Chapter 6 Constraint Satisfaction Problems

CS5811 - Artificial Intelligence

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Outline

- CSP problem definition
- Backtracking search for CSPs
- Problem structure and problem decomposition
A *constraint satisfaction problem* consists of

- a finite set of *variables*, where each variable has a *domain*. Using a set of variables (features) to represent a domain is called a *factored representation*.
- a set of *constraints* that restrict variables or combinations of variables
CSP example: cryptarithmetic

\[
\begin{array}{c}
\text{T W O} \\
+ \text{T W O} \\
\hline
\text{F O U R}
\end{array}
\]

Variables: $F, T, U, W, R, O, X_1, X_2, X_3$
Domains: $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ (same domain for all)
Sample constraints:

- `alldif (F, T, U, W, R, O)`
- or a binary constraint for all, e.g., $F \neq T, F \neq U$.
- A unary constraint: $F \neq 0$
- An n-ary constraint: $O + O = R + 10 \times X_1$
- Can add constraints to restrict the $X_i$'s to 0 or 1.
A *solution* is an assignment to all the variables from their domains so that all the constraints are satisfied. For any CSP, there might be a single solution, multiple solutions, or no solutions at all.
Real-world CSPs

- Assignment problems
  e.g., who teaches what class
- Timetabling problems
  e.g., which class is offered when and where?
- Hardware configuration
- Spreadsheets
- Transportation scheduling
- Factory scheduling
- Floorplanning

Notice that many real-world problems involve real-valued variables
CSPs with discrete variables

- Finite domains
  \( O(d^n) \) complete assignments are possible for \( n \) variables and domain size \( d \)
  e.g., Boolean CSPs, Boolean SATisfiability are \textit{NP-complete}

- Infinite domains (integers, strings, etc.)
  e.g., job scheduling
  variables are start/end days for each job
  \( \text{StartJob}_1 + 5 \leq \text{StartJob}_3 \)
  \textit{linear constraints} are solvable,
  \textit{nonlinear constraints} are \textit{undecidable}
CSPs with continuous variables

- linear constraints solvable in polynomial time by *linear programming (LP)* methods
- e.g., precise start/end times for Hubble Telescope observations with astronomical, precedence, and power constraints
Representing CPSs as canonical search problems

- Standard search problem:
  A \textit{state} is a “black box”, i.e., any old data structure that supports goal test, actions, result, etc.

- CSP:
  - A \textit{state} is defined by \textit{variables} $X_i$ with \textit{values} from \textit{domains} $D_i$
    - e.g., assigned: $\{F = 1\}$,
    - unassigned $\{T, U, W, R, O, X_1, X_2, X_3\}$
  - The \textit{goal test} is that
    - all the variables are assigned
    - all the constraints are satisfied

- Simple example of a \textit{formal representation language}

- Allows useful \textit{general-purpose algorithms} with more power than standard search algorithms:
  Can develop domain-independent heuristics
Working example: map-coloring

Variables: $WA, NT, Q, NSW, V, SA, T$

Domains: $D_i = \{ \text{red}, \text{green}, \text{blue} \}$

Constraints: adjacent regions must have different colors

e.g., $WA \neq NT$ (if the language allows this), or

$(WA, NT) \in \{(\text{red}, \text{green}), (\text{red}, \text{blue}), (\text{green}, \text{red}), (\text{green}, \text{blue}), \ldots \}$
This solution satisfies all the constraints.

\[
\{ \text{WA} = \text{red}, \quad \text{NT} = \text{green}, \quad \text{Q} = \text{red}, \quad \text{NSW} = \text{green}, \\
\text{V} = \text{red}, \quad \text{SA} = \text{blue}, \quad \text{T} = \text{green} \}
\]
In a binary CSP, each constraint relates at most two variables
A binary CSP can be represented as a constraint graph
In the graph, the nodes are variables, the arcs show constraints
General-purpose CSP algorithms use the graph structure to speed up search.
E.g., Tasmania is an independent subproblem
Working with the standard search process

Start with the straightforward approach, then fix it
States are defined by the values assigned so far

Initial state: the empty assignment, $\emptyset$
Actions: Pick an unassigned variable, assign a value that does not conflict with the current assignments
If no assignment is possible, the path is a dead end
Goal test: all the variables have assignments
For a problem with \( n \) variables, every solution appears at depth \( n \)

Depth-first search is a good choice

A node that satisfies the goal test has the complete solution the path is not needed

However, the branching factor is unnecessarily large \((b = (n - 1)d \text{ at depth } l)\)

The search tree gets lots of redundant paths that represent the same solution but the order of assignment is different: \( n!d^n \) leaves are produced
Variable assignments are *commutative*, i.e.,
\[ WA = \text{red} \text{ then } NT = \text{green} \text{ is the same as } NT = \text{green} \text{ then } WA = \text{red} \]

We only need to consider assignments to a single variable at each level
\[ b = d \text{ and there are } d^n \text{ leaves} \]

Depth-first search for CSPs with single-variable assignments is called *backtracking search*

Backtracking search is the basic uninformed algorithm for CSPs

Can solve \( n \)-queens for \( n \approx 25 \)
function Backtracking-Search (csp)
returns a solution, or failure
    return Backtrack({ }, csp)
function **Backtrack** (*assignment*, *csp*)
returns a solution, or failure

if *assignment* is complete then return *assignment*

var ← **Select-Unassigned-Var**(*csp*)

for each *value* in **Order-Domain-Values**(var, *assignment*, *csp*) do

  if *value* is consistent with *assignment* then

    add { var = *value* } to *assignment*

    inferences ← **Inference**(*csp*, var, *value*)

    if inferences ≠ failure then

      add inferences to *assignment*

      result ← **Backtrack** (*assignment*, *csp*)

      if result ≠ failure then return result

    remove { var = *value* } and inferences from *assignment*

return failure
Backtracking example
Backtracking example
Backtracking example
Backtracking example
Improving backtracking efficiency

General-purpose methods can give huge gains in speed:

1. Which variable should be assigned next?
2. In what order should its values be tried?
3. Can we detect inevitable failure early?
4. Can we take advantage of problem structure?
Most constrained variable strategy

*Most constrained variable:* choose the variable with the fewest legal values
Most constraining variable strategy

Tie-breaker among most constrained variables

*Most constraining variable:*
choose the variable with the most constraints on the remaining variables
Least constraining value strategy

Given a variable, choose the *least constraining value*: the one that rules out the fewest values in the remaining variables.

Combining these heuristics makes 1000 queens feasible.
Forward checking

Idea: Keep track of remaining legal values for unassigned variables
Terminate search when any variable has no legal values
Forward checking

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Idea: Keep track of remaining legal values for unassigned variables
Terminate search when any variable has no legal values

WA  NT  Q  NSW  V  SA  T
WA  WA  WA
NT  NT  NT  NT
Q  Q  Q
NSW  NSW  NSW
V  V  V
SA  SA  SA
Forward checking

**Idea:** Keep track of remaining legal values for unassigned variables
Terminate search when any variable has no legal values

<table>
<thead>
<tr>
<th>WA</th>
<th>NT</th>
<th>Q</th>
<th>NSW</th>
<th>V</th>
<th>SA</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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</tbody>
</table>
| ![Map](image.png)

![Map](image.png)
Constraint propagation

Forward checking propagates information from assigned to unassigned variables, but doesn’t provide early detection for all failures:

\[ \text{WA NT Q NSW V SA T} \]

\[ \text{WA WA WA} \]
\[ \text{NT NT NT NT} \]
\[ \text{SA SA SA} \]
\[ \text{V V V} \]
\[ \text{NSW NSW NSW} \]
\[ \text{Q Q Q} \]

\[ \text{NT and SA cannot both be blue!} \]

*Constraint propagation* repeatedly enforces constraints locally
Simplest form of propagation makes each arc **consistent**

$X \rightarrow Y$ is **consistent** iff

for *every* value $x$ of $X$ there is *some* allowed $y$ from $Y$
Simplest form of propagation makes each arc **consistent**

\[ X \rightarrow Y \] is **consistent** iff for **every** value \( x \) of \( X \) there is some allowed \( y \) from \( Y \)
Simplest form of propagation makes each arc **consistent**

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If $X$ loses a value, neighbors of $X$ need to be rechecked
Simplest form of propagation makes each arc **consistent**

$X \rightarrow Y$ is **consistent** iff

for **every** value $x$ of $X$ there is **some** allowed $y$ from $Y$

If $X$ loses a value, neighbors of $X$ need to be rechecked
Arc consistency detects failure earlier than forward checking
Can be run as a preprocessor or after each assignment
Arc consistency algorithm

function AC-3 (csp)
returns false if an inconsistency is found and true otherwise
inputs: csp, a binary CSP with components (X, D, C)
local variables: queue, a queue of arcs, initially all the arcs in csp

while queue is not empty do
    (X_i, X_j) ← REMOVE-FIRST(queue)
    if REVISE(csp, X_i, X_j) then
        if size of D_i = 0 then return false
        for each X_k in X_i. NEIGHBORS-{X_j} do
            add (X_k, X_i) to queue
    return true
function \textsc{Revise} (csp, X_i, X_j)
returns true iff we revise the domain of X_i

\[
\text{revised} \leftarrow \text{false}
\text{for each } x \text{ in } D_i \text{ do}
\text{if no value } y \text{ in } D_j \text{ allows } (x, y) \text{ to satisfy the}
\text{constraint between } X_i \text{ and } X_j
\text{then delete } x \text{ from } D_i
\text{revised} \leftarrow \text{true}
\text{return revised}
\]

\[O(n^2d^3), \text{ can be reduced to } O(n^2d^2)\]
But cannot detect all failures in polynomial time
Problem structure

Tasmania and mainland are *independent subproblems*.
Identifiable as *connected components* of constraint graph.
Suppose each subproblem has $c$ variables out of $n$ total. Worst-case solution cost is $n/c \cdot d^c$, linear in $n$.

E.g., $n = 80$, $d = 2$, $c = 20$

$2^{80} = 4$ billion years at 10 million nodes/sec

$4 \times 2^{20} = 0.4$ seconds at 10 million nodes/sec
Theorem: if the constraint graph has no loops, the CSP can be solved in $O(n d^2)$ time.

Compare to general CSPs, where worst-case time is $O(d^n)$.

This property also applies to logical and probabilistic reasoning: an important example of the relation between syntactic restrictions and the complexity of reasoning.
Algorithm for tree-structured CSPs

1. Choose a variable as root, order variables from root to leaves such that every node’s parent precedes it in the ordering

2. For \( j \) from \( n \) down to 2, apply \text{Make-Arc-Consistent}(\text{Parent}(X_j), X_j) \) (will remove inconsistent values)

3. For \( i \) from 1 to \( n \), assign \( X_i \) consistently with \( \text{Parent}(X_i) \)
function Tree-CSP-Solver (csp)
returns a solution, or failure
inputs: csp, a binary CSP with components \((X, D, C)\)

\(n \leftarrow \text{number of variables in } X\)
assignment \(\leftarrow \text{an empty assignment}\)
root \(\leftarrow \text{any variable in } X\)
\(X \leftarrow \text{TopologicalSort}(X, \text{root})\)
for \(j = n \text{ down to } 2\) do
    Make-Arc-Consistent(\(\text{Parent}(X_j), X_j\))
    if it cannot be made consistent then return failure
for \(i = 1 \text{ to } n\) do
    assignment \([X_i]\) \(\leftarrow \text{any consistent value from } D_i\)
    if there is no consistent value then return failure
return assignment
Nearly tree-structured CSPs

*Conditioning*: instantiate a variable, prune its neighbors’ domains
**Cutset conditioning**: instantiate (in all ways) a set of variables such that the remaining constraint graph is a tree.

Cutset size $c \quad \longrightarrow \quad$ runtime $O(d^c \cdot (n - c)d^2)$, very fast for small $c$.
CSPs are a special kind of problem: states defined by values of a fixed set of variables goal test defined by constraints on variable values

Backtracking = depth-first search with one variable assigned per node

Variable ordering and value selection heuristics help significantly

Forward checking prevents assignments that guarantee later failure
Constraint propagation (e.g., arc consistency) does additional work to constrain values and detect inconsistencies.

The CSP representation allows analysis of problem structure.

Tree-structured CSPs can be solved in linear time.

(Iterative min-conflicts is usually effective in practice.)
Sources for the slides

- AIMA textbook (3rd edition)
- AIMA slides (http://aima.cs.berkeley.edu/)