Nonlinear Stability of the Classical Nusselt Problem of Film Condensation and Wave Effects

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1 Introduction

The Nusselt problem [1] of film condensation of quiescent saturated vapor on a vertical wall has been extensively studied—analytically, computationally, and experimentally. For this problem, the state of empirical knowledge with regard to typical wave effects on the heat transfer rates is quite good (see [2]). Despite this, a good understanding of noise effects and instability mechanisms for the flow has been lacking. With the help of first-principles-based computational simulations, this paper explores these issues and presents new results and understanding. Furthermore, compatibility with the well-known results for this problem provides a test of the efficacy of the first-principles-based simulation methodology employed here. This benchmark study, along with some experimental results, also strengthens the confidence in other convergent computational solutions obtained by essentially the same simulation methodology for internal condensing flows studied elsewhere [3–5].

The well-known analytical solution [1] of the Nusselt problem was improvised by Rohsenow [6] to account for the effects of the energy convection term. Subsequently, Sparrow and Gregg [7] provided a similarity solution under the assumption of zero interfacial shears. Vapor shear effects were accounted for, by an integral method, in the work of Chen [8], and, by a similarity solution technique, in the work of Koh et al. [9]. Dhir and Lienhard [10] applied/generalized the solution for situations involving varying gravitational inclinations. The computational solution of the steady problem that has been presented in this paper is consistent with the well-known Nusselt solution and its improvements. A good review of the criteria for the range of applicability of the Nusselt solution and its modifications is available in Armas et al. [11]. More specifically, this paper solves the steady Nusselt problem without making any of the usual approximations for the governing equations and yet yields the solutions which are in a good agreement with the Nusselt solution [1]. The paper also shows that, for steady solutions, it is only the near-interface vapor pressure field \((p_2 - p_0)\) that is significantly affected by the presence or absence of surface tension.

The unsteady solution for this problem—after ignoring the restriction based on the continuity of tangential velocities at the interface—has been attempted by Miyara [12]. This work has tried to improve upon earlier related computational efforts [13] for this problem. But these computational results suffer from the fact that the three different ways of computing interfacial mass flux values (from considerations of the relative velocity of vapor at the interface, the relative velocity of liquid at the interface, and the heat transfer across the interface) are not equal to one another. In this regard, the simulation results presented here are accurate and are shown to satisfy this and all the remaining interface conditions.

Wave initiation mechanisms can, in principle, also be understood by linear or nonlinear stability analyses. The linearized stability analyses of Unsal and Thomas [14] and Spindler [15] yield results that are mutually consistent but do not satisfy the well-known experimental results that are associated with laminar-to-turbulent transition. The experimental observance of this laminar-to-turbulent transition is believed to be related to Tollmien-Schlichting-type instability waves [16] that are suitably modified by free surface phenomena and mass transfer across the interface. The paper shows that the instability mechanism for this problem is necessarily a nonlinear phenomenon in time (as opposed to nonlinearities due to the size of the amplitudes alone) and, therefore, cannot be identified by either the linearized stability analyses assumption [(14,15)] or partial nonlinear analyses [(17)] that employ two term expansions in wave number and wave amplitude. This full nonlinear stability analysis presented here does achieve agreement with the reported values of \(Re_{DL} \approx 30\).

Our results are in basic agreement with the known experimental result that laminar wavy flows occur over a zone for which, approximately, \(Re_\theta \approx 30\) (see \(Re_\theta\) definition in the Nomenclature and Incropera et al. [2]) and that the waves are typically small to nonexistent over a zone for which, approximately, \(Re_\theta \leq 30\). Experimentally obtained local heat transfer coefficients for the wavy regime have been proposed by Kutateladze [18], Chun and Seban [19], etc., and are also given in Incropera et al. [2]. The results given here are consistent with the range of heat transfer enhancements that are expected under typical wavy conditions.

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Accurate steady and unsteady numerical solutions of the full two-dimensional (2D) governing equations for the Nusselt problem (film condensation of quiescent saturated vapor on a vertical wall) are presented and related to known results. The problem, solved accurately up to film Reynolds number of 60 \((Re_f \approx 60)\), establishes various features of the well-known steady solution and reveals the interesting phenomena of stability, instability, and nonlinear wave effects. It is shown that intrinsic flow instabilities cause the wave effects to grow over the well-known experiments-based range of \(Re_\theta \approx 30\). The wave effects due to film flow’s sensitivity to ever-present minuscule transverse vibrations of the condensing surface are also described. The results suggest some ways of choosing wall noise—through suitable actuators—that can dampen or enhance or dampen wave fluctuations and thus increase or decrease heat transfer rates over the laminar-to-turbulent transition zone. [DOI: 10.1115/1.2198249]
Unlike the linearized stability analyses, the accurate simulations reported here also suggest an insignificant role of surface tension on the value of $Re_{dc}c=30$. This is in agreement with experiments for this gravity driven flow but not in agreement with the results based on the linearized stability analyses given in [14,15].

This paper also clearly identifies how significantly wall noise interacts with Tollmien-Schlichting-type growing waves in the wavy laminar regime (i.e., transition to turbulence regime, which is typically characterized by $30\% Re=1800$). It is shown that wall noise—modeled as a linear superposition of transverse displacement standing waves of different amplitude, frequency, and wavelength—can either diminish or accentuate the wave effects. The enhancement of wave effects can be ensured if one can place suitably chosen noise sources (e.g., actuators) that satisfy a certain resonance condition. This result highlights the fact that knowledge of control and wall noise are important issues in ensuring repetitiveness of wall effects in the transition-to-turbulence regime.

2 Governing Equations

The liquid and vapor phases in the flow (e.g., Fig. 1) are denoted by a subscript $l$: $l=1$ for liquid and $l=2$ for vapor. The fluid properties (density $\rho$, viscosity $\mu$, specific heat $C_p$, and thermal conductivity $k$) with subscript “$l$” are assumed to take their representative constant values for each phase ($l=1$ or 2). Let $T_l$ be the temperature fields, $p_l$ the pressure fields, $T_s(p)$ the saturation temperature of the vapor as a function of local pressure $p$, $\Delta$ be the film thickness, $\dot{m}$ be the local interfacial mass flux, $T_0(x)$ be the physical value of the steady Nusselt film thickness $\Delta N(x)$ at $x=X_c$, where $X_c$ is a known numerical multiple of the well known ([1]) physical value of the steady Nusselt film thickness $\Delta N(x)$ at $x=X_c$. That is, $Y_c=c\cdot\Delta N(x_c)$, where $c$ is a known number (e.g., $c=47$ for all cases shown here). This makes $Y_c$ a priori known and sufficiently large to capture all the relevant vapor flow. While other choices of characteristic length $Y_c$ are possible (e.g., $Y_c=\Delta Y(x_c)$), this choice is convenient for implementing the computational approach employed here. Furthermore, let the characteristic speed $U$ be the average value, at $x=X_c$, of the $x$ component of the liquid speed obtained from the well-known [1] Nusselt solution. That is, $U=g(\rho_1-p_2)\cdot\Delta N(x)Y_c^2/3\mu_1$. The above choices of characteristic length and speed are used for defining the nondimensional variables whose computationally obtained values are reported in this paper. As needed, this choice is related to other results obtained from other commonly used choices of characteristic length and speed. Let $g_0$ and $g_1$ be the components of gravity along $x$ and $y$ axes, $p_0$ be the pressure of the far-field quiescent vapor, $\Delta T=T_s(p_0)-T_s(0)$ be the representative controlling temperature difference between the vapor and the bottom plate, and $h_{fs}$ be the heat of vaporization at saturation temperature $T_s(p)$. With $\epsilon$ representing the physical time, we introduce a new list of fundamental nondimensional variables through the following definitions:

$$
\begin{align*}
\{x, y, \delta, u, \dot{m}\} &= \left\{ \frac{x}{\sqrt{\epsilon}} \frac{y}{\sqrt{\epsilon}} \frac{\delta}{\delta_l} \frac{u}{U} \frac{\dot{m}}{\rho U^2} \right\} \\
\{v, \theta, \pi, \epsilon\} &= \left\{ \frac{v}{U} \frac{\theta}{T} \frac{\pi}{\rho_1 U^2} \frac{\epsilon}{(Y/U)} \right\}
\end{align*}
$$

(1)

2.1 Interior Equations. The nearly exact interface conditions (Delhaye [20]) for condensing flows, with some approximations, are given in Narain et al. [3]—see their Appendix Eqs. (A1)–(A7). Utilizing a superscript “$i$” for values of the flow variables at the interface given by $H=\rho-i\cdot(x, y, \epsilon)$, the nondimensional forms of the interface conditions are given below.

- The nondimensional form of the requirement of continuity of tangential component of velocities (Eq. (A2) of [3]) becomes:

$$
\dot{u}_l^{(i)} = \dot{u}_l^{(i)} - \delta_l (\dot{u}_l^{(i)} - \dot{u}_l^{(i)})
$$

(2)

- The nondimensional form of the normal component of momentum balance at the interface (Eq. (A3) of [3]) becomes:

$$
\frac{\pi^{(i)}_l}{\pi^{(i)}_l} = \frac{1}{\frac{\rho_1}{\rho_1} + \frac{\sigma}{\sigma T}} + \frac{\dot{m}^2}{\rho_1 U^2} - \frac{1}{\rho_1 U^2}
$$

(3)

where $\sigma=1/\rho_1$, surface tension $\sigma=\sigma T$ with $T$ being the interface temperature.

- The tangential component of momentum balance at the interface (Eq. (A4) of [3]) becomes:

$$
\frac{\dot{u}_l^{(i)}}{\dot{u}_l^{(i)}} = \frac{\mu_2}{\mu_1} \frac{\dot{u}_l^{(i)}}{\dot{u}_l^{(i)}} + \frac{[I]}{[I]}
$$

(4)

The term $[I]$ used here is defined by Eq. (A6) of the Appendix.

- The nondimensional forms of non-zero-temperature mass fluxes $m_{l_k}$ and $m_{v_k}$ (defined in Eq. (A5) of [3]) impose kinematic constraints on the interfacial values of the liquid and vapor velocity fields and are given by:

$$
\begin{align*}
\dot{m}_{l_k} &= [u^l_1(\delta l\delta l) - (\delta l - \delta l\delta l)]/(1 + (\delta l\delta l)^2) \\
\dot{m}_{v_k} &= \left(\frac{p_2}{p_1}\right)[u^v_1(\delta l\delta l) - (\delta l - \delta l\delta l)]/(1 + (\delta l\delta l)^2)
\end{align*}
$$

(5)

- The nondimensional form of non-zero-interfacial mass flux $m_{\text{Energy}}$ (as given by Eq. (A6) of [3]) represents the constraint imposed on mass flux by the balance equation for the net energy transfer across the interface, and is given by:

Fig. 1 Cooled vertical plate in a quiescent (far field) vapor flow—geometry used for simulations
\[
m_{\text{Energy}} = J_a / (R_e, P_r_1) \left[ \frac{\partial d}{\partial t} \right] - (k_j / k_i) \frac{\partial \theta_j}{\partial t} \right]
\]

where \( J_a = C_p \Delta T / T_0 \) and \( h_{fg} = h_{fg} / T_s (p_{sat}) \equiv h_{fg} / T_s (p_{sat}) \).

2. Interfacial mass balance requires that the net mass flux (in kg/m²/s) at a point on the interface, as given by Eq. (A7) of [3], be single valued regardless of which physical process is used to obtain it. The nondimensional form of this requirement becomes:

\[
m_{\text{LK}} = m_{\text{VK}} = m_{\text{Energy}} = m
\]

(7)

It should be noted that negligible interfacial thermal resistance and equilibrium thermodynamics on either side of the interface are assumed to hold for x values downstream of the origin (i.e., second or third computational cell onward). And hence, as in Nusselt [1] solution and as per discussions leading to Eq. (A8) in [3], no nonequilibrium thermodynamic model for the interfacial mass flux \( m \) is needed to obtain a solution.

- The nondimensional thermodynamic restriction on interfacial temperatures (as given by Eq. (A8) in [3]) becomes:

\[
\theta_i = \theta_y = T_s (p_{sat}) / T = \theta_s (\pi_2)
\]

(8)

Within the vapor domain, for any of the typical refrigerants (such as R113 considered here) changes in absolute pressure relative to the interface pressure are sufficient to affect vapor motion, but, at the same time, they are too small to affect saturation temperatures. This allows the approximation:

\[
\theta_i (\pi_2) = \theta_s (0)
\]

2.3 Boundary Conditions. Since the vapor is stationary at locations far away from the condensate, the appropriate far field vapor boundary conditions are prescribed along lines AB (z = 0), BC (\( \gamma = y_c \)), and CD (z = X_c or \( x = x_c = y_c / Y_c \)) in Fig. 1. These are:

- \( \pi_2 (0,y,t) = 0 \) and \( \partial \pi_2 / \partial x (0,y,t) = 0 \)

- \( \pi_2 (x,1,t) = 0 \) and \( \partial \pi_2 / \partial y (x,1,t) = 0 \)

- \( \pi_2 (x,y,t,0) = 0 \) and \( \partial \pi_2 / \partial x (x,y,t,0) = 0 \)

(9)

At the condensing surface (y = 0), we have:

\[
u_t (x,0,t) = \nu_t (x,0,t) = 0 \quad \text{and} \quad \theta_t (x,0,t) = \theta_s (T_s / \Delta T)
\]

Furthermore, vapor can be assumed to be at uniform saturation temperature—i.e., \( \theta_s (x,y,t) = \theta_s (0) \) at all locations in the vapor domain. This is reasonable because effects of superheat \( \Delta T_{\text{sat}} \) (in the typical 5–10°C range) are verifiably negligible for the typically small values of vapor Jacob number \( J_a = C_p \Delta T_{\text{sat}} / h_{fg} \) encountered for most vapor flow conditions studied here. The point D at \( x = x_c \) is considered to be slightly but sufficiently above the interface and no exit condition is imposed along ED. However, the mass flow over ED—which specifically includes the liquid portion \( 0 \leq \gamma \leq \delta (x_c) \)—is required to satisfy the overall mass balance for a control volume formed by the bounding surfaces \( x = 0, x = X_c, \gamma = 0 \), and \( \gamma = y_c \).

2.4 Initial Conditions. The above described continuum equations do not model and incorporate various intermolecular forces that are important in determining the time evolution of very thin (10–100 nm) condensate film thickness \( \delta (x,t) \). As a result, \( t = 0 \) cannot be chosen to be the time when saturated vapor first comes in contact with and condenses on the dry subcooled \( T_s (x) < T_s (p_{sat}) \) vertical/inclined wall. With the above modeling limitations, the strategy here is to start at a time \( t = 0 \) for which one has a sufficiently thick steady solution of the continuum equations
\[ \bar{z}_i(x) = \frac{1}{\Delta} \int_{\Delta}^{0} \frac{g(p_1 - p_2)(\Delta y(x))^2}{3\mu_1} \, dy \]

\[ \text{Re}_i(x) = \frac{p_i g(p_1 - p_2)(\Delta y(x))^2}{3\mu_1} \]

\[ \text{Re}_j(x) = \frac{4p_i U \Delta y(x)}{\mu_1} \]

(14)

3 Computational Approach

Most of the details of the 2D steady/unsteady approach are the same as described in Narain et al. [3], Liang et al. [4], and Liang [5]. However, unlike the internal condensing flows considered in the earlier papers, the external condensing flow of this paper employs a different computational approach for the implementation of the pressure boundary condition described in Eq. (9). Since the boundary conditions that need to be imposed along lines AB, BC, and CD in Fig. 1 are much like prescribing the shear and pressure on the interface, the “tau-p” approach for the interface (see “tau-p” method described in Liang [5], Narain et al. [3], and Yu [21]—instead of other available approaches [22]—was adapted to satisfy the convergent boundary conditions in Eq. (9).

The solutions’ convergence in the interior of each phase, grid independence, and satisfaction of interface and boundary conditions are better (i.e., within 5%—see a representative case in Table 2) than what were reported (within 7% on average) in [3–5]. On any interface with propagating waves, the critical and difficult to satisfy requirement is Eq. (7)—the requirement of the equality of three different computed/obtained values of interfacial mass flux (this is known to be difficult for the more general interface capturing techniques such as level set [23] or Volume of Fluid [24]). However, this requirement is met by the interface tracking approach employed in this paper (see Table 2). As shown in [3,4], one of the interface conditions [viz. \(m_{LK} = m_{Energy}\) in Eq. (7)] yields the interface tracking equation that is used in this paper. This equation is of the hyperbolic form:

\[ \frac{\partial \delta}{\partial t} + \bar{u}(x,t) \frac{\partial \delta}{\partial x} = \bar{v}(x,t) \]

(15)

where the characteristic speed is \( \bar{u} = u_i + \frac{[\rho_i/\text{Re}_1 \rho_i] \partial \delta}{\partial x} \) and the forcing function is \( \bar{v} = v_i + \frac{[\rho_i/\text{Re}_1 \rho_i] \partial \phi_i}{\partial x} \). The spatial and temporal grid spacings and total lengths impose a restriction on wavelength \( \lambda \) and frequency \( f \) that can be adequately resolved. If the maximum spacing of the grid in the \( x \) direction is \( \Delta x \), and its total length is \( L \), while the total time duration is \( t \), and is divided in equal intervals of duration \( \Delta t \); the restrictions imposed by Nyquist criteria [25] are well satisfied for \( \lambda \approx 4\Delta x \) and \( f \approx (4\Delta t)^{-1} \) and the restrictions imposed by the domain lengths are well satisfied for \( \lambda \approx x \) and \( f \approx 2/t \). The initial \(( t=0) \) spatial and temporal grids are defined by \(( n_i \times n_j \times n_k), \) where \( n_i \) is the total number of initial grid points along \( y \) (0 to 1), and \( n_j \) is number of time steps with equal intervals \( (\Delta t) \). Typical values of \( n_i \) and \( n_j \) were used 30–40 and 50–70 for typical maximum values of \( x \) and \( y \) of 50 and 1. Attainable values of \( n_i \) depend on \( n_j, n_y, \Delta t, \) and the available computer memory for the storage of flow variables. For the cases reported here, typical maximum \( n_i \times \Delta t \) (\( n \Delta t \)) values are in the range from 30 \times 5 (=150) to 36 \times 7.5 (=270).

4 Computational Results

4.1 Simulation Results for the Steady Problem. The classical Nusselt solution [1] was improvised by Rohsenow et al. [6] to account for the effects of the neglected convection term in the energy equation. Subsequently, Sparrow and Gregg [7] and Koh et al. [9] accounted for the effects of convection and inertia terms within the framework of the boundary layer and small slope approximations \((\partial \phi_i/\partial x \approx \partial \phi_i/\partial y \) and \( \phi^2 \ll 1 \)) of the governing equations.

For a specific case (see Table 1 and [26]), Figure 2(a) demonstrates not only the ability of our computational approach to solve the problem as posed by the Nusselt formulation [1] but, also, to solve the full steady problem without the Nusselt approximations. Since a significant amount of parametric study has already been done (see [7,9], etc.) for this problem, no parametric study has been done here. This is because this paper limits itself to a qualitative understanding of the steady base flow and has a greater focus on an understanding of the superimposed wave effects.

In Fig. 2(a) and its inset, for the conditions described in Table 1, “Curve 1” represents the classical analytical solution of Nusselt plotted as \( \delta = \Delta x/y \) against \( x = \Delta x/y \), where \( \Delta x \) is defined in Eq. (14). “Curve 2” represents the solution under all Nusselt approximations except one—namely, in the interior equations, \( \partial \phi_i/\partial x \) terms have been retained. “Curve 3” is for the case which retains the negligible vapor viscosity and negligible vapor motion assumptions (resulting in \( \partial \phi_i/\partial x \approx 0 \) and \( \phi^2 \approx 0 \) but does not neglect liquid inertia and convection. “Curve 4” is for the full problem—discussed below for “Curve 5”—except that the surface tension term on the right side of Eq. (3) has been dropped. Curve 5 is for the full steady problem posed by Eqs. (A1)–(A4) under interface conditions (2)–(8) and boundary conditions (9) and (10). The effects of the presence or absence of surface tension, as shown in Figs. 2(c) and 2(d), are negligible on the motion of the vapor but are noticeable on the interfacial values of the pressures.

\[ \begin{array}{cccccccc}
\text{Table 1: Specification of a flow situation involving saturated R-113 (ASHRAE [26])} \\

<table>
<thead>
<tr>
<th>( p_w ) (kPa)</th>
<th>( T_w (\degree C) )</th>
<th>( \Delta T (\degree C) )</th>
<th>( p_1/\rho_1 )</th>
<th>( \rho_1/\mu_1 )</th>
<th>( \text{We} )</th>
<th>( \text{Pr}_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>108.85</td>
<td>49.5</td>
<td>5</td>
<td>0.005</td>
<td>0.020</td>
<td>67.9</td>
<td>7.3</td>
</tr>
</tbody>
</table>
\end{array} \]

| Table 2: Representative interfacial values of nondimensional variables show satisfaction of all interface conditions (for flow exposed to initial disturbance with \( \varepsilon = 0.7 \) and \( \Delta \alpha = 15 \)) because appropriate contiguous columns are nearly equal to one another—as required by Eqs. (2)–(4), (7), and (8) |

<table>
<thead>
<tr>
<th>( x )</th>
<th>( m_{LK} )</th>
<th>( m_{VK} )</th>
<th>( m_{Energy} )</th>
<th>( \bar{u}_1 )</th>
<th>( \bar{u}_2 )</th>
<th>( \bar{a}_1 )</th>
<th>( \bar{a}_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>5.37E–05</td>
<td>5.33E–05</td>
<td>5.34E–05</td>
<td>1.86E–04</td>
<td>1.86E–04</td>
<td>1.15E–02</td>
<td>0.099621</td>
</tr>
<tr>
<td>8</td>
<td>3.55E–05</td>
<td>3.52E–05</td>
<td>3.53E–05</td>
<td>–2.22E–06</td>
<td>–2.22E–06</td>
<td>2.96E–02</td>
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</tr>
<tr>
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<td>3.07E–05</td>
<td>3.05E–05</td>
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<td>2.74E–05</td>
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<td>9.88E–07</td>
<td>5.36E–02</td>
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<td>5.57E–07</td>
<td>6.34E–02</td>
<td>0.011717</td>
</tr>
</tbody>
</table>

Note: \( \phi = \frac{\delta}{u_2} + \phi \left( \frac{u_1}{u_2} \right) \)}
Fig. 2  (a) For the case of R-113 (see Table 1) experiencing film condensation on a vertical plate, the figure above shows steady film thickness values under different approximations. Here, \( x_c = X_c / Y_c = 50 \) and \( c = Y_c / \Delta \varphi (X_c) = 47 \). (b) For the case of R-113 (see Table 1) experiencing film condensation on a vertical plate, the figure above shows the streamline pattern and contour zones depicting a range of \(|\psi|\) values. (c) For Curves 4 and 5 of the base flow in (a), the figure above shows the \( u(x', y) \) versus \( y \) for \( x' = 20 \). (d) For the case of R-113 (see Table 1) experiencing film condensation on a vertical plate, the figure above shows the contour zones for temperature and a representative plot of \( \theta(x', y) \) for \( x' = 20 \). (e) For Curves 4 and 5 of the base flow in (a), the figure above shows the \( \pi(x', y) \) versus \( y \) for \( x' = 20 \).
and Fig. 3

**Figure 3** (a) For the base flow in Fig. 2(a), the figure above shows the stable response \((\Delta t=7.5, t=150)\) of the film thickness \(\delta(x,t)\) as a result of an initial disturbance \(\delta(x,0) = \delta_{\text{steady}}(x) [1+\varepsilon \delta'(x,0)], \) where \(\delta'(x,0)=\sin(2\pi x/\lambda_0), \varepsilon=0.15\) and \(\lambda_0=7\). (b) For the base flow in Fig. 2(a), the figure above shows the unstable response \((\Delta t=7.5, t=247.5)\) of the film thickness \(\delta(x,t)\) as a result of an initial disturbance \(\delta(x,0) = \delta_{\text{steady}}(x) [1+\varepsilon \delta'(x,0)], \) where \(\delta'(x,0)=\sin(2\pi x/\lambda_0), \varepsilon=0.15\) and \(\lambda_0=15\).

Generally, for all the cases considered here, the Jacob number \((Ja)\) is small and hence all predictions of \(\delta(x)\) are within 2.5% of the Nusselt solution.

**4.2 Simulation Results for the Unsteady Problem.**

**4.2.1 Nonlinear Stability and the Effects of Initial Disturbances.** The stable response of the flow in Fig. 2(a) to an initial sinusoidal disturbance of wavelength \(\lambda=7\) is shown in Fig. 3(a). Here, by stability, it is meant that as waves travel forward to downstream locations, the amplitudes of the waves diminish—with respect to the initial amplitude. Similarly, by instability, it is meant that as waves travel forward to downstream locations, the amplitudes of the waves increase and become significantly larger than the initial amplitudes. Figure 3(b) shows an unstable response of the flow in Fig. 2(a) to an initial sinusoidal disturbance of wavelength \(\lambda=15\). Clearly, in Figs. 3(a) and 3(b), the stability/instability phenomena manifests only after a certain downstream distance marked by \(x=x_{cr}\). As indicated by the dual labeling of the \(x\) axis in Fig. 4, for any given flow, there is a one to one correspondence between \(Re(x)\) and \(Re_{11}x=x_{cr}\).

Figure 4 shows the general response, at \(t=247.5\) of the flow in Fig. 2(a) to initial disturbances of various wavelengths—viz. \(\lambda =5, 9, 15, \) and 23. Generally, for \(\lambda<\lambda_{cr}\) (here, \(\lambda_{cr}=11.5\)), the response is stable and for \(\lambda>\lambda_{cr}\), the response is unstable. The larger wavelength waves are clearly manifested for \(x>x_{cr}=18\) (i.e., \(Re(x)>Re_{11}x=x_{cr}=7\) or, equivalently, \(Re_{s}>Re_{11}x=x_{cr}=28\)).

**4.2.2 Effects of Surface Tension.** The above described instability mechanisms—of the Tollmien-Schlichting [16] type—are only mildly affected by surface tension. Figure 5 shows that the waves are gravity dominated and surface tension effects are negligible. For the flow considered in Fig. 5, cases \(\sigma=\sigma^*\) and \(\sigma=2.5\sigma^*\), when compared to the \(\sigma=0\) case, indicate that surface tension only slightly assists in steepening the front of the wave.

**4.2.3 Computed Values of \(Re_{11}x\).** A compilation of various experimental results (see Incropera and DeWitt [2] and Kutateladze [18]) for steam and common refrigerants have led to the commonly used estimate of \(Re_{11}x=30\) and a subsequent laminar-to-turbulent transition regime that is characterized by \(30<Re_{11}x<1800\). It is found that in the parameter set in Eq. (12), \(\rho_1\) and \(\mu_1\) are important parameters affecting the value of \(Re_{11}x\) because of their appearance in the definition of \(Re_{1}\) and in the nondimensionalization process itself (e.g., \(m=\omega_{1}l_{1}/\rho_{1}U\)). However, the remaining nondimensional parameters such as \(\rho_2/\rho_1, \mu_2/\mu_1, \) etc., are unimportant because, as shown later in Fig. 8,
vapor motion does not significantly influence the wave phenomena. In addition, Table 3 shows that effects of changes in $\Delta T$ are small on the values of $Re_{cl}$, but not on $x_c$.

Since, for common fluids, an increase in $\rho_1$ typically accompanies an increase in $\mu_1$, typical changes in the kinematic viscosity $\nu_1$ ($=\mu_1/\rho_1$) values for nonmetallic vapors are limited. For this reason, as shown in Table 4, the changes in $Re_{cl}$ with the changes in the fluid are not much and $Re_{cl}$ remains in the 27–35 range. This justifies the continued use of the estimate $Re_{cl}=30$. However, for uncommon or specially designed fluids, one may be able to change $\rho_1$ without significantly changing $\mu_1$ (Table 5) or change $\mu_1$ without significantly changing $\rho_1$ (see Table 6). Under these special conditions, one can expect a more significant departure from “30” in the estimated value of $Re_{cl}$ (e.g., in Table 6, one finds a case for which $Re_{cl}\approx 9$).

### 4.2.4 Comparison of Exact Nonlinear Stability Analyses Results Presented Here With the Known Linearized Stability Results

Linearized stability analyses and associated results by Unsal and Thomas [14] and Spindler [15] for the Nusselt solution are available in the literature. However, their results, though mutually compatible, are not compatible with the above-described experimental estimates of $Re_{cl}=30$. Subsequent attempts at nonlinear analysis for this problem (see, e.g., Unsal and Thomas [17]) indicate the partially correct view that the instabilities predicted by linear analyses are not going to work and the problem needs some sort of non-linear analysis.

To facilitate a comparison with the known results, we present here, for the first time, the nearly exact nonlinear stability analyses and associated results. It should be noted that the predictions of the time-dependent disturbances $\delta'(x,t)=\delta(x,t)-\delta_{steady}(x)$ in Figs. 3 and 4, associated with an initial nonzero disturbance $\delta'(x,0)$, are obtained by nearly exact nonlinear simulations (see Table 2 for representative satisfaction of all the interface conditions) and, therefore, these simulations are capable of providing good results from the point of view of both linear and nonlinear stability analyses. The initial disturbances in Figs. 3 and 4 are sinusoidal in nature and are given by the relation:

$$\delta'(x,0) = e \cos(\theta(x,0)) = e \cos\left(\frac{2\pi}{\lambda_0}(x-x^*)\right),$$  \hspace{1cm} (16)

over a suitable range of $x$ values with $e>0$ and $x=x^*$ being a location where the phase angle $\theta(x,0)$ corresponds to a positive peak and is defined to be zero.

The time evolution $\delta'(x,t)$ of the initial disturbance in Eq. (16) can be characterized by a sinusoidal Fourier component—associated with the disturbance in Eq. (16)—and is given as:

$$\delta'(x,t) = a(t) \cdot \cos(\theta(x,t))$$  \hspace{1cm} (17)

However the linear and nonlinear stability analyses of condensing flow in the literature [14–17] and other air-water free-surface flows [see 27] make the common, but restrictive, assumption that Eq. (17) can have a special simplified form, viz.:

$$\delta'(x,t) = Re\left[\epsilon \exp(\alpha x - C \cdot t)\right]$$  \hspace{1cm} (18)

where $\alpha = 2\pi/\lambda_0$ is a wave number (with $\lambda_0$ being the constant wavelength) independent of $x$. $\epsilon$ is the complex number evaluated by an $\epsilon=1$ and $C=C_r+iC_i$ is a complex number dependent only on wave-length $\lambda_0$ (or wave number $\alpha$). This means that the popular analyses restrict the amplitude and phase-angle variations in Eq. (17) to:

$$a(t) = \epsilon \exp(\alpha C \cdot t)$$

The restriction in Eq. (19) is particularly severe for free surface problems (such as this problem and other air/water and evaporating flow free-surface problems) because, as shown in Eq. (27) of Narain et al. [3], the equation governing $\delta'(x,t)$ is given by:

$$\frac{\partial \delta'}{\partial t} + \frac{\partial \delta'}{\partial x} = \bar{u}$$  \hspace{1cm} (20)

where $\bar{u}(x,t) = \bar{u}(x,t) - \bar{u}_{steady}(x) - \left[\bar{u} - \bar{u}_{steady}(x)\right] \frac{d \delta_{steady}}{dx}$, $\bar{u}(x,t) = v_1 + i\alpha Re_{cl}Pr_1 \left(\partial \theta_1/\partial \gamma\right)$, as per its definition in Eq. 27 of [3].

As a result of the above, it is easy to see that waves travel along a family of characteristics curves $x=x_c(t)$, where $x_c(t)$ satisfies Eq. (25) of [3]. That is:

$$\frac{dx_c}{dt} = \bar{u}[x_c(t),t]$$

$$x_c(0) = x^*$$

$$x_c(t') = 0$$  \hspace{1cm} (21)

where $x^*$ is any given value of $x^*>0$ and $t'$ is any given time $t>0$. For the no disturbance $[\bar{u}(x,t) = \bar{u}_{steady}]$ and large disturbance

<table>
<thead>
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<th>$\Delta T$</th>
<th>$x_c$</th>
<th>$Re_s$</th>
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<tr>
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<td>31.15</td>
</tr>
<tr>
<td>20</td>
<td>6.25</td>
<td>33.27</td>
</tr>
</tbody>
</table>

Table 3 Effects of changes in $\Delta T$

Note: All fluids considered here have $\rho_1/\rho_1=0.00525$, $\mu_1/\mu_1=0.0289$, $We=402.133$, and $Pr_1=7.223$.

<table>
<thead>
<tr>
<th>$\rho_1$</th>
<th>$\mu_1$</th>
<th>$\nu_1=\mu_1/\rho_1$</th>
<th>Fluid name for same $\nu_1$</th>
<th>$x_c$</th>
<th>$Re_s$</th>
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</thead>
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<td>R 12</td>
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<td>3.44168E−07</td>
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<td>19.6</td>
<td>27.72</td>
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</tbody>
</table>

Table 4 Effects of changes in viscosity ($\nu_1=\mu_1/\rho_1$)

Note: All fluids considered here have $\rho_1/\rho_1=0.00525$, $\mu_1/\mu_1=0.0289$, $We=402.133$, and $Pr_1=7.223$.

<table>
<thead>
<tr>
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<th>$\mu_1$</th>
<th>$\nu_1=\mu_1/\rho_1$</th>
<th>Fluid name for same $\nu_1$</th>
<th>$x_c$</th>
<th>$Re_s$</th>
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</tbody>
</table>

Table 5 Effects of changing $\rho_1$, without significantly changing $\mu_1$

Note: All fluids considered here have $\rho_1/\rho_1=0.00525$, $\mu_1/\mu_1=0.0289$, $We=402.133$, and $Pr_1=7.223$.

<table>
<thead>
<tr>
<th>$\rho_1$</th>
<th>$\mu_1$</th>
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cases shown in Fig. 6(a), representative plots of \( \bar{u}(x,t) \) are shown in Fig. 6(b). The linearized stability analyses assumption leading to Eq. (19) is \( \bar{u}(x,t)=C_1(\lambda_0) \), where \( \lambda_0 \) is independent of \( x \). The plots in Fig. 6(b) clearly indicate that \( \bar{u}(x,t) \neq C_1(\lambda_0) \) and hence this linearized stability assumption is not appropriate. Furthermore, this assumption implies that the characteristics governing the problem are a set of parallel straight lines \( \{ x_t=x_0(0)+C_1(\lambda_0) \} \). However, a fourth-order Runge-Kutta solution of Eq. (21) for \( \bar{u} = \bar{u}_{\text{steady}} \) or \( \bar{u}(x,t) \) leads to a set of characteristics whose slopes either change gradually with \( x \) (this is the case when \( \bar{u} = \bar{u}_{\text{steady}} \)) or have some superimposed oscillations on these gradually varying slopes [as shown for, \( \bar{u}=\bar{u}(x,t) \), in Fig. 6(b)]. The characteristics associated with \( \bar{u}=\bar{u}(x,t) \) in Fig. 6(b) are shown in Fig. 6(c). Clearly, the actual characteristics do not agree with the linearized stability assumption of them being a set of parallel straight lines.

In fact, with \( \bar{u}'(x,0) \) given by Eq. (16), it is easily seen that the characteristics speed \( \bar{u} \) in Eq. (21) is also the phase-speed and waves traveling along the characteristics have a constant phase-angle \( \theta(x,t) \), provided \( \theta(x,t) \) is given by:

\[
\theta(x,t) = \frac{2\pi}{\lambda_0} \left( x - x_0 - \int_0^t \bar{u}'[x_0(\tau), \tau] \, d\tau \right)
\]  

(22)

where \( x_0(\tau) \) is the characteristic \( 0 \leq \tau \leq t \) which, at time \( t \), passes through the point \( x \) and thus satisfies \( d\theta/dt = 0 \) along a characteristic \( x = x_0(t) \). Also, Eq. (22) is compatible with the correct phase angle associated with the initial \( (at t=0) \) disturbance in Eq. (16).

Therefore, all along \( x_0(\tau) \quad (0 \leq \tau \leq \omega) \), we have \( \theta(x,t) = \theta[x_0(\tau), \tau] = \theta[x_0(0),0] = 2\pi/\lambda_0[x_0(0)-x_0]=\text{constant} \).

Substitution of Eq. (22) in the definition of local wavelength \( \lambda(x,t) = 2\pi|\partial \theta / \partial x|^{-1} \) and local time-period \( T(x,t) = -2\pi|\partial \theta / \partial x|^{-1} \) (see Eq. 1.28 in [28]) imply nonconstant wavelengths (unlike the assumption in linear stability analyses) and frequencies (or time periods \( T \)) given by:

\[
\lambda(x,t) = \lambda_0 \left[ 1 - \int_0^T \left( \frac{\partial \theta}{\partial x} \right)_x \right]^{-1}
\]  

(23)

\[
T(x,t) = \lambda(x,t)/\bar{u}(x,t)
\]

Also, substitution of Eq. (17) [with \( \theta(x,t) \) given by Eq. (22)] in to the governing equations [Eqs. (20) and (21)] imply that, along the characteristics, the amplitude \( a(t) \) grows according to the equation:

\[
\frac{da}{dt} = \frac{1}{\cos[\theta[x_0(0),0]]} \phi \delta
\]  

(24)

where \( \phi = \bar{u}'[x_0(\tau), \tau] \) is the value of \( \bar{u}' \) in Eq. (20) along a characteristic curve. An integration of Eq. (24) along a characteristic yields:
Fig. 7 Stability boundaries as obtained by Unsal and Thomas [14] and Spindler [15], and this work

\[
\frac{a(t)}{a(0)} = 1 + \frac{1}{a(0)\cos(\theta|x(0),0|)} \cdot \int_{0}^{t} \tilde{b}(\tau) \cdot d\tau
\]  

(25)

For the flow cases considered in Figs. 6(a)–6(c), a plot of \( \frac{a(t)}{a(0)} \) for different values of initial disturbance wavelength \( \lambda_0 \) [see Eq. (16)], with \( x(0) = 6.65 \), is shown in Fig. 6(d).

From Fig. 6(d), it is clear that the long time growth of \( a(t) \) requires the full nonlinear analysis result in Eq. (25) and cannot be captured by the linearized stability assumption [14,15] of \( a(t) \) being given by the exponential function in Eq. (19).

For the above reasons, for the flow considered in Figs. 6(a)–6(d), the simulation-based approximate stability boundary depicted in Fig. 7 is far more trustworthy (it is compatible with the experimental estimate on \( \text{Re}_d = 30 \)) than the Unsal [14] and Spindler [15] results, based on the inappropriate assumptions in Eqs. (18) and (19).

It should be further noted that the nonlinear instability mechanisms illustrated by the full nonlinear solutions in Fig. 4—besides being in agreement with experiments regarding \( \text{Re}_d = 30 \)—also show two other regularities. These regularities are: (i) As shown in Fig. 8 and its caption, the vapor motion or its fluctuations do not play a significant role in the evolution of gravity dominated waves, and (ii) the results in Fig. 5 indicate negligible impact of surface tension and this is consistent with experimental results but not with the linearized stability theory’s result of a strong dependence of \( x_{cr} \) on the presence or absence of surface tension (see Eq. (32) in [14] that states \( x_{cr} \sim a^{1/2} \)).

4.2.5 Different Wave Mechanisms in Different Zones. As marked in Fig. 4, for \( x < x_{cr} \), the initial disturbances may persist but the zone is considered stable because the growth of the waves is considered small. However, at longer lengths in Fig. 6(d), there is a loss of stability, that is, amplitudes for \( \lambda_0 > \lambda_{cr} \) are eventually sufficiently large. The largeness of amplitude \( a(t) \) is defined here to mean that, for \( x > x_{cr} \), \( a(t)/a(0) \approx 2.5 \) at large \( t \) and, at the same time, the peaks of the resulting disturbances are off by more than 15% of the initial undisturbed film thickness values at the current locations of the waves (see Fig. 4). In the above definition of \( x_{cr} \), the large amplitude waves at \( x > x_{cr} \) (see Fig. 4) arise from initial disturbances at \( x \sim 0 \), where their amplitudes are sufficiently small (less than 10% of the small steady film thickness at \( x \sim 0 \)). Another feature of \( x_{cr} \) being defined this way is that for \( \lambda_0 < \lambda_{cr} \), the amplitude ratio \( a(t)/a(0) \) is eventually less than 1.0 [i.e., \( a(t)/a(0) < 1 \)] at large \( t \). Under the above definition, the longer length flows lose stability for \( \text{Re}_d = 30 \).

The wall noise is assumed to be a superposition of standing waves of the form given by:

\[
\nu_1(x,0,t) = \varepsilon_b \sin(2\pi x/\lambda_{cr}) \sin(2\pi \tau/T_b) \]

(26)

As shown in [3], one expects a resonance condition to hold if the time period of oscillations \( T_b \) in the wall noise of Eq. (26) is such that the forward travelling component of the standing noise has approximately the same phase speed as \( \overline{u}(x,t) \) (which is nearly equal to \( \overline{u}_{read} \) for small amplitude interfacial waves). This means, for resonance with large amplitude waves, \( T_b = T_b(x,t) \) must satisfy:

\[
\lambda_b \quad \text{or} \quad \overline{u}_{read}(x) \]

(27)

or \( T_b = T_b(x) \) must satisfy:

\[
\lambda_b \quad \text{or} \quad \overline{u}_{read}(x) \]

(28)

For nonresonant single frequency (\( T_b = \) constant) wall noise, as shown in Figs. 9(a) and 9(b), depending on the value of \( T_b \), one may have either a negligible (if \( T_b \) is nowhere in the resonant range) or a constructive (if \( T_b \) is somewhere in the resonant range) interference between wall noise and intrinsic waves. As shown in Fig. 10(a), for the unstable \( \lambda_0 = 15 > \lambda_{cr} \) case (where \( \lambda_{cr} \) is same as the one obtained from the earlier initial disturbance analysis), the resonant wall noise [in the sense of Eqs. (27) and (28)] interacts with the intrinsic waves and sustains a large amplitude travelling wave. Differently, but interestingly, as shown in Fig. 10(b), for the damped or stable \( \lambda_b = 5 < \lambda_{cr} \) case, the resonant bottom wall noise governed by Eq. (28) interacts with the intrinsic waves and sustains a travelling wave which “beats” in the sense that the noises are alternately damped over a period of time and then regained over a subsequent period of time.

The above shows that wave effects will, in general, be present. They may be intrinsic in the absence of wall noise (i.e., due to initial disturbances alone) or they may be interactive in the presence of wall noise. Furthermore, the extent of the wave effects on heat transfer rates and interfacial shear will, in general, remain nondeterministic unless, experimentally, one can use wall noise actuation to have a well-defined, enhanced, or diminished form of wave effects.

4.2.6 Impact on the Heat Transfer Rates and Shear Stress. Effect of waves on the wall heat transfer rate and wall shear stress, for the \( \lambda > \lambda_{cr} \) initial disturbance case of Fig. 4, is shown in Figs. 11(a) and 11(b). This confirms the well known [2] fact that, in the presence of gravity-assisted drainage, intrinsic waves enhance
both of these important parameters. The extent of the heat transfer enhancements is typically found to be within the range (i.e. 1.2 times the values associated with smooth-interface heat transfer rates) suggested by empirical correlations (see [2,18]). This enhancement can be actively increased or reduced with the help of resonant or nonresonant wall noises.

5 Conclusions

- The computational approach presented here accurately solves the steady Nusselt problem and produces results in agreement with the Nusselt solution.
- The results considered thus far affirm the experimental result that instability mechanisms associated with laminar to turbulence transitions can typically be estimated to occur around $Re_{Gr}=30$.
- The waves and the wave effects are quite sensitive to the presence or absence of the wall noise. This sensitivity to the frequency and wavelength spectrum of the wall noise can be exploited either to suppress or enhance the wave effects.
- As is well known for this problem, the results affirm that heat transfer rates and shear rates are significantly enhanced by the presence of waves.

Acknowledgment

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Nomenclature

$X_c$ = distance from leading edge (BC in Fig. 1), m
$Y_c$ = characteristics length (AB in Fig. 1), m

Greek Symbols

$\pi$ = nondimensional pressure
$\theta$ = nondimensional temperature
The wall heat flux $q_{lat}$ is given by

$$q_{lat} = \frac{\partial T}{\partial x}$$

Fig. 11 (a) For the base flow in Fig. 2(a), the figure above shows the wall shear stress for cases with and without initial disturbance $\delta \times, 0 = \delta_{steady}(x) \times [1 + \epsilon \delta \times (x, 0)]$, where $\delta \times (x, 0) = \sin(2\pi x/\lambda_o), \epsilon = 0.15, \lambda_o = 15, \text{and } \Delta t = 7.5$. (b) For the base flow in Fig. 2(a), the figure above shows the wall shear stress for cases with and without initial disturbance $\delta \times, 0 = \delta_{steady}(x) \times [1 + \epsilon \delta \times (x, 0)]$, where $\delta \times (x, 0) = \sin(2\pi x/\lambda_o), \epsilon = 0.15, \lambda_o = 15, \text{and } \Delta t = 7.5$.

$$\rho = \text{density, } \text{kg/m}^3$$
$$\mu = \text{viscosity, Pa s}$$
$$\Delta = \text{physical value of condensate thickness, m}$$
$$\Delta_N = \text{physical value of Nusselt film thickness at } x, \left[\left(k_2/\mu_2\right)^{1/4}/\left(h/\mu_2(\mu_2/\mu_1)g^{1/4}\right)\right], \text{m}$$
$$\delta = \text{nondimensional value of condensate thickness}$$
$$\nu = \text{kinematic viscosity } \mu/\rho, \text{m}^2/s$$
$$\sigma = \text{surface tension, N/m}$$
$$\epsilon = \text{amplitude of nondimensional disturbances representing values of } \delta \times (x, 0)$$
$$\epsilon_b = \text{amplitude of nondimensional bottom wall vibrations sensed through } v_1(x, 0, t)$$
$$\lambda = \text{nondimensional wavelength}$$
$$\lambda_o = \text{nondimensional wavelength for the initial disturbance } \delta \times (x, 0)$$
$$\alpha = \text{wave number, } 2\pi/\lambda$$

Subscripts

$l = \text{it takes a value of 1 for liquid phase and 2 for vapor phase}$
$s = \text{saturation condition}$
$w = \text{wall}$
$b = \text{bottom wall}$

Superscripts

$i = \text{value of a variable at an interface location}$

Appendix

The differential forms of mass, momentum (x and y components), and energy equations in terms of nondimensional variables for flows in the interior of either of the phases ($I = 1$ or 2) for this external flow are given as

$$\frac{\partial \theta_i}{\partial t} + u_{i1} \frac{\partial \theta_i}{\partial x} + u_{i2} \frac{\partial \theta_i}{\partial y} = \frac{1}{Re_i Pr_i} \left( \frac{\partial^2 \theta_i}{\partial x^2} + \frac{\partial^2 \theta_i}{\partial y^2} \right)$$

$$\frac{\partial \theta_i}{\partial t} + u_{i1} \frac{\partial \theta_i}{\partial x} + u_{i2} \frac{\partial \theta_i}{\partial y} = \frac{1}{Re_i Pr_i} \left( \frac{\partial^2 \theta_i}{\partial x^2} + \frac{\partial^2 \theta_i}{\partial y^2} \right)$$

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$$\frac{\partial \theta_i}{\partial t} + u_{i1} \frac{\partial \theta_i}{\partial x} + u_{i2} \frac{\partial \theta_i}{\partial y} = \frac{1}{Re_i Pr_i} \left( \frac{\partial^2 \theta_i}{\partial x^2} + \frac{\partial^2 \theta_i}{\partial y^2} \right)$$

$$\frac{\partial \theta_i}{\partial t} + u_{i1} \frac{\partial \theta_i}{\partial x} + u_{i2} \frac{\partial \theta_i}{\partial y} = \frac{1}{Re_i Pr_i} \left( \frac{\partial^2 \theta_i}{\partial x^2} + \frac{\partial^2 \theta_i}{\partial y^2} \right)$$

$$\frac{\partial \theta_i}{\partial t} + u_{i1} \frac{\partial \theta_i}{\partial x} + u_{i2} \frac{\partial \theta_i}{\partial y} = \frac{1}{Re_i Pr_i} \left( \frac{\partial^2 \theta_i}{\partial x^2} + \frac{\partial^2 \theta_i}{\partial y^2} \right)$$

$$\frac{\partial \theta_i}{\partial t} + u_{i1} \frac{\partial \theta_i}{\partial x} + u_{i2} \frac{\partial \theta_i}{\partial y} = \frac{1}{Re_i Pr_i} \left( \frac{\partial^2 \theta_i}{\partial x^2} + \frac{\partial^2 \theta_i}{\partial y^2} \right)$$

$$\frac{\partial \theta_i}{\partial t} + u_{i1} \frac{\partial \theta_i}{\partial x} + u_{i2} \frac{\partial \theta_i}{\partial y} = \frac{1}{Re_i Pr_i} \left( \frac{\partial^2 \theta_i}{\partial x^2} + \frac{\partial^2 \theta_i}{\partial y^2} \right)$$

where $Re_i = \mu_i U_{i1}/\nu_i, Pr_i = \mu_i C_p/k_i, Fr_{x1} = g_0 Y_{i1}/U_{i1}^2$, and $Fr_{y1} = g_0 Y_{i2}/U_{i2}^2$.

Under negligible inertia, negligible convection, and boundary layer (9/ax ≈ 9/ax) approximations for thin condensate, Nusselt formulation [1] effectively replaces, for $I = 1, (A2)-(A4)$ above by:

$$0 \equiv -\frac{\partial \theta_1}{\partial x} + Fr_{x1} - 1 + \frac{1}{Re_i Pr_i} \frac{\partial^2 \theta_1}{\partial y^2}$$

$$0 \equiv -\frac{\partial \theta_2}{\partial x} + Fr_{y1} - 1 + \frac{1}{Re_i Pr_i} \frac{\partial^2 \theta_2}{\partial y^2}$$

For $I = 2$, the additional Nusselt [1] approximation of negligible vapor viscosity and saturated vapor eliminates any consideration of vapor motion and vapor temperature variation for obtaining the steady liquid condensate solution. Therefore, one does not need to consider the vapor ($I = 2$) Eqs. (A1)-(A4) in order to obtain the Nusselt ([1]) solution.

The term $[r]$ on the right side of Eq. (4) is given by:

$$[r] = \left\{ \begin{array}{l}
\left( \frac{\mu_2}{\mu_1} \frac{\partial v_2}{\partial x} \right) - \left( \frac{\partial v_1}{\partial x} \right) \\
+ 2 \delta_1 \left( \frac{\partial \theta_1}{\partial x} \right) - \left( \frac{\partial \theta_1}{\partial x} \right)
\end{array} \right\}$$

References

Laminar Film Condensation,” ASME J. Heat Transfer, 81, pp. 13–18.