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INTEGRAL DEFERRED CORRECTION METHODS CONSTRUCTED WITH HIGH ORDER RUNGE–KUTTA INTEGRATORS.

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Abstract. Spectral deferred correction (SDC) methods for solving ordinary differential equations (ODEs) were introduced by Dutt, Greengard and Rokhlin [4]. It was shown in [4] that SDC methods can achieve arbitrary high order accuracy and possess nice stability properties. Their SDC methods are constructed with low order integrators, such as forward Euler or backward Euler, and are able to handle stiff and non-stiff terms in the ODEs. In this paper, we use high order Runge-Kutta (RK) integrators to construct a family of related methods, which we refer to as integral deferred correction (IDC) methods. The distribution of quadrature nodes is assumed to be uniform and the corresponding local error analysis is given. The smoothness of the error vector associated with an IDC method, measured by the discrete Sobolev norm [7, 17], is a crucial tool in our analysis. The expected order of accuracy is demonstrated through several numerical examples. Superior numerical stability and accuracy regions are observed when high order RK integrators are used to construct IDC methods.

1. Introduction

In this paper, we consider integral deferred correction methods (IDC) [4] for solving initial value problems (IVP) consisting of ordinary differential equations (ODEs). Note that in the literature, this family of methods is referred to as spectral deferred correction (SDC) methods if Gaussian quadrature nodes are used to compute a correction to the defect or error. Compared to traditional multi-step methods, e.g., Adams-Bashforth methods (AB) and multi-stage methods, e.g., Runge-Kutta (RK) methods, IDC methods are able to achieve arbitrary high order accuracy without tedious algebraic computations, and are able to handle stiff and non-stiff terms in the ODEs. IDC methods also maintain reasonably large stability regions as the order of the schemes increase.

As discussed in [4], there are various decisions to be made when constructing IDC methods; much work has been done in the literature to explore these choices. The selection of quadrature nodes is discussed in [11], while [15] uses semi-implicit schemes to handle temporal multi-scale problems. The authors in [13, 12] also study the choice of predictors and correctors to construct semi-implicit SDC methods. In [8, 9], Krylov subspace methods are used to accelerate the convergence

of SDC methods. In [7, 6], the smoothness of the error vector associated with an SDC method is introduced, and is used to discuss the convergence of SDC schemes. Their analysis and concept of smoothness of the error vector motivated this paper. Theoretical convergence results for SDC method constructed using low order integrators are discussed in various papers [5, 1, 18]. Application of SDC methods to PDEs, through the method of lines approach, can be found in [16, 10, 2, 18, 14].

The main focus of this paper is to study IDC methods constructed using high order RK integrators. Specifically, we will prove under mild conditions, that using an rth order RK integrator to solve the error equation in a correction loop, increases the order of accuracy of an IDC method by r orders. The smoothness of the error vector associated with an IDC method, measured by a discrete Sobolev norm (introduced and used in [7, 17]), is a crucial concept in the local error analysis. In contrast to the SDC methods in [4], we assume that the quadrature nodes are uniformly distributed. For the case of a non-uniform distribution of quadrature nodes, including the Gaussian quadrature nodes discussed in [4], preliminary numerical results indicate that a corresponding relation for the order increase does not hold. We address these issues in [3].

The main part of the paper is organized into six sections. §2 is a review of IDC methods described in [4], while §3 introduces the concept of discrete smoothness, measured by a discrete Sobolev norm. §4 gives an analysis of the local error of IDC methods constructed with forward Euler integrators; this section sets a framework for the analysis of IDC methods constructed using high order RK methods in §5. In §6, a numerical example is provided to support the theoretical results in §4 and §5. Superior stability and accuracy properties of IDC methods constructed with high order RK integrators are also demonstrated. Concluding remarks are given §7.

2. Review of IDC methods

This section is a review of IDC methods from [4]. Our discussion on these methods is based on notations introduced below. We consider an IVP consisting of a system of ODEs and initial conditions,

$$(2.1) \qquad \begin{cases} y'(t) = f(t, y), & t \in [0, T], \\ y(0) = y_0. \end{cases}$$

The time domain, $[0, T]$, is discretized into intervals,

$$0 = t_1 < t_2 < \cdots < t_n < \cdots < t_N = T,$$

and each interval, $I_n = [t_n, t_{n+1}]$, is further discretized into sub-intervals,

$$(2.2) \qquad t_n = t_{n,0} = t_{n,1} < \cdots < t_{n,m} < \cdots < t_{n,M} = t_{n+1}.$$

The IDC method on each time interval $[t_n, t_{n+1}]$ is described below. We drop the subscript n, e.g., $t_0 := t_{n,0}$ in (2.2), with the understanding that the IDC method is described for one time interval. We also refer to $t_m := t_{n,m}$ as grid points or quadrature nodes, whose index m runs from 0 to M, and denote the sub-interval sizes as $h_m = t_m - t_{m-1}$, $m = 1, \ldots, M$.

- (prediction step) Use an $(r_0)^{th}$ order numerical method to obtain a numerical solution, $\vec{\eta}^{[0]} = (\eta_0^{[0]}, \eta_1^{[0]}, \ldots, \eta_m^{[0]}, \ldots, \eta_M^{[0]})$, which is an r_0^{th} order approximation to $\vec{y} = (y_0, y_1, \ldots, y_m, \ldots, y_M)$, where $y_m = y(t_m)$ is the

exact solution at t_m. For example, applying a first order forward Euler method to (2.1) gives $\eta^{[0]}_{m+1} = \eta^{[0]}_m + h_{m+1}f(t, \eta^{[0]}_m)$, $m = 0, \ldots, M-1$.

- (correction loop) Use the error function to improve the accuracy of the scheme at each iteration.

  For $k = 1, \ldots, k_l$ (k_l is number of correction steps)

  (1) Denote the *error function* from the previous step as

$$(2.3) \qquad e^{(k-1)}(t) = y(t) - \eta^{(k-1)}(t),$$

  where $y(t)$ is the exact solution and $\eta^{(k-1)}(t)$ is an Mth degree polynomial interpolating $\vec{\eta}^{[k-1]}$. Note that the error function, $e^{(k-1)}(t)$, is not a polynomial in general.

  (2) Compute the *residual function*, $\epsilon^{(k-1)}(t) = (\eta^{(k-1)})'(t) - f(t, \eta^{(k-1)}(t))$. In the literature, the residual function is often called the pointwise, or differential defect.

  (3) Compute the *numerical error vector*, $\vec{\delta}^{[k]} = (\delta^{[k]}_0, \ldots, \delta^{[k]}_m, \ldots, \delta^{[k]}_M)$, using an $(r_k)^{th}$ order numerical method to discretize the integral form of the *error equation*,

$$(2.4a) \qquad \left(e^{(k-1)} + \int_0^t \epsilon^{(k-1)}(\tau)\,d\tau\right)'(t) = f(t, \eta^{(k-1)}(t) + e^{(k-1)}(t)) - f(t, \eta^{(k-1)}(t))$$

$$(2.4b) \qquad \qquad \qquad \doteq F(t, e^{(k-1)}(t)),$$

  where $F(t, e(t)) = f(t, \eta(t) + e(t)) - f(t, \eta(t))$. $\vec{\delta}^{[k]}$ is an $(r_k)^{th}$ order approximation to $\vec{e}^{[k-1]} = (e^{[k-1]}_0, \ldots, e^{[k-1]}_m, \ldots, e^{[k-1]}_M)$, and $e^{[k-1]}_m = e^{(k-1)}(t_m)$ is the value of the exact error function at t_m. For example, applying a first order forward Euler method to (2.4) gives,

$$(2.5)$$

$$\delta^{[k]}_{m+1} = \delta^{[k]}_m + h_{m+1}(f(t_m, \eta^{[k-1]}_m + \delta^{[k]}_m) - f(t_m, \eta^{[k-1]}_m)) - \int_{t_m}^{t_{m+1}} \epsilon^{(k-1)}(t)\,dt$$

$$\approx \delta^{[k]}_m + h_{m+1}(f(t_m, \eta^{[k-1]}_m + \delta^{[k]}_m) - f(t_m, \eta^{[k-1]}_m)) - \sum_{j=0}^{M} \alpha_{m,j} f(t_j, \eta^{[k-1]}_j),$$

$$m = 0, \ldots, M-1,$$

  where we have approximated the integral by interpolatory quadrature formulas, as in [4].

  (4) Update the numerical solution $\vec{\eta}^{[k]} = \vec{\eta}^{[k-1]} + \vec{\delta}^{[k]}$.

Notationally, superscripts with a round bracket, e.g., (k), denote a function, while superscripts with a square bracket, e.g., $[k]$, denote a vector at the kth correction step. English letters are reserved for functions or vectors in the exact solution space, e.g., an exact solution $y(t)$ and an exact error function $e(t)$, while Greek letters denote functions or vectors in the numerical solution space, e.g., a numerical solution $\eta(t)$, and a numerical error function $\delta(t)$.

Remark 2.1. There are various construction decisions for generating an IDC method. For example, the distribution of grid points (2.2), or the choice of different integrators. In this paper, we would like to make the statements "$(r_0)^{th}$ *order approximation*" and "$(r_k)^{th}$ *order approximation*" in our description of IDC methods

mathematically rigorous. We will focus our discussion on a uniform distribution of quadrature nodes. The non-uniform case, including Gaussian quadrature nodes, will be addressed in [3].

Remark 2.2. Deferred correction methods described in reference [4] and cited therein differ from the spectral deferred correction in that a differential form of the error equation is formulated,

$$(2.6) \qquad (e^{(k-1)})'(t) = f(t, \eta^{(k-1)}(t) + e^{(k-1)}(t)) - \frac{\partial}{\partial t}\eta^{(k-1)}(t),$$

instead of (2.4). In this paper, we focus on IDC methods and briefly address the corresponding results in deferred correction methods.

3. Mathematical Preliminaries

Definition 3.1. (smoothness of a function): A function $f(t)$, $t \in [0, T]$ possesses S degrees of smoothness, if $\|d^s f\|_\infty := \left\|\frac{\partial^s}{\partial t^s} f\right\|_\infty$ is bounded, for $s = 0, 1, 2, ..., S$, where $\|f\|_\infty \doteq \max_{t \in [0,T]} |f(t)|$.

We would like to establish the concept of smoothness for a discrete data set, analog to that of a function. Consider the discrete data set,

$$(3.1) \qquad (t, f) = \{(t_0, f_0), ..., (t_M, f_M)\},$$

where $f_m = f(t_m)$ and $t_m = mh$, $m = 0, \ldots, M$ are equi-spaced points. The smoothness of a discrete data set is established in the limiting process of $h \to 0$.

Definition 3.2. (discrete differentiation): Given a discrete data set, (t, f), defined in (3.1), denote L^M as the usual Lagrange interpolant, an Mth degree polynomial that interpolates (t, f),

$$(3.2) \qquad L^M(t, f) = \sum_{m=0}^{M} c_m(t) f_m, \quad \text{where} \quad c_m(t) = \prod_{n \neq m} \frac{t - t_n}{t_m - t_n}.$$

An sth degree discrete differentiation is a linear mapping that maps $\vec{f} = (f_0, f_1, ..., f_M)$ into $\overrightarrow{\hat{d}_s f}$, where $(\hat{d}_s f)_m = \frac{\partial^s}{\partial t^s} L^M(t, f)|_{t=t_m}$. This linear mapping can be represented by a matrix multiplication $\overrightarrow{\hat{d}_s f} = D_s \cdot \vec{f}$, where $D_s \in \mathcal{R}^{(M+1) \times (M+1)}$, and $(D_s)_{mn} = \frac{\partial^s}{\partial t^s} c_n(t)|_{t=t_m}$, $m, n = 0, ..., M$.

Definition 3.3. The $(\hat{S}, \infty)$ Sobolev norm of the discrete data set (t, f), is defined to be

$$\left\|\vec{f}\right\|_{\hat{S},\infty} = \sum_{s=0}^{S} \left\|\overrightarrow{\hat{d}_s f}\right\|_\infty = \sum_{s=0}^{S} \left\|D_s \cdot \vec{f}\right\|_\infty,$$

where $\overrightarrow{\hat{d}_0 f} = Id \cdot \vec{f}$ is the identity matrix operating on $\vec{f}$.

Definition 3.4. (smoothness of a discrete data set): A discrete data set (3.1), possesses S ($S \leq M$) degrees of smoothness, if $\|\vec{f}\|_{\hat{S},\infty}$ is bounded as $h \to 0$.

The smoothness of a discrete data set can also be measured by divided difference approximations to the derivative of the discrete data set.

Definition 3.5. (divided differences): Given a discrete data set (t, f), (3.1), a divided difference of the discrete data set is a linear mapping that maps $\vec{f} = (f_0, f_1, ..., f_M)$ into $\overrightarrow{d_1 f}$, with $size(\overrightarrow{d_1 f}) = M$ and $(d_1 f)_m = \frac{f_{m+1} - f_m}{t_{m+1} - t_m}$ for $m = 0, ..., M - 1$. An sth degree divided difference is a linear mapping that maps $\vec{f}$ into $\overrightarrow{d_s f}$, with $size(\overrightarrow{d_s f}) = M - s + 1$ and $(d_s f)_m = \frac{(d_{s-1} f)_{m+1} - (d_{s-1} f)_m}{t_{m+1} - t_m}$, for $m = 0, ..., M - s$.

Definition 3.6. The (S, ∞) Sobolev norm of (t, f) is defined to be

$$\left\| \vec{f} \right\|_{S, \infty} = \sum_{s=0}^{S} \left\| \overrightarrow{d_s f} \right\|_{\infty},$$

where $\overrightarrow{d_0 f} = Id \cdot f$.

Proposition 3.7. *The $(\hat{S}, \infty)$ Sobolev norm defined in Definition 3.3 is equivalent to the (S, ∞) Sobolev norm defined in Definition 3.6.*

Proof. We will prove the equivalence of the $(\hat{1}, \infty)$ and $(1, \infty)$ norm; the equivalence of the general $(\hat{S}, \infty)$ and (S, ∞) norm can be proved in a similar fashion. For any discrete data set (3.1), let $L^M(t, f)$ be the corresponding Lagrange interpolant (3.2). By Definition 3.2, $(\hat{d}_1 f)_m = \frac{\partial}{\partial t} L^M(t, f)|_{t=t_m}$. $(d_1 f)_m$ is a first order approximation to $\frac{\partial}{\partial t} L^M(t, f)|_{t=t_m}$, i.e.,

$$(3.3) \qquad (d_1 f)_m = \frac{\partial}{\partial t} L^M(t, f)|_{t=t_m} + \mathcal{O}(h) = (\hat{d}_1 f)_m + \mathcal{O}(h).$$

From (3.3), if $(\hat{1}, \infty)$ norm is bounded and independent of H, then so is the $(1, \infty)$ norm, and vice versa. This proves the equivalence of the two norms. $\qquad \square$

Utilizing Proposition 3.7, the smoothness of a discrete data set can also be measured by divided differences of the discrete data set.

Definition 3.8. (smoothness of a discrete data set) A discrete data set (3.1), possesses S ($S \leq M$) degrees smoothness, if $\left\| \vec{f} \right\|_{S, \infty}$ is bounded, as $h \to 0$.

Remark 3.9. We require $S \leq M$ in Definition 3.4 and Definition 3.8, because $\overrightarrow{\hat{d}_{M+1} f} = \vec{0}$ and $size(\overrightarrow{d_{M+1} f}) = 0$, for any discrete data set.

Example 3.10. (a not so smooth discrete data set): The discrete data set,

$$(t, f) = \{(0, 0), (h, h), (2h, 2h), (3h, h), (4h, 0)\},$$

with $H = 4h$ has only one degree of smoothness in the discrete sense, according to either Definition 3.4 or 3.8.

We will use the definitions of smoothness, Definition 3.4 and Definition 3.8 interchangeably as convenient. Propositions 3.11 through 3.15 describe basic properties associated with the smoothness of discrete data sets. We omit the proofs for brevity.

Proposition 3.11. *If a discrete data set (t, f), as defined in (3.1), has S degrees of smoothness, then $(t, h^p f) = \{(t_m, h^p \cdot f_m)\}_{m=0}^{M}$ has $\min(S + p, M)$ degrees of smoothness.*

Proposition 3.12. *If discrete data sets (t, f) and (t, g) have S_f and S_g degrees of smoothness respectively, then $(t, f + g) = \{(t_m, f_m + g_m)\}_{m=0}^M$ and $(t, f \cdot g) = \{(t_m, f_m \cdot g_m)\}_{m=0}^M$ have $\min(S_f, S_g)$ degrees of smoothness.*

Proposition 3.13. *If a function $f(t)$, $t \in [0, H]$, has S degrees of smoothness in the continuous sense, then the discrete data set $(t, f) = \{(t_m, f(t_m))\}_{m=0}^M$ has $\min(S, M)$ degrees of smoothness in the discrete sense.*

Proposition 3.14. *If a function $f(t)$, $t \in [0, H]$, has S degrees of smoothness in the continuous sense, then the discrete data set $(t, H^p \cdot \frac{d^q f}{dt^q}) = \{(t_m, H^p \frac{d^q f}{dt^q}(t_m))\}_{m=0}^M$ has $\min(S - q + p, M)$ degrees of smoothness in the discrete sense with $q \leq S$. This is a direct consequence from Proposition 3.13 and Proposition 3.11.*

Proposition 3.15. *If a function $f(t)$ has S_f degrees of smoothness in the continuous sense and the discrete data set $(t, g) = \{t_m, g_m\}_{m=0}^M$ has S_g degrees of smoothness in the discrete sense, then the discrete data set $(t, f \circ g) = \{t_m, f(g_m)\}_{m=0}^M$ has $\min(S_f, S_g)$ degrees of smoothness in the discrete sense.*

Finally, we establish the relationship between the smoothness of the error vector associated with an IDC method, measured by properties of its discrete derivative or its divided difference, and the derivative of the corresponding error function.

Proposition 3.16. *Let $y(t)$ in IVP (2.1) have $S \geq M + 2$ degrees of smoothness in the continuous sense, and let η, be the numerical solution computed using an IDC method constructed with $(M+1)$ equispaced quadrature nodes in each sub interval. If the error vector associated with an IDC method, $\vec{e} = \vec{y} - \vec{\eta}$, satisfies $\|\vec{e}\|_\infty \sim \mathcal{O}(h^{r+1})$ with $r \leq M$, and the corresponding error function from interpolation errors, $e(t) = y(t) - L^M(t, \eta)$, satisfies $e_i = e(t = t_i)$, (note that $e(t)$ is not a polynomial in general), then*

- *$e(t) \sim \mathcal{O}(h^{r+1})$, for $t \in [0, H]$.*
- *Let $\tilde{\vec{e}} = \frac{1}{h^r} \vec{e}$ and $\tilde{e}(t) = \frac{1}{h^r} e(t)$. If $\tilde{\vec{e}}$ has $S_{\tilde{e}} \leq M + 1 - r$ degrees of smoothness in the discrete sense, then*

$$(3.4) \qquad \frac{d^i}{dt^i} \tilde{e}(t) \sim \begin{cases} \mathcal{O}(1), & \text{if } 0 \leq i \leq S_{\tilde{e}} \\ \mathcal{O}(\frac{1}{h^{i - S_{\tilde{e}}}}), & \text{if } S_{\tilde{e}} \leq i \leq M + 1 - r, \end{cases} \qquad \forall t \in [0, H].$$

- *The vector $(\frac{d^i \tilde{e}(t)}{dt^i}|_{t=t_0}, \dots, \frac{d^i \tilde{e}(t)}{dt^i}|_{t=t_M})$ has $S_{\tilde{e}} - i$ degrees of smoothness in the discrete sense for $i \leq S_{\tilde{e}}$.*

The proof of this proposition can be inferred from well known properties of the derivatives in Lagrange interpolation. Note that these error estimates for polynomial interpolation do not require that the grid be equispaced.

4. LOCAL ERROR OF IDC METHODS CONSTRUCTED WITH FORWARD EULER INTEGRATORS

In this section, we provide local error estimates for IDC methods constructed using forward Euler integrators. Our approach differs from the convergence and local error estimates discussed in [8, 18] respectively. This section introduces the framework which allows for the analysis of IDC methods constructed using high order RK methods, discussed in Section 5. In the following theorems and lemmas, we will assume that function $f(t, y)$ in IVP (2.1) has $S - 1$ degrees of smoothness if the solution $y(t)$ has S degrees of smoothness.

Theorem 4.1. *Let $y(t)$, the solution to the IVP (2.1), have at least $S \geq M + 2$ degrees of smoothness in the continuous sense. Then, the local error for an IDC method constructed using $(M + 1)$ uniformly distributed nodes, $(t_m = mh, m = 0, \ldots, M)$, and forward Euler integrators for the prediction and $k_l \leq M$ correction loops, is $\mathcal{O}(h^{k_l+2})$.*

Proof. The proof of Theorem 4.1 follows from two lemmas discussed in depth below. We will show in Lemma 4.2 that the theorem holds for $k = 0$, and in Lemma 4.3 that an inductive argument is satisfied. Note that this theorem is also implied by discussions in [5]. $\qquad\square$

Lemma 4.2. *(prediction step): Consider an IDC method constructed using $(M+1)$ uniformly distributed nodes, and a forward Euler integrator for the prediction step. Let $y(t)$, the solution to the IVP (2.1), have at least $S \geq M + 2$ degrees of smoothness, and let $\vec{\eta}^{[0]} = (\eta_0^{[0]}, \ldots, \eta_m^{[0]}, \ldots, \eta_M^{[0]})$, be the numerical solution computed after the prediction step. Then the error vector, $\vec{e}^{[0]} = \vec{y} - \vec{\eta}^{[0]}$, satisfies $\|\vec{e}^{[0]}\|_\infty \sim \mathcal{O}(h^2)$, and the rescaled error vector, $\vec{\tilde{e}}^{[0]} = \frac{1}{h}\vec{e}^{[0]}$, has M degrees of smoothness in the discrete sense.*

Proof. We drop the superscript [0] as there is no ambiguity. Since $\eta_{m+1} = \eta_m + hf(t_m, \eta_m)$, the error at t_{m+1}, $e_{m+1} = y_{m+1} - \eta_{m+1}$, satisfies

$$e_{m+1} = e_m + h(f(t_m, y_m) - f(t_m, \eta_m)) + \sum_{i=2}^{S-1} \frac{h^i}{i!} y^{(i)}(t_m) + \mathcal{O}(h^S),$$

where we have performed a Taylor expansion of y_{m+1} about $t = t_m$. Let $u_m = f(t_m, y_m) - f(t_m, \eta_m)$, and $r_m = \frac{h^2}{2!} y^{(2)}(t_m) + \cdots + \frac{h^{S-1}}{(S-1)!} y^{(S-1)}(t_m)$. Notice that

$$u_m = e_m f_y(t_m, y_m) + \cdots + \frac{(-1)^{S-1}(e_m)^{S-2}}{(S-2)!} f_{y^{S-2}}(t_m, y_m) + \mathcal{O}((e_m)^{S-1}),$$

where we have performed a Taylor expansion of $f(t, \eta_m)$ about $y = y_m$. We are now ready to bound $\|\vec{e}^{[0]}\|_\infty$ by induction. By definition, $e_0 = 0$, so certainly, $e_0 \sim \mathcal{O}(h^2)$. Assume that $e_m \sim \mathcal{O}(h^2)$. Since $u_m \sim \mathcal{O}(e_m) \sim \mathcal{O}(h^2)$, we have

$$e_{m+1} = e_m + hu_m + r_m + \mathcal{O}(h^S) \sim \mathcal{O}(h^2),$$

which completes the inductive proof that $\|\vec{e}\|_\infty \sim \mathcal{O}(h^2)$. Note that the inductive proof was with respect to m, the index of the grid points.

To prove the smoothness of the rescaled error vector, we will again use an inductive approach, but this time, with respect to s, the degree of smoothness. First, note that a divided difference approximation to the derivative of the rescaled error vector gives,

(4.1) $$(d_1\tilde{e})_m = \frac{\tilde{e}_{m+1} - \tilde{e}_m}{h} = \tilde{u}_m + \frac{r_m}{h^2} + \mathcal{O}(h^{S-2}),$$

where

$$\tilde{u}_m = \frac{u_m}{h} = \sum_{i=1}^{S-2} (-1)^{i+1} \frac{h^{i-1}}{i!} f_{y^i}(t_m, y_m)(\tilde{e}_m)^i + \mathcal{O}(h^{2S-3}).$$

We are now ready to prove that $\vec{\tilde{e}}$ has M degrees of smoothness by induction. Since $\|\vec{\tilde{e}}\|_\infty \sim \mathcal{O}(h)$, $\vec{\tilde{e}}$ has at least zero degrees of smoothness in the discrete sense. Assume that $\vec{\tilde{e}}$ has $s \leq M - 1$ degrees of smoothness. We will show that $\overrightarrow{d_1\tilde{e}}$ has

s degrees of smoothness, from which we can conclude that $\vec{\tilde{e}}$ has $(s+1)$ degrees of smoothness.

Since f_{y^i} has $(S-i-1)$ degrees of smoothness in the continuous sense, $\overrightarrow{f_{y^i}} = [f_{y^i}(t_0, y_0), \ldots, f_{y^i}(t_M, y_M)]$ has $(S-i-1)$ degrees of smoothness in the discrete sense. Consequently, $h^{i-1}\overrightarrow{f_{y^i}}$ has $(S-2)$ degrees of smoothness, which implies that $\vec{u}$ has $\min(S-2, s)$ degrees of smoothness. Similarly, $\frac{\vec{r}}{h^2}$ has $(S-2)$ degrees of smoothness in the discrete sense. Hence $\overrightarrow{d_1\tilde{e}}$ has s degrees of smoothness $\implies \vec{\tilde{e}}$ has $(s+1)$ degrees of smoothness. Since this argument holds for $S \geq M+2$, we can conclude that $\vec{\tilde{e}}$ has M degrees of smoothness. $\square$

Lemma 4.3. *(correction step): Consider an IDC method constructed using $(M+1)$ uniformly distributed nodes, and forward Euler integrators for the prediction and $k_l \leq M$ correction loops. Also, let $y(t)$, the solution to the IVP (2.1), have at least $S \geq M+2$ degrees of smoothness, and let $\vec{\eta}^{[k]} = (\eta_0^{[k]}, \ldots, \eta_m^{[k]}, \ldots, \eta_M^{[k]})$, be the numerical solution computed after the kth correction loop. If $\vec{e}^{[k-1]} \sim \mathcal{O}(h^{k+1})$ and $\vec{\tilde{e}}^{[k-1]} = \frac{1}{h^k}\vec{e}^{[k-1]}$ has $(M-k+1)$ degrees of smoothness, then $\|\vec{e}^{[k]}\|_\infty \sim \mathcal{O}(h^{k+2})$, and $\vec{\tilde{e}}^{[k]} = \frac{1}{h^{k+1}}\vec{e}^{[k]}$ has $(M-k)$ degrees of smoothness in the discrete sense.*

Proof. The error equation, (2.4), integrated over $[t_m, t_{m+1}]$ gives

$$(4.2) \qquad e_{m+1}^{[k-1]} = e_m^{[k-1]} + \int_{t_m}^{t_{m+1}} F(t, e^{(k-1)}(t))\, dt - \int_{t_m}^{t_{m+1}} \epsilon^{(k-1)}(t)\, dt$$

$$= e_m^{[k-1]} + hF(t_m, e_m^{[k-1]}) + \sum_{i=1}^{M} \frac{h^{i+1}}{(i+1)!}\frac{d^i}{dt^i}F(t_m, e_m^{[k-1]}) + \mathcal{O}(h^{M+2}) - \int_{t_m}^{t_{m+1}} \epsilon^{(k-1)}(t)\, dt.$$

Recall from equation (2.5), that the numerical error vector, which arises from utilizing a forward Euler integrator to solve the error equation, satisfies

$$\delta_{m+1}^{[k]} = \delta_m^{[k]} + h\left(f(t_m, \eta_m^{[k-1]} + \delta_m^{[k]}) - f(t_m, \eta_m^{[k-1]})\right) - \left(\int_{t_m}^{t_{m+1}} \epsilon^{(k-1)}(t)\, dt + \mathcal{O}(h^{M+2})\right).$$

Subtracting the numerical error vector from the error equation gives

$$e_{m+1}^{[k]} = e_m^{[k]} + h\left(f(t_m, y_m) - f(t_m, \eta_m^{[k-1]} + \delta_m^{[k]})\right) + \sum_{i=1}^{M} \frac{h^{i+1}}{(i+1)!}\frac{d^i}{dt^i}F(t_m, e_m^{[k-1]}) + \mathcal{O}(h^{M+2})$$

$$= e_m^{[k]} + hu_m^{[k]} + r_m^{[k-1]} + \mathcal{O}(h^{M+2}),$$

where $r_m^{[k-1]} = \sum_{i=1}^{M} \frac{h^{i+1}}{(i+1)!}\frac{d^i}{dt^i}F(t_m, e_m^{[k-1]})$, and $u_m^{[k]} = f(t_m, y_m) - f(t_m, \eta_m^{[k-1]} + \delta_m^{[k]})$. Taylor expanding $f(t_m, \eta_m^{[k-1]} + \delta_m^{[k]})$ about $y = y_m$, gives,

$$(4.3) \qquad u_m^{[k]} = \sum_{i=1}^{S-2} \frac{(-1)^{i+1}}{i!} f_{y^i}(t_m, y_m)(e_m^{[k]})^i + \mathcal{O}((e_m^{[k]})^{S-1}).$$

We will also need a Taylor expansion of $F(t, e^{(k-1)}(t)) = f(t, y(t)) - f(t, y(t) - e^{(k-1)}(t))$,

$$F(t, e^{(k-1)}t) = \sum_{j=1}^{S-2} \frac{(-1)^{j+1}}{j!} f_{y^j}(t, y(t))(e^{(k-1)}(t))^j + \mathcal{O}((e^{(k-1)}(t))^{S-1}).$$

Since $\vec{\tilde{e}}^{[k-1]}$ has $(M-k+1)$ degrees of smoothness in the discrete sense, $\frac{d}{dt}\tilde{e}^{(k-1)}(t) \sim \mathcal{O}(1)$ from Proposition 3.16. Thus,

$$\frac{dF}{dt} = \sum_{j=1}^{S-2} \frac{(-1)^{j+1}h^{kj}}{j!} \left(\frac{df_{y^j}}{dt}(\tilde{e}^{(k-1)})^j + \frac{d(\tilde{e}^{(k-1)})^j}{dt}f_{y^j} \right) + \mathcal{O}(h^{(k+1)(S-1)-1}) \sim \mathcal{O}(h^k).$$

We are now ready to bound $\|\vec{e}^{[k]}\|_\infty$ by induction. By definition, $e_0^{[k]} = 0$, so certainly, $e_0^{[k]} \sim \mathcal{O}(h^{k+2})$. Assume that $e_m^{[k]} \sim \mathcal{O}(h^{k+2})$. Then,

$$\begin{aligned} e_{m+1}^{[k]} &= e_m^{[k]} + hu_m^{[k]} + r_m^{[k-1]} + \mathcal{O}(h^{M+2}) \\ &\sim \mathcal{O}(h^{k+2}) + \mathcal{O}(h^{k+3}) + \mathcal{O}(h^{k+2}) + \mathcal{O}(h^{M+2}) \sim \mathcal{O}(h^{k+2}), \end{aligned}$$

since each term of $r_m^{[k-1]} \sim \mathcal{O}(h^{k+2})$, and $u_m^{[k-1]}$ is bounded in (4.3). This completes the inductive proof for the error bound.

To prove the smoothness of the rescaled error vector, we use an inductive argument based on s, the degree of smoothness of $\vec{\tilde{e}}^{[k]}$. This is similar in spirit to the proof of Lemma 4.2. First, the rescaled error vector, $\vec{\tilde{e}}^{[k]}$, has at least 0 degrees of smoothness since $\left\|\vec{\tilde{e}}^{[k]}\right\|_\infty \sim \mathcal{O}(h)$ is bounded. Assume that $\vec{\tilde{e}}^{[k]}$ has $s < M - k$ degrees of smoothness. We will prove that $\overrightarrow{d_1\tilde{e}^{[k]}}$ has s degrees of smoothness, from which we can conclude that $\vec{\tilde{e}}^{[k]}$ has $(s+1)$ degrees of smoothness in the discrete sense. The divided difference approximation to the derivative of the rescaled error vector can be expressed as,

$$(d_1\tilde{e}^{[k]})_m = \frac{\tilde{e}_{m+1}^{[k]} - \tilde{e}_m^{[k]}}{h} = \tilde{u}_m^{[k]} + \tilde{r}_m^{[k-1]} + \mathcal{O}(h^{M-k}),$$

where

$$\tilde{u}_m^{[k]} = \sum_{i=1}^{S-2} \frac{(-1)^{i+1}h^{(k+1)(i-1)}}{i!}f_{y^i}(t_m, y_m)(\tilde{e}_m^{[k]})^i + \mathcal{O}(h^{(S-2)(k+2)+1}),$$

has s degrees of smoothness, and

$$\begin{aligned} \tilde{r}_m^{[k-1]} &= \frac{r_m^{[k-1]}}{h^{k+2}} = \sum_{i=1}^{M} \frac{h^{i-1}}{(i+1)!}\frac{d^{i-1}}{dt^{i-1}} \left(\frac{1}{h^k}\frac{d}{dt}F(t_m, e^{(k-1)}(t_m)) \right) \\ &= \sum_{i=1}^{M} \frac{h^{i-1}}{(i+1)!}\frac{d^{i-1}}{dt^{i-1}} \sum_{j=1}^{S-2} \frac{(-1)^{j+1}}{j!} \left(\left(h^{k(j-1)}\frac{df_{y^j}}{dt}\right)(\tilde{e}^{(k-1)})^j + \frac{d(\tilde{e}^{(k-1)})^j}{dt}(h^{k(j-1)}f_{y^j}) \right), \end{aligned}$$

has at least $\min(S - (j+1) + k(j-1) - 1, M - k) \geq s$ degrees of smoothness. Since $\overrightarrow{d_1\tilde{e}^{[k]}}$ has s degrees of smoothness, we can conclude that $\vec{\tilde{e}}^{[k]}$ has $(M+1-k)$ degrees of smoothness. □

Remark 4.4. For the case of non-uniformly distributed quadrature nodes, the rescaled error vector $\vec{\tilde{e}}^{[0]}$ possesses only one degree of smoothness. An example is given below. The lack of smoothness also holds for the rescaled error vector obtained after an RK prediction step in Section 5. Consider the case of four quadrature nodes distributed such that $h_1 = \frac{H}{6}$, $h_2 = \frac{2H}{3}$ and $h_3 = \frac{H}{6}$. Then, $\tilde{u}_m$ and $\frac{r_m}{h_m^2}$ in equation

(4.1) only possess at most one degree of smoothness. For example, if $y(t) = t^3$ then

$$\frac{r_m}{h_m^2} = \frac{1}{2!}y^{(2)}(t_m) + \frac{h_m}{3!}y^{(3)}(t_m) + \cdots.$$

has only one degree of discrete smoothness.

Remark 4.5. Theorem 4.1 can be extended to the case of non-uniformly distributed quadrature nodes, despite the reduced smoothness of the error vector. The proof is similar in spirit to that of Theorem 4.1, since the rescaled error vector only requires one degree of smoothness to show that $\|\vec{e}^{[k]}\| \sim \mathcal{O}(h^{k+2})$. This robust behavior for the Euler case is in sharp contrast to the behavior of defect correction methods on a non-uniform grid, which show no increase in order at all. For related results, see [1].

5. Local error of IDC methods constructed with high order RK integrators

In this section, we first review properties of RK integrators, then provide the local error estimates for IDC methods which utilize high order RK methods in the prediction and correction steps. The outline for the error estimates is similar to the local error estimates for IDC methods constructed with forward Euler integrators, presented in the previous section. A key set of equations, (5.8), is derived for applying a high order RK correction. In the following theorems and lemmas, we will assume that function $f(t, y)$ in IVP (2.1) has $S - 1$ degrees of smoothness if the solution $y(t)$ has S degrees of smoothness.

Definition 5.1. A p-stage explicit RK method can expressed in the form

$$\eta_{i+1} = \eta_i + \sum_{j=1}^{p} b_j k_j, \quad k_j = hf\left(t_i + c_j h, \eta_i + \sum_{l=1}^{j-1} a_{jl} k_l\right) \quad j = 1, 2, 3, \ldots, p,$$

where a_{jl}, b_j, and c_j are real coefficients. An RK method is of order r if $f(t, y)$ in (2.1) is sufficiently smooth, and $\|y(t_i + h) - \eta_{i+1}\| \leq Kh^{r+1}$, for some constant $K > 0$, i.e., the Taylor series expansion for the exact solution, $y(t_i + h)$, and η_{i+1}, coincide up to and including the term h^r.

Proposition 5.2. *Let $y(t)$, the solution to IVP (2.1), have $S \geq r$ degrees of smoothness, and let η be the numerical solution obtained using an rth order RK method. If $t_{m+1} - t_m = h$, then*

$$(5.1) \qquad \eta_{m+1} = \eta_m + \sum_{j=}^{S-1} \frac{h^j}{j!} E_j(t_m, \eta_m) + R_r(t_m, \eta_m) + \mathcal{O}(h^S),$$

where the functions $E_j(t, y)$ are the elementary differentials of y,

$$(5.2) \qquad y^{(j)}(t) := E_j(t, y(t)) = \sum_{q_y^i + q_t^i \leq j-1} \alpha_{q_y^1 q_t^1 \cdots q_y^{n_i} q_t^{n_i}}^{w_1 \cdots w_{n_i} w_f} \prod_{i=1}^{n_i} (f_{t^{q_t^i} y^{q_y^i}})^{w_i} f^{w_f}.$$

and the remainder term,

$$R_r(t, \eta) = \sum_{j=r+1}^{S-1} h^j \left(\sum_{q_y^i + q_t^i \leq j-1} \beta_{q_t^1 q_y^1 \cdots q_t^{n_i} q_y^{n_i} q_f}^{w_1 w_2 \cdots w_{n_i} w_f} \prod_{i=1}^{n_i} (f_{t^{q_t^i} y^{q_y^i}})^{w_i}(t_m, \eta_m) f^{w_f}(t_m, \eta_m) \right),$$

has constant coefficients $\beta_{q_y^1 q_t^1 \cdots q_y^{n_i} q_t^{n_i}}^{w_1 w_2 \cdots w_{n_i} w_f}$ *determined by the specific RK method.*

Proof. Equation (5.1) comes from a Taylor expansion of η_{m+1} about $t = t_m$. Note that the first $r + 1$ terms coincide with the Taylor expansion of the exact solution,

$$(5.3) \qquad y(t_{m+1}) = y(t_m) + \sum_{j=1}^{S-1} \frac{h^j}{j!} y^{(j)}(t_m) + \mathcal{O}(h^S).$$

The remainder term can be proved by induction. Once again, the range of summation, $q_t^i + q_y^i \le j - 1$, is not restrictive since some of the coefficients $\beta_{q_y^1 q_t^1 \cdots q_y^{n_i} q_t^{n_i}}^{w_1 w_2 \cdots w_{n_i} w_f}$ will be zero. $\qquad\square$

Theorem 5.3. *Let $y(t)$, the solution to the IVP (2.1), have at least $S \ge M + 2$ degrees of smoothness in the continuous sense. Then, the local error for an IDC method constructed using $(M + 1)$ uniformly distributed nodes, $(t_m = mh, m = 0, \ldots, M)$, an $(r_0)^{th}$ order RK method in the prediction step and $(r_1, r_2, \ldots, r_{k_l})^{th}$ order RK methods, is $\mathcal{O}(h^{(s_{k_l}+1)})$, where $s_{k_l} = \sum_{j=0}^{k_l} r_j \le M + 1$.*

Proof. The proof of Theorem 5.3 follows from two lemmas discussed in this section. We will show in Lemma 5.4 that the theorem holds for $k = 0$, and in Lemma 5.5 that an inductive argument is satisfied. $\qquad\square$

Lemma 5.4. *(prediction step): Consider an IDC method constructed using $(M+1)$ uniformly distributed nodes, and an r_0^{th} order RK method for the prediction step. Let $y(t)$, the solution to IVP (2.1), have at least $S \ge M + 2$ degrees of smoothness in the continuous sense, and let $(\eta_0^{[0]}, \ldots, \eta_m^{[0]}, \ldots, \eta_M^{[0]})$, be the numerical solution computed after the prediction step. Then, the error vector, $\vec{e}^{[0]} = \vec{y} - \vec{\eta}^{[0]}$, satisfies $\|\vec{e}^{[0]}\|_\infty \sim \mathcal{O}(h^{r_0+1})$, and the rescaled error vector, $\vec{\tilde{e}}^{[0]} = \frac{1}{h^{r_0}} \vec{e}^{[0]}$, has $\min(S-r_0, M)$ degrees of smoothness in the discrete sense.*

Proof. We drop the superscript [0] since there is no ambiguity. First, note that a Taylor expansion of the error, $e_{m+1} = y_{m+1} - \eta_{m+1}$, about $t = t_m$, (see equations (5.3) and (5.1)), results in

$$e_{m+1} = e_m + u_m + r_{1,m} - r_{2,m} + \mathcal{O}(h^S),$$

where

$$u_m = \sum_{i=1}^{r_0} \frac{h^i}{i!} (E_i(t_m, y_m) - E_i(t_m, \eta_m))$$

$$= \sum_{i=1}^{r_0} \frac{h^i}{i!} \sum_{j=1}^{S-1-i} \frac{(-1)^{j-1}(e_m)^j}{j!} \frac{\partial^j E_i}{\partial y^j}(t_m, y_m) + \mathcal{O}((e_m)^{S-i}),$$

$$r_{1,m} = \sum_{i=r_0+1}^{S-1} \frac{h^i}{i!} y^{(i)}(t_m),$$

$$r_{2,m} = \sum_{j=r_0+1}^{S-1} h^j \sum_{q_t^i + q_y^i \le j-1} \beta_{q_t^1 q_y^1 \cdots q_t^{n_i} q_y^{n_i} q_f}^{w_1 w_2 \cdots w_{n_i} w_f} \prod_{i=1}^{n_i} (f_{t^{q_t^i} y^{q_y^i}})^{w_i}(t_m, \eta_m) f^{w_f}(t_m, \eta_m).$$

We are now ready to bound $\|\vec{\tilde{e}}^{[0]}\|_\infty$ by induction. By definition, $e_0 = 0$, so certainly, $e_0 \sim (h^{r_0+1})$. Assume that $e_m \sim \mathcal{O}(h^{r_0+1})$. Since $u_m \sim \mathcal{O}(h^{r_0+2})$, $r_{1,m} \sim \mathcal{O}(h^{r_0+1})$

and $r_{2,m} \sim \mathcal{O}(h^{r_0+1})$, we have $e_{m+1} \sim \mathcal{O}(h^{r_0+1})$, which completes the inductive proof.

To prove the smoothness of the rescaled error vector, we will again use an inductive approach, but this time, with respect to s, the degree of smoothness. First, note that a divided difference approximation to the derivative of the rescaled error vector gives

$$(d_1\tilde{e})_m = \frac{\tilde{e}_{m+1} - \tilde{e}_m}{h} = \tilde{u}_m + \tilde{r}_{1,m} - \tilde{r}_{2,m} + \mathcal{O}(h^{S-r_0-1}),$$

where

$$\tilde{u}_m = \frac{u_m}{h^{r_0+1}} = \sum_{i=1}^{r_0} \frac{1}{i!} \sum_{j=1}^{S-1-i} \frac{(-1)^{j-1}h^{i+r_0(j-1)-1}}{j!} \frac{\partial^j E_i}{\partial y^j}(t_m, y_m)(\tilde{e}_m)^j + \mathcal{O}(h^{S-r_0-1}),$$

$$\tilde{r}_{1,m} = \frac{r_{1,m}}{h^{r_0+1}} = \sum_{i=r_0+1}^{S-1} \frac{h^{i-r_0-1}}{i!} y^{(i)}(t_m),$$

$$\tilde{r}_{2,m} = \frac{R_{r_0}(t_m, \eta_m)}{h^{r_0+1}} = \sum_{j=0}^{S-r_0-2} h^j \sum_{q_t^i + q_y^i \le j+r_0} \beta_{q_t^1 q_y^1 \cdots q_t^{n_i} q_y^{n_i} q_f}^{w_1 w_2 \cdots w_{n_i} w_f} \prod_{i=1}^{n_i} (f_{t^{q_t^i} y^{q_y^i}})^{w_i}(t_m, \eta_m) f^{w_f}(t_m, \eta_m).$$

We are now ready to prove that $\vec{\tilde{e}}$ has $\min(S - r_0, M)$ degrees of smoothness by induction. Since $\|\vec{\tilde{e}}\|_\infty \sim \mathcal{O}(h)$ is bounded, $\vec{\tilde{e}}$ has at least zero degrees of smoothness in the discrete sense. Assume that $\vec{\tilde{e}}$ has $s < \min(S - r_0, M)$ degrees of smoothness in the discrete sense. We will prove that $\overrightarrow{d_1\tilde{e}}$ has s degrees of smoothness, from which we can conclude $\vec{\tilde{e}}$ has $(s + 1)$ degrees of smoothness in the discrete sense. From a similar argument in the proof of Lemma 4.2, $\vec{\tilde{u}}$ has s degrees of smoothness in the discrete sense. Assuming that $y(t)$ has S degrees of smoothness, $\overrightarrow{\tilde{r}}_1$ has $(S - r_0 - 1)$ degrees of smoothness. Since $\vec{\tilde{e}}$ has s degrees of smoothness, $\vec{e}$ has $\min(s + r_0, M)$ degrees of smoothness and $\vec{\eta} = \vec{y} - \vec{e}$ has $\min(s + r_0, M)$ degrees of smoothness. Thus, $\overrightarrow{\tilde{r}_2}$ has $\min(s+r_0, S-r_0-1, M)$, at least s, degrees of smoothness in the discrete sense. Therefore $\vec{\tilde{e}}$ has $(s + 1)$ degrees of smoothness. We can now conclude that $\vec{\tilde{e}}$ has $\min(S - r_0, M)$ degrees of smoothness in the discrete sense. $\square$

Lemma 5.5. *(correction step): Let $y(t)$, the solution to IVP (2.1), have at least $S \ge M + 2$ degrees of smoothness in the continuous sense. Consider an IDC method, constructed using $(M + 1)$ uniformly distributed nodes, an $(r_0)^{th}$ order RK method in the prediction step and $(r_1, r_2, \ldots, r_{k_l})^{th}$ order RK methods in the respective correction loops. If the error vector after the $(k - 1)^{st}$ loop satisfies $\vec{e}^{[k-1]} \sim \mathcal{O}(h^{s_{k-1}+1})$, and the rescaled error vector, $\vec{\tilde{e}}^{[k-1]} = \frac{1}{h^{s_{k-1}}}\vec{e}^{[k-1]}$, has $(M + 1 - s_{k-1})$ degrees of smoothness in the discrete sense, then, the updated error vector, $\vec{e}^{[k]}$, satisfies $\|\vec{e}^{[k]}\|_\infty \sim \mathcal{O}(h^{s_k+1})$, and the rescaled error vector, $\vec{\tilde{e}}^{[k]} = \frac{1}{h^{s_k}}\vec{e}^{[k]}$, has $(M + 1 - s_k)$ degrees of smoothness in the discrete sense.*

The proof of this lemma is very technical and involved. First, we discuss properties of the error function, $e^{(k-1)}(t)$. Then, the actual construction of IDC methods using RK integrators within the correction loops is described. A discussion on the properties of the numerical error vector is given before Lemma 5.5 is proved. Note

that for a non-uniform distribution of nodes, $\vec{\tilde{e}}^{[0]}$ has only one degree of smoothness (as discussed in Remark 4.4), hence, Lemma 5.5 does not apply.

5.1. **Properties of the error function.** We write the error equation (2.4) as

$$(5.4) \qquad (Q^{(k-1)})'(t) = F(t, Q^{(k-1)}(t) - E^{(k-1)}(t)) \doteq G^{(k-1)}(t, Q^{(k-1)}(t)),$$

where $Q^{(k-1)}(t) = e^{(k-1)}(t) + E^{(k-1)}(t)$, and $E^{(k-1)}(t) = \int_0^t \epsilon^{(k-1)}(\tau) \, d\tau$. Our analysis for the error vector will rely on this form of error equation, (5.4), or the scaled variant,

$$(5.5) \qquad (\tilde{Q}^{(k-1)})'(t) = \tilde{G}^{(k-1)}(t, \tilde{Q}^{(k-1)}(t)),$$

where $\tilde{Q}^{(k-1)}(t) = \frac{Q^{(k-1)}(t)}{h^{s_{k-1}}}$, and $\tilde{G}^{(k-1)}(t, \tilde{Q}^{(k-1)}(t)) = \frac{G^{(k-1)}(t, h^{s_{k-1}} \tilde{Q}^{(k-1)}(t))}{h^{s_{k-1}}}$.

Proposition 5.6. *If $e^{(k-1)}(t) \sim \mathcal{O}(h^{s_{k-1}+1})$, then $Q^{(k-1)}(t) \sim \mathcal{O}(h^{s_{k-1}+1})$, and $G^{(k-1)}(t, Q^{(k-1)}(t)) \sim \mathcal{O}(h^{s_{k-1}+1})$.*

Proof. From the above definition of G,

$$G^{(k-1)}(t, Q^{(k-1)}(t)) = F(t, e^{(k-1)}(t))$$

$$= \sum_{i=1}^{S-2} \frac{(-1)^{i+1}}{i!} \frac{\partial^i f}{\partial y^i}(t, y(t))(e^{(k-1)}(t))^i + \mathcal{O}(h^{(s_{k-1}+1)(S-1)}) \sim \mathcal{O}(h^{s_{k-1}+1}),$$

and $Q^{(k-1)}(t) = \int_0^t G^{(k-1)}(\tau, Q^{(k-1)}(\tau)) \, d\tau \sim \mathcal{O}(h^{s_{k-1}+1})$. □

Proposition 5.7. *Suppose $\vec{e}^{[k-1]} \sim \mathcal{O}(h^{s_{k-1}+1})$ and the rescaled error vector, $\vec{\tilde{e}}^{[k-1]}$, has $(M + 1 - s_{k-1})$ degrees of smoothness in the discrete sense. Then*

$$(5.6) \qquad \frac{d^l}{dt^l} \tilde{Q}^{(k-1)}(t) \sim \mathcal{O}(1), \quad if \quad l \leq M + 1 - s_{k-1}, \quad \forall t \in [0, H].$$

Proof. Since the rescaled error vector, $\vec{\tilde{e}}^{[k-1]}$, has $(M + 1 - s_{k-1})$ degrees of smoothness in the discrete sense, from Proposition 3.16, $\frac{d^l}{dt^l} \tilde{e}^{(k-1)}(t) \sim \mathcal{O}(1)$, for $l \leq M + 1 - s_{k-1}, \forall t \in [0, H]$. From Proposition 5.6,

$$Q^{(k-1)}(t) = \int_0^t G^{(k-1)}(\tau, Q^{(k-1)}(\tau)) \, d\tau$$

$$= \int_0^t \sum_{i=1}^{S-2} \frac{(-1)^{i+1}}{i!} \frac{\partial^i f}{\partial y^i}(\tau, y(\tau))(e^{(k-1)}(\tau))^i \, d\tau + \mathcal{O}(h^{(s_{k-1}+1)(S-1)+1}),$$

we have

$$\frac{d}{dt} \tilde{Q}^{(k-1)}(t) = \sum_{i=1}^{S-2} \frac{(-1)^{i+1}}{i!} h^{s_{k-1}(i-1)} \frac{\partial^i f}{\partial y^i}(t, y(t)) (\tilde{e}^{(k-1)}(t))^i + \mathcal{O}(h^{(s_{k-1}+1)(S-2)+1})$$

$$\sim \mathcal{O}(1),$$

$$\frac{d^l}{dt^l} \tilde{Q}^{(k-1)}(t) \sim \frac{d^{l-1}}{dt^{l-1}} (\tilde{e}^{(k-1)}(t)) \sim \mathcal{O}(1), \quad for \quad l = 2, \ldots, M + 1 - s_{k-1}.$$

Note that if $\vec{\tilde{e}}^{[k-1]}$ does not have the required degrees of smoothness (e.g., if a non-uniform mesh is used to construct the IDC method), then the discrete derivative of $\frac{d^l}{dt^l} \tilde{Q}^{(k-1)}$ is no longer $\mathcal{O}(1)$. □

Proposition 5.8. $\tilde{Q}^{(k-1)}(t)$ *satisfies*

$$\tilde{Q}_{m+1}^{[k-1]} = \tilde{Q}_m^{[k-1]} + \sum_{i=1}^{M-s_{k-1}} \frac{h^i}{i!} \tilde{G}_{i-1}^{(k-1)}(t_m, \tilde{Q}_m^{[k-1]}) + \mathcal{O}(h^{M+1-s_{k-1}}),$$

or, equivalently,

(5.7)

$$e_{m+1}^{[k-1]} = e_m^{[k-1]} + h^{s_{k-1}} \sum_{i=1}^{M-s_{k-1}} \frac{h^i}{i!} \tilde{G}_{i-1}^{(k-1)}(t_m, \tilde{Q}_m^{[k-1]}) + \mathcal{O}(h^{M+1}) - \int_{t_m}^{t_{m+1}} \epsilon^{(k-1)}(\tau)\, d\tau,$$

where

$$\tilde{G}_{i-1}^{(k-1)}(t, \tilde{Q}) := \sum_{q_t^i + q_{\tilde{Q}}^i \leq i-1} \gamma_{q_t^1 q_{\tilde{Q}}^1 \cdots q_t^{n_i} q_{\tilde{Q}}^{n_i}}^{w_1 \cdots w_{n_i} w_f} \prod_{i=1}^{n_i} (\tilde{G}_{t^{q_t^i} \tilde{Q}^{q_{\tilde{Q}}^i}}^{(k-1)})^{w_i} (\tilde{G}^{(k-1)})^{w_f}, \quad i \leq M - s_{k-1}.$$

As before, $\gamma_{q_t^1 q_{\tilde{Q}}^1 \cdots q_t^{n_i} q_{\tilde{Q}}^{n_i}}^{w_1 \cdots w_{n_i} w_f}$ *are constant coefficients, and,* w_i, w_f, q_t^i, $q_{\tilde{Q}}^i$ *and* n_i *are non-negative integers. Note that*

$$\tilde{G}_{i-1}^{(k-1)}(t, \tilde{Q}^{(k-1)}(t)) := \frac{d^{i-1}\tilde{G}^{(k-1)}}{dt^{i-1}} = \frac{d^i}{dt^i}\tilde{Q}^{(k-1)}(t) \sim \mathcal{O}_h(1), \qquad i \leq M - s_{k-1}.$$

5.2. RK integrators within the correction loop of IDC methods.

A p-stage, r^{th} order RK method applied to (5.5) gives,

$$\tilde{k}_1 = \tilde{G}^{(k-1)}(t_0, \tilde{Q}_0^{[k-1]})$$

$$\tilde{k}_2 = \tilde{G}^{(k-1)}(t_0 + c_2 h, (\tilde{Q}_0^{[k-1]} + h a_{2,1}\tilde{k}_1))$$

$$\cdots$$

$$\tilde{k}_p = \tilde{G}^{(k-1)}(t_0 + c_p h, (\tilde{Q}_0^{[k-1]} + h(a_{p,1}\tilde{k}_1 + \ldots + a_{p,p-1}\tilde{k}_{p-1})))$$

$$\tilde{\Omega}_1^{[k]} = \tilde{Q}_0^{[k-1]} + h(b_1 \tilde{k}_1 + \ldots + b_p \tilde{k}_p),$$

where Ω denotes the solution in numerical space. In the actual implementation, we discretize the error equation (2.4) as follows,

$$k_1 = F(t_0, e_0^{[k-1]})$$

$$k_2 = F\left(t_0 + c_2 h, e_0^{[k-1]} + h a_{2,1} k_1 - \int_{t_0}^{t_0 + c_2 h} \epsilon^{(k-1)}(\tau)\, d\tau\right)$$

$$\cdots$$

$$k_p = F\left(t_0 + c_p h, e_0^{[k-1]} + h \sum_{i=1}^p a_{pi} k_i - \int_{t_0}^{t_0 + c_p h} \epsilon^{(k-1)}(\tau)\, d\tau\right)$$

(5.8) $\qquad \delta_1^{[k]} = e_0^{[k-1]} + h \sum_{i=1}^p b_i k_i - \int_{t_0}^{t_0+h} \epsilon^{(k-1)}(\tau)\, d\tau.$

Proposition 5.9. *The Taylor series for* $\tilde{e}^{(k-1)}(t_0 + h) = \frac{1}{h^{s_{k-1}}} e^{(k-1)}(t_0 + h)$ *and for* $\frac{1}{h^{s_{k-1}}}\delta_1^{[k]}$ *above, coincide up to and including the term* h^r*, for a sufficiently smooth error function* $\tilde{e}^{(k-1)}(t)$.

Proof. First, we prove by induction that

(5.9) $$h^{s_{k-1}}\tilde{k}_i = k_i, \quad \forall i = 1, \ldots, p.$$

Since

$$h^{s_{k-1}}\tilde{k}_1 = G^{(k-1)}(t_0, Q_0^{[k-1]}) = F(t_0, e^{(k-1)}(t_0)) = k_1,$$

the claim is true for $i = 1$. Assume that (5.9) is true for $1, \ldots, i$. Then,

$$h^{s_{k-1}}\tilde{k}_{i+1} = G^{(k-1)}\left(t_0 + c_{i+1}h, h^{s_{k-1}}(\tilde{Q}^{(k-1)}(t_0) + h\sum_{j=1}^{i} a_{i+1,j}\tilde{k}_j)\right)$$

$$= G^{(k-1)}\left(t_0 + c_{i+1}h, Q^{(k-1)}(t_0) + h\sum_{j=1}^{i} a_{i+1,j}k_j\right)$$

$$= F\left(t_0 + c_{i+1}h, Q^{(k-1)}(t_0) + h\sum_{j=1}^{i} a_{i+1,j}k_j - E^{(k-1)}(t_0 + c_{i+1}h)\right)$$

$$= F\left(t_0 + c_{i+1}h, e_0^{[k-1]} + h\sum_{j=1}^{i} a_{i+1,j}k_j - \int_{t_0}^{t_0+c_{i+1}h} \epsilon^{(k-1)}(\tau)\,d\tau\right) = k_{i+1},$$

which completes the inductive proof. Also,

$$\delta_1^{[k]} := h^{s_{k-1}}\tilde{\Omega}_1^{[k]} - E(t_1)$$

$$= Q^{(k-1)}(t_0) - E(t_1) + h(b_1 k_1 + \ldots + b_p k_p)$$

$$= e_0^{[k-1]} + h(b_1 k_1 + \ldots + b_p k_p) - \int_{t_0}^{t_0+h} \epsilon^{(k-1)}(\tau)\,d\tau.$$

Since the Taylor series for $\tilde{Q}^{(k-1)}(t_0 + h)$ and $\tilde{\Omega}_1^{[k]}$ coincide, up to and including, the term h^r, the Taylor series for $\frac{1}{h^{s_{k-1}}}e^{(k-1)}(t_0 + h)$ and $\frac{1}{h^{s_{k-1}}}\delta_1^{[k]}$ also coincide, up to and including, the term h^r. □

Proposition 5.10. *Suppose that the error equation* (2.4) *is discretized using the algorithm in* (5.8). *Then, the numerical error vector satisfies*

(5.10) $$\delta_{m+1}^{[k]} = \delta_m^{[k]} + \sum_{i=1}^{r_k} \frac{h^{s_{k-1}+i}}{i!}\tilde{G}^{(k-1)}(t_m, \tilde{\Omega}_m^{[k]}) + h^{s_{k-1}}\tilde{R}_{r_k+1}(t_m, \tilde{\Omega}_m^{[k]})$$

$$- \int_{t_m}^{t_{m+1}} \epsilon^{(k-1)}(\tau)\,d\tau + \mathcal{O}(h^{M+1}),$$

where $\tilde{\Omega}_m^{[k]} = \frac{1}{h^{s_{k-1}}}\left(\delta^{[k]}(t_m) + \int_0^{t_m} \epsilon^{(k-1)}(\tau)\,d\tau\right)$ *and*

$$\tilde{R}_{r_k+1}(t, \tilde{\Omega}) = \sum_{q=r_k+1}^{M-s_{k-1}} h^q \sum_{q_t^i + q_{\tilde{Q}}^i \leq q-1} \zeta_{q_t^1 q_{\tilde{Q}}^1 \cdots q_t^{n_i} q_{\tilde{Q}}^{n_i}}^{w_1 w_2 \cdots w_{n_i} w_f} \prod_{i=1}^{n_i} (\tilde{G}_{t^{q_t^i}\tilde{Q}^{q_{\tilde{Q}}^i}}^{(k-1)}(t, \tilde{\Omega}))^{w_i} (\tilde{G}^{(k-1)}(t, \tilde{\Omega}))^{w_f}.$$

Here $\zeta_{q_t^1 q_{\tilde{Q}}^1 \cdots q_t^{n_i} q_{\tilde{Q}}^{n_i}}^{w_1 w_2 \cdots w_{n_i} w_f}$ *are constant coefficients determined by* (5.8). *Note that the* $\mathcal{O}(h^{M+1})$ *truncation error arises from numerically computing* $\int_{t_m}^{t_{m+1}} \epsilon^{(k-1)}(\tau)\,d\tau$,

and that $\tilde{G}^{(k-1)}_{t^{q_t^i}\tilde{Q}^{q_{\tilde{Q}}^i}}(t,\tilde{\Omega}) \sim \mathcal{O}_h(1)$ for $q_t^i + q_{\tilde{Q}}^i \leq M - s_{k-1}$. We omit the proof, which is similar in spirit to the proof of Proposition 5.2.

Remark 5.11. In algorithm (5.8), an evaluation of $f(t,\eta(t))$ is needed at intermediate stages, e.g., at $t = t_0 + c_p h$; this results in additional function evaluations of $f(t,\eta(t))$. Since function evaluations are usually the most computationally intensive portion of an algorithm, we perform polynomial interpolations from known values instead. Specifically, given $\vec{f} = (f(t_0,\eta_0),\ldots,f(t_m,\eta_m),\ldots,f(t_M,\eta_M))$, an Mth degree Lagrange interpolant, $L^M(t,f)$, (3.2), is constructed and used to approximate $f(t,\eta(t))|_{t=t_0+c_p h}$. This approximation has truncation error, $\mathcal{O}(h^{M+2})$.

Remark 5.12. To evaluate the integral term in algorithm (5.8), we integrate the corresponding Lagrange interpolant, $\int_{t_0}^{t_0+c_i h} L^M(t,\epsilon^{(k-1)})\,dt$, $i = 1,\ldots,p$, with truncation error, $\mathcal{O}(h^{M+2})$.

Proof of Lemma 5.5. Subtracting the numerical error vector, (5.10), from the integrated error equation, (5.7),

(5.11)

$$
\begin{aligned}
e^{[k]}_{m+1} &= e^{[k-1]}_{m+1} - \delta^{[k]}_{m+1} \\
&= e^{[k]}_m + h^{s_{k-1}} \sum_{i=r_k+1}^{M+1-s_{k-1}} \frac{h^i}{i!} \tilde{G}^{(k-1)}_{i-1}(t_m, \tilde{Q}^{[k-1]}_m) \\
&\quad + h^{s_{k-1}} \sum_{i=1}^{r_k} \frac{h^i}{i!} \left(\tilde{G}^{(k-1)}_{i-1}\left(t_m, \frac{e^{[k-1]}_m + E^{(k-1)}(t_m)}{h^{s_{k-1}}}\right) - \tilde{G}^{(k-1)}_{i-1}\left(t_m, \frac{\delta^{[k]}_m + E^{(k-1)}(t_m)}{h^{s_{k-1}}}\right) \right) \\
&\quad - h^{s_{k-1}} \tilde{R}_{r_k+1}\left(t_m, \frac{\delta^{[k]}_m + E^{(k-1)}(t_m)}{h^{s_{k-1}}}\right) + \mathcal{O}(h^{M+2}) \\
&= e^{[k]}_m + u^{[k]}_m + r^{[k-1]}_{1,m} - r^{[k]}_{2,m} + \mathcal{O}(h^{M+2}),
\end{aligned}
$$

where

(5.12)

$$
\begin{aligned}
u^{[k]}_m &= h^{s_{k-1}} \sum_{i=1}^{r_k} \frac{h^i}{i!} \left(\tilde{G}_{i-1}\left(t_m, \frac{e^{[k-1]}_m + E^{(k-1)}(t_m)}{h^{s_{k-1}}}\right) - \tilde{G}_{i-1}\left(t_m, \frac{\delta^{[k]}_m + E^{(k-1)}(t_m)}{h^{s_{k-1}}}\right) \right) \\
&= \sum_{i=1}^{r_k} \frac{h^i}{i!} \frac{d^{i-1}}{dt^{i-1}} \left(G^{(k-1)}\left(t_m, e^{[k-1]}_m + E^{(k-1)}(t_m)\right) - G^{(k-1)}\left(t_m, \delta^{[k]}_m + E^{(k-1)}(t_m)\right) \right) \\
&= \sum_{i=1}^{r_k} \frac{h^i}{i!} \frac{d^{i-1}}{dt^{i-1}} \left(f\left(t_m, y(t_m)\right) - f\left(t_m, \eta^{(k)}(t_m)\right) \right) \\
&= \sum_{i=1}^{r_k} \frac{h}{i!} h^{i-1} \frac{d^{i-1}}{dt^{i-1}} \left(\sum_{j=1}^{S-2} \frac{(-1)^{j+1}}{j!} \frac{\partial^j f}{\partial y^j}(t_m, y(t_m))(e^{[k]}_m)^j + \mathcal{O}((e^{[k]}_m)^{S-1}) \right),
\end{aligned}
$$

and since $s_k = s_{k-1} + r_k$,

$$(5.13) \qquad r_{1,m}^{[k-1]} = h^{s_{k-1}} \sum_{i=r_k+1}^{M+1-s_{k-1}} \frac{h^i}{i!} \tilde{G}_{i-1}(t_m, \tilde{Q}_m^{[k-1]})$$

$$= h^{s_k+1} \sum_{i=0}^{M-s_k} \frac{1}{(r_k+i+1)!} h^i \frac{d^i}{dt^i} \left(\frac{d^{r_k}}{dt^{r_k}} \tilde{G}(t_m, \tilde{Q}_m^{[k-1]}) \right),$$

$$(5.14) \quad r_{2,m}^{[k]} = h^{s_{k-1}} \tilde{R}_{r_k+1} \left(t_m, \frac{\delta_m^{[k]} + E^{(k-1)}(t_m)}{h^{s_{k-1}}} \right)$$

$$= h^{s_k+1} \sum_{i=0}^{M+1-s_k} h^i \sum_{q_{\tilde{Q}}^i + q_t^i \le i + r_k} \zeta_{q_{\tilde{Q}}^1 q_t^1 \cdots q_{\tilde{Q}}^{n_i} q_t^{n_i}}^{w_1 w_2 \cdots w_{n_i} w_f} \prod_{i=1}^{n_i} \left(\tilde{G}_{\tilde{Q}^{q_{\tilde{Q}}^i} t^{q_t^i}}(t, \tilde{\Omega}_m^{[k]}) \right)^{w_i} \tilde{G}^{w_f}(t, \tilde{\Omega}_m^{[k]}).$$

We are now ready to bound $\|\vec{e}^{[k-1]}\|_\infty$ using an inductive argument. By definition, $e_0^{[k]} = 0$, so certainly $e_0^{[k]} \sim \mathcal{O}(h^{s_k+1})$. Assume that $e_m^{[k]} \sim \mathcal{O}(h^{s_k+1})$. From (5.12), $u_m^{[k]} \sim \mathcal{O}(h^{s_k+2})$. By Proposition 5.7, $\frac{d^{r_k}}{dt^{r_k}} \tilde{G}(t_m, \tilde{Q}_m^{[k-1]}) = \frac{d^{r_k+1}}{dt^{r_k+1}} \tilde{Q}^{(k-1)}(t)|_{t=t_m} \sim \mathcal{O}(1)$. Therefore, $r_{1,m}^{[k-1]} \sim \mathcal{O}(h^{s_k+1})$ from (5.13), and $r_{2,m}^{[k]} \sim \mathcal{O}(h^{s_k+1})$ from (5.14) and Proposition 5.10. Thus, $e_{m+1}^{[k]} \sim \mathcal{O}(h^{s_k+1})$, completing the inductive proof.

As before, we will prove the smoothness of the rescaled error vector using an inductive argument based on s, the degree of smoothness of $\vec{\tilde{e}}^{[k]}$. First, the rescaled error vector has at least 0 degrees of smoothness since $\|\vec{\tilde{e}}^{[k]}\|_\infty \sim \mathcal{O}(h) < \infty$. Assume that $\vec{\tilde{e}}^{[k]}$ has $s < M + 2 - s_k$ degrees of smoothness in the discrete sense. We will prove that $\overrightarrow{d_1 \tilde{e}^{[k]}}$ has s degrees of smoothness, from which we can then conclude that $\vec{\tilde{e}}^{[k]}$ has $(s+1)$ degrees of smoothness. Using (5.11), the divided difference approximation to the derivative of the rescaled error vector satisfies

$$(d_1 \tilde{e}^{[k]})_m = \frac{\tilde{u}_m^{[k]}}{h} + \frac{1}{h} \tilde{r}_{1,m}^{[k-1]} - \frac{1}{h} \tilde{r}_{2,m}^{[k]} + \mathcal{O}(h^{M+2-s_k}),$$

where

$$\frac{\tilde{u}_m^{[k]}}{h} = \sum_{i=1}^{r_k} \frac{h^{i-1}}{i!} \frac{d^{i-1}}{dt^{i-1}} \sum_{j=1}^{S-2} \frac{(-1)^{j+1} h^{s_k(j-1)}}{j!} \frac{\partial^j f}{\partial y^j}(t_m, y(t_m)(\tilde{e}_m^{[k]})^j + \mathcal{O}(h^{(s_k+1)(S-2)-1}),$$

$$\frac{\tilde{r}_{1,m}^{[k-1]}}{h} = \sum_{i=0}^{S-r_k-1} \frac{h^i}{(r_k+i+1)!} \frac{d^i}{dt^i} \left(\frac{d^{r_k}}{dt^{r_k}} \tilde{G}(t_m, \tilde{Q}_m^{[k-1]}) \right),$$

$$\frac{\tilde{r}_{2,m}^{[k]}}{h} = \sum_{i=0}^{M+2-s_k} h^i \sum_{q_{\tilde{Q}}^i + q_t^i \le i + r_k} \zeta_{q_{\tilde{Q}}^1 q_t^1 \cdots q_{\tilde{Q}}^{n_i} q_t^{n_i}}^{w_1 w_2 \cdots w_{n_i} w_f} \prod_{i=1}^{n_i} \left(\tilde{G}_{\tilde{Q}^{q_{\tilde{Q}}^i} t^{q_t^i}}(t, \tilde{\Omega}_m^{[k]}) \right)^{w_i} \tilde{G}^{w_f}(t, \tilde{\Omega}_m^{[k]}).$$

are computed from (5.12), (5.13), and (5.14).

Using a similar argument from before, $\frac{1}{h} \vec{\tilde{u}}^{[k]}$ has s degrees of smoothness in the discrete sense. Similarly, $\frac{1}{h} \vec{\tilde{r}}_1^{[k]}$ has $(M - s_k)$ degrees of smoothness since $\frac{d^{r_k}}{dt^{r_k}} \tilde{G}(t, \tilde{Q}^{(k-1)})|_{t=t_m} = \frac{d^{r_k+1}}{dt^{r_k+1}} \tilde{Q}^{(k-1)}(t)|_{t=t_m}$, has $(M - s_k)$ degrees of smoothness. Since $\frac{1}{h} \vec{\tilde{r}}_2^{[k]}$ also has $\min(M - s_k, s)$ degrees of smoothness in the discrete sense, we can conclude that $\overrightarrow{d_1 \tilde{e}^{[k]}}$ has s degrees of smoothness in the discrete sense, which

implies that $\vec{\tilde{e}}^{[k]}$ has $(M + 1 - s_k)$ degrees of smoothness in the discrete sense, completing the inductive proof. $\qquad\square$

Theorem 5.13. *(deferred correction): Suppose $y(t)$, the solution to IVP (2.1), has $S \geq M + 1$ degrees of smoothness. Consider a deferred correction method constructed with $(M + 1)$ uniformly distributed nodes, $t_m = mh, m = 0, \ldots, M$, an r_0^{th} order RK method in the prediction step, and $(r_1, r_2, \ldots, r_{k_l})^{th}$ order RK methods for k_l correction loops. The local error for the above deferred correction method is $\mathcal{O}(h^{s_{k_l}+1})$, where $s_{k_l} = \sum_{j=0}^{k_l} r_j \leq M + 1$.*

Proof. The proof of this theorem is similar to that of Theorem 5.3. The difference involves using the differential form of the error equation (2.6), instead of the Picard integral equation in the correction steps. This leads to a difference in the maximum achievable order. Specifically, deferred correction methods approximate $\frac{\partial}{\partial t}\eta^{(k-1)}(t)$, (2.6), with at most M^{th} order accuracy. Thus a deferred correction method can achieve M^{th} order accuracy at best, unlike the IDC methods which can achieve $(M + 1)^{st}$ order accuracy. $\qquad\square$

6. Numerical examples

We test the order of accuracy for various IDC methods constructed using a variety of integrators in the prediction and correction steps. Our numerical runs are in agreement with the analysis in Sections 4 and 5. Then, we compare the stability and accuracy of these methods. Superior stability and accuracy properties are observed for IDC8-RK4 versus IDC8-RK2 and IDC8-FE.

Example 6.1. Consider the IVP,

$$(6.1) \qquad y'(t) = -2\pi \sin 2\pi t - 2(y - \cos 2\pi t), \quad y(0) = 1,$$

which has an exact solution, $y(t) = \cos 2\pi t$. We solve this IVP numerically using IDC methods constructed with eight uniformly distributed nodes and various integrators. Specifically, given a final integration time, T, and number of intervals N, the IDC method is iterated completely in each interval, $[t_{i-1}, t_i] = T/N, i = 1, \ldots, N$ using the quadrature nodes $t_{i,j} = t_i + jh, h = H/7, j = 0, \ldots, 7$. This defines the starting value for the next interval. In the table, IDC8-FE denotes the IDC method constructed using eight uniformly distributed nodes and forward Euler integrators for the prediction and correction loops, IDC8-RK2 denotes the IDC method constructed using eight uniformly distributed nodes and RK2 integrators for the prediction and correction loops, etc. Eighth order convergence is observed for all IDC8 schemes tested, in agreement with the analysis presented in Section 4 and 5. Interestingly, the error of the numerical solution obtained using IDC8-RK4, is consistently one order of magnitude smaller than that of those obtained using IDC8-FE and IDC8-RK2. More analysis and discussion on this observation are provided in [3]. The results are summarized in Table 1.

In Table 2, the computations are repeated with deferred correction methods. DC8-FE denotes a deferred correction method constructed using eight uniformly distributed nodes and forward Euler integrators for the prediction and correction loops, DC8-RK2 denotes a deferred correction method with eight uniformly distributed nodes and RK2 integrators for the prediction and correction loops etc. The expected seventh order convergence is observed for DC8-FE. The eighth order convergence observed for DC8-RK2 and DC8-RK4 is puzzling. This behavior does

TABLE 1. The error at $T = 20$ and the corresponding order of accuracy for various IDC methods used to solve IVP (6.1). Note that the number of steps, N, is inversely proportional to the interval size, $H = T/N$, and the sub-interval size is $H/7$.

–	IDC8-FE		IDC8-RK2		IDC8-RK4	
steps (N)	error	order	error	order	error	order
40	5.47E-6	–	5.48E-6	–	4.49E-7	–
80	1.49E-8	8.52	1.49E-8	8.52	1.17E-9	8.57
120	5.42E-10	8.17	5.43E-10	8.16	4.27E-11	8.18
160	5.30E-11	8.08	5.31E-11	8.08	4.16E-12	8.10
200	8.79E-12	8.05	8.80E-12	8.05	6.83E-13	8.10

not appear if an odd number of quadrature nodes are used to construct the DC method.

TABLE 2. The error at $T = 20$ and the corresponding order of accuracy for various DC methods used to solve IVP (6.1). Note that the number of steps, N, is inversely proportional to the interval size, $H = T/N$, and the sub-interval size is $H/7$.

–	DC8-FE		DC8-RK2		DC8-RK4	
steps (N)	error	order	error	order	error	order
40	3.89E-5	–	5.72E-6	–	5.87E-7	–
80	3.30E-7	6.88	2.60E-8	7.79	2.54E-9	7.85
120	2.15E-8	6.74	1.02E-9	8.00	9.83E-11	8.02
160	2.91E-9	6.94	1.02E-10	8.00	9.81E-12	8.01
200	6.11E-10	6.99	1.70E-11	8.00	1.64E-12	8.01

Definition 6.2. The amplification factor for a numerical method, $Am(\lambda)$, can be interpreted as the numerical solution to

(6.2) $$y'(t) = \lambda y(t), \quad y(0) = 1,$$

after one time step of size 1 for $\lambda \in \mathbb{C}$, i.e., $Am(\lambda) = y(1)$.

Definition 6.3. The stability region, S, for a numerical method, is the subset of the complex plane $\mathbb{C}$, consisting of all λ such that $Am(\lambda) \leq 1$,

$$S = \{\lambda : Am(\lambda) \leq 1\}.$$

In Figure 1(a), the stability regions for IDC8-FE, IDC8-RK2 and IDC8-RK4 are computed numerically and plotted. Interestingly, the area of the stability regions increases with the order of the embedded integrator. This isn't overly surprising since the regions of absolute stability for RK methods of order one (Euler's method) through four increases with the order. A similar observation is made in Figure 1(b) for twelfth order IDC methods constructed using twelve interior points and various integrators.

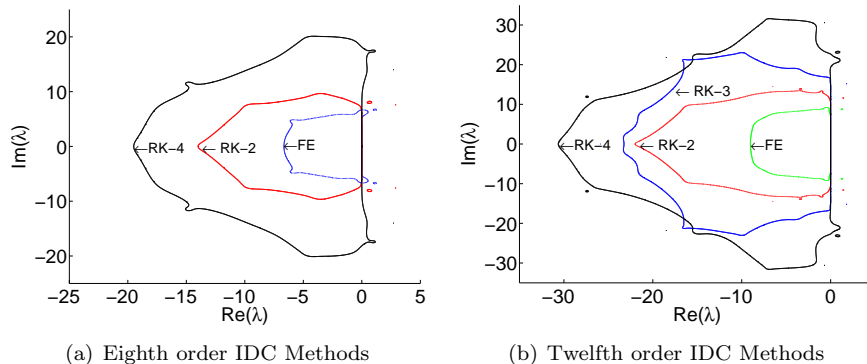(a) Eighth order IDC Methods   (b) Twelfth order IDC Methods

FIGURE 1. (a) Stability regions for eighth order IDC methods constructed using eight uniformly distributed points and various integrators. The regions of absolute stability increase with the order of the embedded integrator. (b) A similar observation is shown for twelfth order IDC methods.

Definition 6.4. Let $e(\lambda)$ be the error at $T = 1$, obtained using a numerical method to solve IVP (6.2), $\lambda \in \mathbb{C}$, with a fixed number of function evaluations (i.e., dt is chosen so that the total number of function evaluations for the method can be controlled). Then, the accuracy plot for that numerical method is defined to be a contour plot of the error, $e(\lambda)$.

Generating the accuracy plots for different methods, where the same number of function evaluations is used to generate each plot, gives us a way to qualitatively, and quantitatively compare the performance of the different schemes; i.e., for the *same* amount of work, how accurate are the methods?

In Figures 2(a) and 2(b), the accuracy plots for a generic RK4 method are shown generated width 56 and 560 function evaluations respectively. In Figures 2(c) and 2(d), accuracy plots for IDC8-FE are generated with 57 and 570 function evaluations, while Figures 2(e) and 2(f) show the accuracy plots for IDC8-RK4 generated with 57 and 570 function evaluations. For the same number of function evaluations, the accuracy plots show that IDC8-RK4 performs better than RK4 and IDC8-FE. This is particularly evident when $|\lambda| \ll 1$.

7. CONCLUSIONS

In this paper, a local error analysis is given for SDC methods constructed using general high order RK methods and a uniform distribution of quadrature nodes. Similar arguments apply for the deferred correction method. Numerical examples are in agreement with our analysis.

Our analysis does not extend for non-uniform distributions of quadrature nodes; in fact, preliminary numerical experiments show that the accuracy order of an SDC method, constructed with a non-uniform distribution of quadrature nodes, doesn't always increase with r orders, even when an rth order RK method is applied. Investigations on using non-uniform quadrature nodes and other high order methods, e.g., multi-step methods, are on-going research topics.
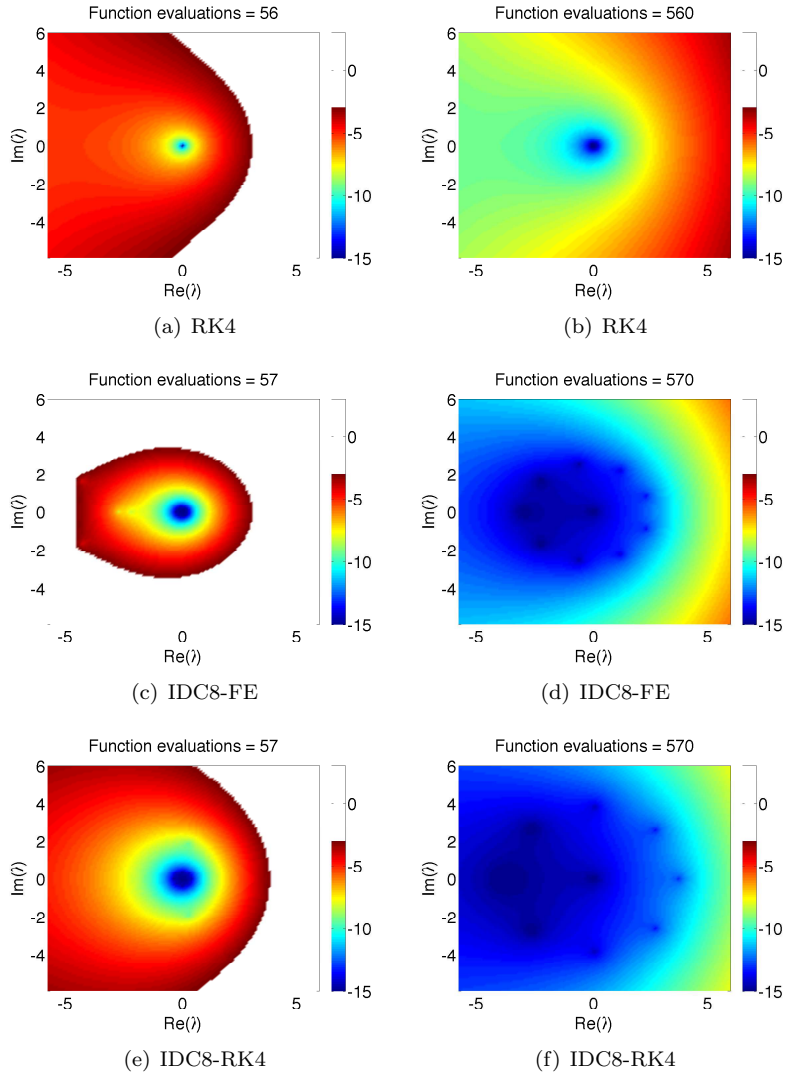
FIGURE 2. The accuracy plots for RK4, IDC8-FE and IDC8-RK4 are generated for ~ 56 function evaluations and ~ 560 function evaluations.

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