1 \Leftrightarrow s(s-1)m^2 \equiv 0 \pmod 6$, $(s-1)m \equiv 0 \pmod 2$, and $(m,s) \neq (1,7)$

- Conditions $\Rightarrow$ can partition the transverse triples of type m^s into disjoint transverse STS

$$T_1, T_2, \ldots T_{(s-2)m}.$$

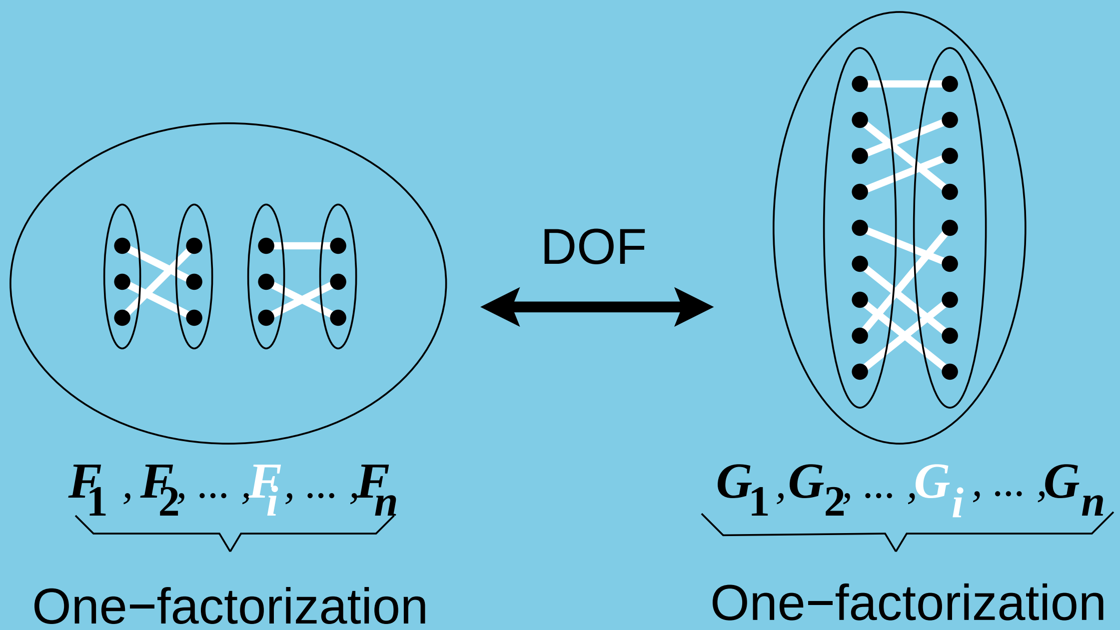
- Add hole of new points

$$x_1, x_2, \ldots x_{(s-2)m}.$$

- Join x_i to each triple in T_i.

The DOF construction

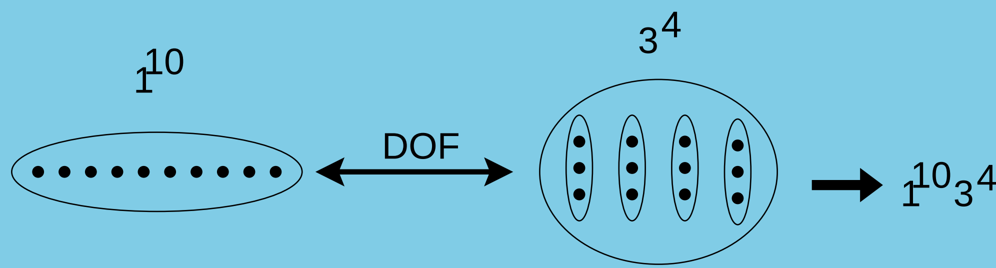
Cover type (2,1) & (1,2) triples with type (2,2) quads.



$$F_i \leftrightarrow G_i = \{e \cup e' : e \in E(F_i), e' \in E(G_i)\}$$

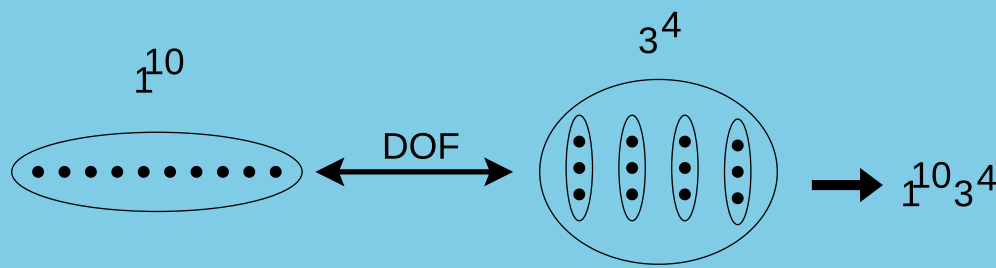
DOF applications

T.5

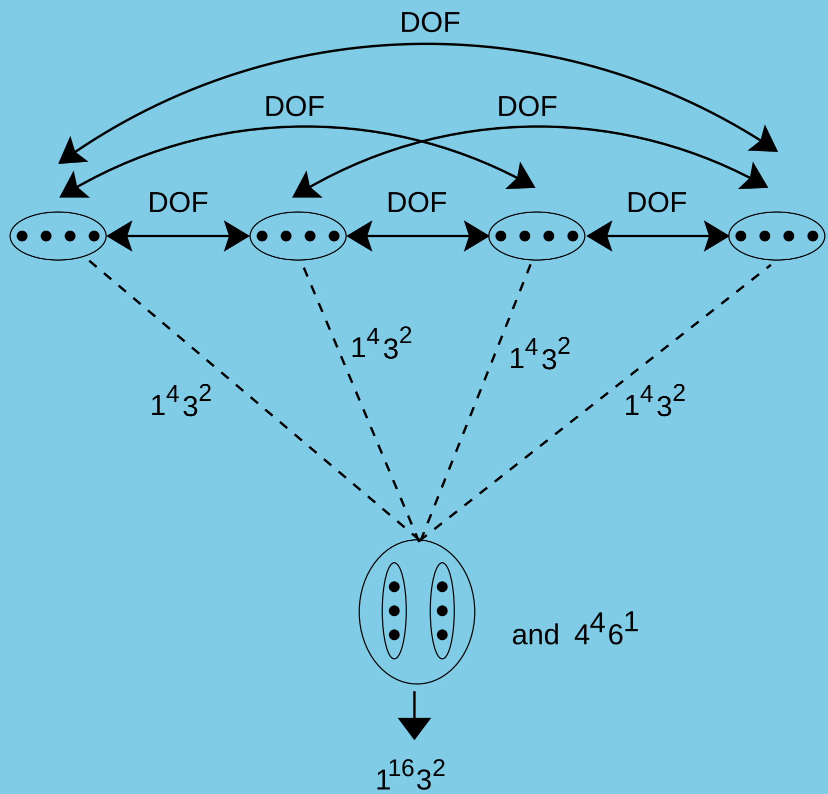


DOF applications

T.5



T.7



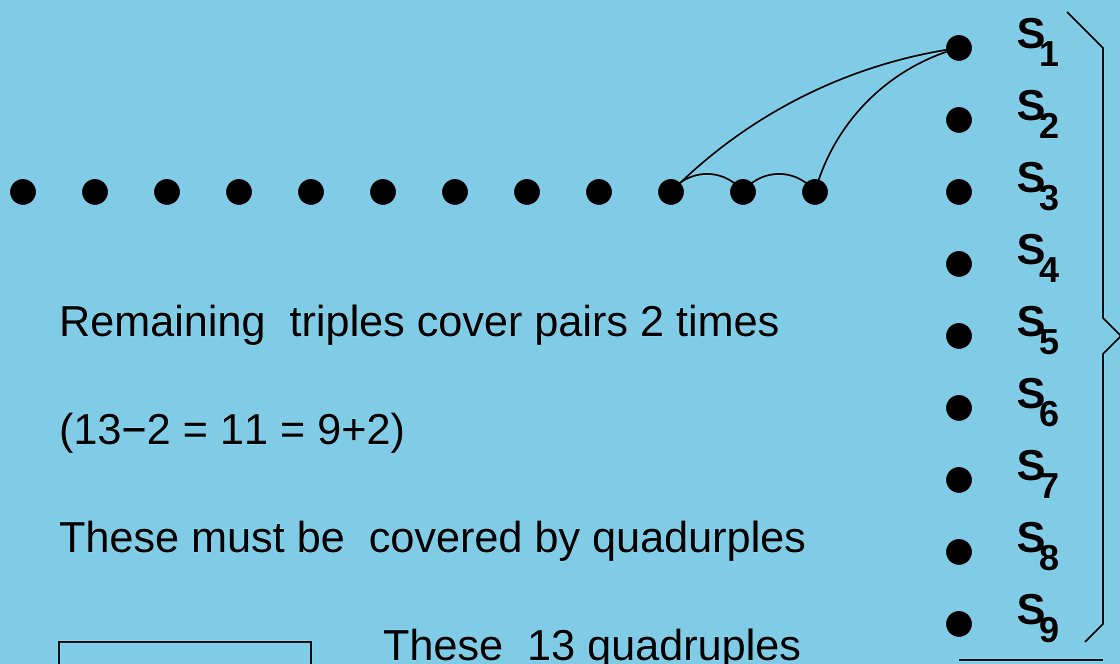
Results on Transverse SQS $4 \leq v \leq 22$

v	type	existence	remarks
4	1^4	Yes	**T.1**
8	1^8	Yes	**T.1**
	2^4	Yes	**T.1**
10	1^{10}	Yes	**T.1**
	2^5	No	**T.1**
	$1^4 3^2$	Yes	Computed or **T.6**
12	3^4	Yes	**T.1**
	$2^4 4^1$	Yes	Computed
	$1^7 5^1$	No	**T.4**
14	1^{14}	Yes	**T.1**
	2^7	Yes	**T.1**

v	type	existence	remarks
16	1^{16}	Yes	**T.1**
	2^{8}	Yes	**T.1**
	$1^{13}3^{1}$	Yes	Computed
	$1^{10}3^{2}$	Yes	Computed
	$1^{7}3^{3}$	Yes	Computed
	$1^{4}3^{4}$	Yes	Computed
	$1^{1}3^{5}$	No	**T.8**
	4^{4}	Yes	**T.1**
	$1^{9}7^{1}$	Yes	**T.4**
18	3^{6}	Yes	**T.1**
	$2^{7}4^{1}$	Yes	Computed
	$2^{1}4^{4}$	Yes	Computed
	$1^{13}5^{1}$	?	?

20	1^{20}	Yes	**T.1**
	$1^{13}7^{1}$	?	?
	2^{10}	Yes	**T.1**
	4^{5}	Yes	**T.1**
	5^{4}	Yes	**T.1**
	$2^{4}6^{2}$	Yes	Computed or **T.6**
	$2^{6}8^{1}$	Yes	**T.4**
	$2^{7}6^{1}$	?	?
	$3^{5}5^{1}$	?	?
22	1^{22}	Yes	**T.1**
	$1^{6}3^{3}7^{1}$	?	?
	$1^{10}3^{4}$	Yes	**T.5**
	$1^{12}3^{1}7^{1}$	?	?
	$1^{13}9^{1}$	?	?
	$1^{16}3^{2}$	Yes	**T.7**
	$1^{4}3^{6}$	Yes	Computed
	$1^{8}7^{2}$	Yes	Computed or **T.6**
	2^{11}	Yes	**T.1**
	$2^{7}8^{1}$	Yes	Computed
	$3^{5}7^{1}$	?	?
	$4^{4}6^{1}$	Yes	Computed

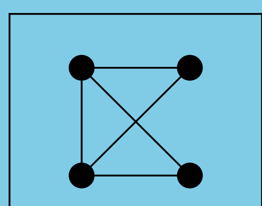
An interesting side problem: $1^{13}9^1$



Remaining triples cover pairs 2 times

(13−2 = 11 = 9+2)

These must be covered by quadurples



These 13 quadruples cover each pair once. So, they are a projective plane of order 3.

Can the non-collinear triples of a projective plane of order 3 be partitioned into disjoint Steiner Triple systems?

Exit Kimberly, enter Melissa

Melissa Keranen's goal: Investigate transverse $\mathrm{SQS}$of type $g^t u^1$.

Theorem: (M.S. Keranen) *If there is a* $\mathrm{OA}(3, k+1, n)$*, and a* $w^k x^1$ *for each* $x \in \mathcal{U}$*,then there is a transverse* SQS*of type* $(wn)^k u^1$ *for every* $u = u_1 + u_2 + \cdots + u_n$*, with* $u_i \in \mathcal{U}$*,* $i = 1, 2, \ldots, n$*.*

Theorem: (K.A. Bush 1952) *There is a* $\mathrm{OA}(3, k+1, n)$ *for all* $n = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$*, when* $p_i^{e_i} \geq k$ *is a prime power.*

Theorem: (M.S. Keranen) *There are transverse* SQS*of type* 2^k*,* 2^{k+1} *and* $2^k 4^1$ *for all* $k \equiv 1 \pmod 3$*.*

Corollary: (M.S. Keranen) *There is Transverse* SQS*of type* $(2n)^k u^1$ *for all* $u \leq 4n$*,* $k \equiv 1 \pmod 3$ *and* $n = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$*, when* $p_i^{e_i} \geq k$ *is a prime power.*

Specialize to $t = 4$, i.e. g^4, u^1

Theorem: (M.S. Keranen)

1. *If a transverse* SQS *exists of type* g^4u^1, *then* g, u *are even and* $u \leq 2g$.

2. *A a transverse* SQS *exists of type* $(2n)^4u^1$, *when* $u \equiv 0 \pmod 4$, $u \leq 4n$ *and* $n \not\equiv 3, 6 \pmod 9$.

3. *A a transverse* SQS *exists of type* $(4n)^4u^1$, *when* $u \equiv 0 \pmod 2$, $u \leq 8n$ *and* $n \not\equiv 3, 6 \pmod 9$.