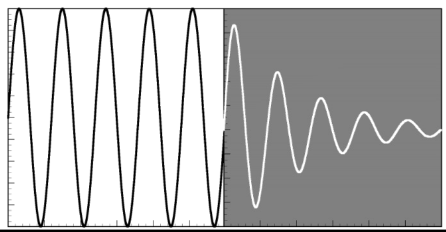
Mathematical description of EM waves

- Use of complex numbers to represent EM waves
- The complex refractive index
  - Scattering = real part
  - Absorption = imaginary part
- Absorption and skin depth
  - Beer's Law



1

What is a wave?

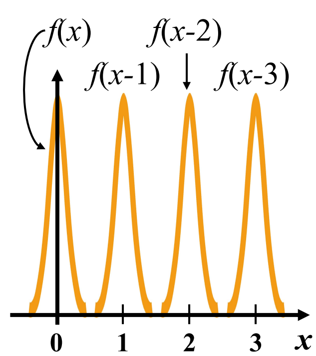
A wave is anything that moves.

To displace any function $f(x)$ to the right, just change its argument from x to $x-a$, where a is a positive number.

If we let $a = v\,t$, where v is positive and t is time, then the displacement will increase with time.

So $f(x - v\,t)$ represents a rightward, or forward, propagating wave.

Similarly, $f(x + v\,t)$ represents a leftward, or backward, propagating wave, where v is the velocity of the wave.



For an EM wave, we could have $\mathbf{E} = f(x \pm vt)$

2

The one-dimensional wave equation

The one-dimensional wave equation for scalar (i.e., non-vector) functions, f:

$$\frac{\partial^2 f}{\partial x^2} - \frac{1}{\mathrm{v}^2}\frac{\partial^2 f}{\partial t^2} = 0$$

where v will be the velocity of the wave.

The wave equation has the simple solution:

$$f(x,t) = f(x \pm \mathrm{v}t)$$

where $f(u)$ can be any twice-differentiable function.

3

What about a harmonic wave?

$$E = E_0 \cos k(x - ct)$$

E_0 = wave amplitude (related to the energy carried by the wave).

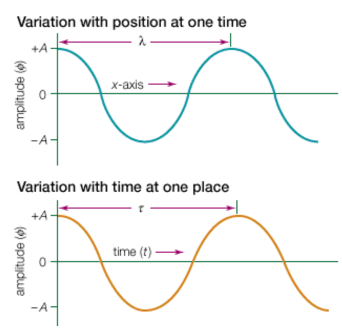
$k = \frac{2\pi}{\lambda} = 2\pi\tilde{\nu}$ = angular wavenumber

(λ = wavelength; $\tilde{\nu}$ = wavenumber = $1/\lambda$)

Alternatively:

$$E = E_0 \cos(kx - \omega t)$$

Where $\omega = kc = 2\pi c/\lambda = 2\pi f$ = angular frequency (f = frequency)

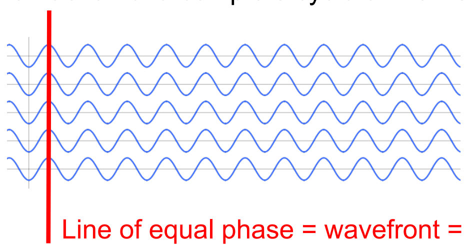


5

What about a harmonic wave?

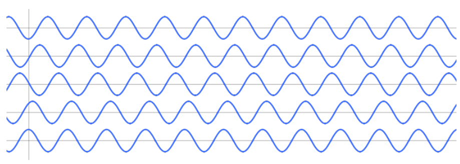
$$E = E_0 \cos k(x - ct); \quad \phi = k(x - ct)$$

The argument of the cosine function represents the *phase* of the wave, ϕ, or the fraction of a complete cycle of the wave.



In-phase waves

Line of equal phase = wavefront = contours of maximum field



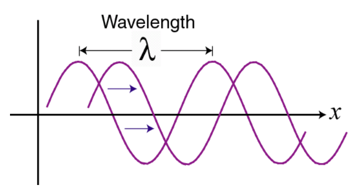
Out-of-phase waves

6

The Phase Velocity

How fast is the wave traveling?

Velocity is a reference distance divided by a reference time.



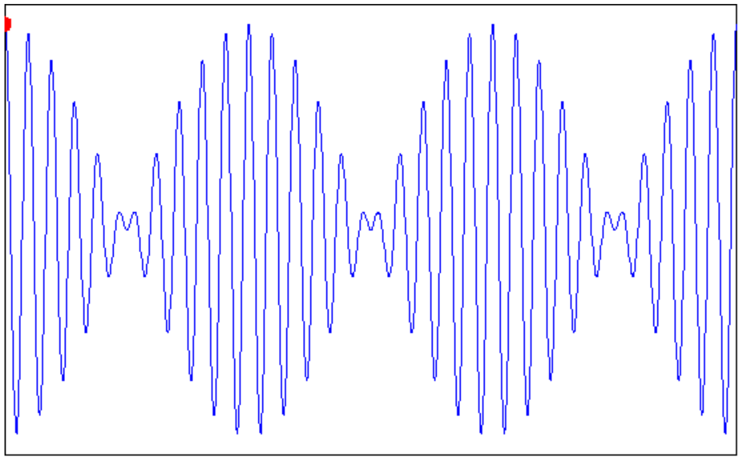
The phase velocity is the wavelength / period: $v = \lambda / \tau$

Since $f = 1/\tau$: $v = \lambda f$

In terms of k, $k = 2\pi / \lambda$, and the angular frequency, $\omega = 2\pi / \tau$, this is: $v = \omega / k$

7

The Group Velocity

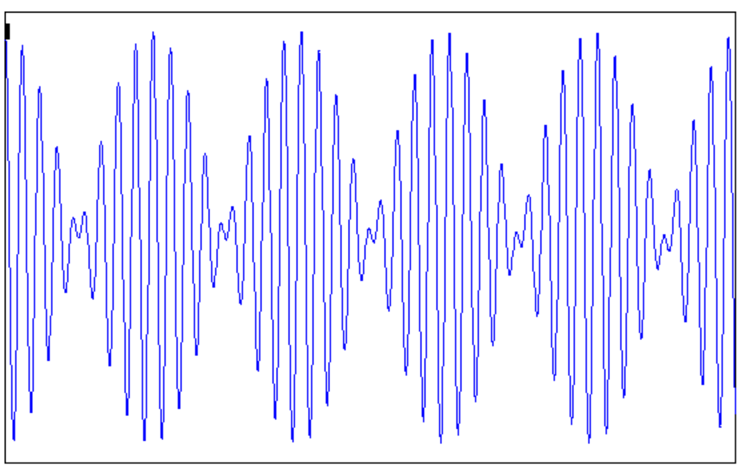


This is the velocity at which the overall shape of the wave's amplitudes, or the wave 'envelope', propagates. (= *signal velocity*)

Here, phase velocity = group velocity (the medium is *non-dispersive*)

8

Dispersion: phase/group velocity depends on frequency

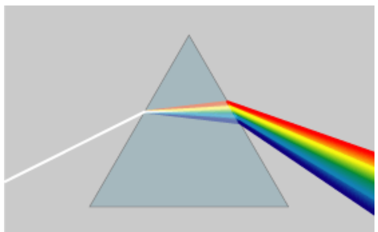


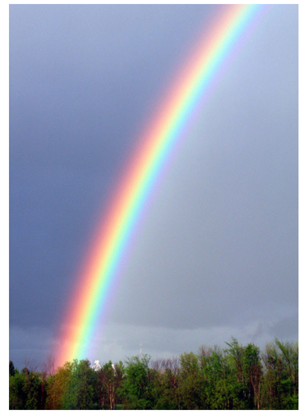
Black dot moves at phase velocity. Red dot moves at group velocity.

This is *normal dispersion* (refractive index *decreases with increasing* λ)

9

Normal dispersion of visible light



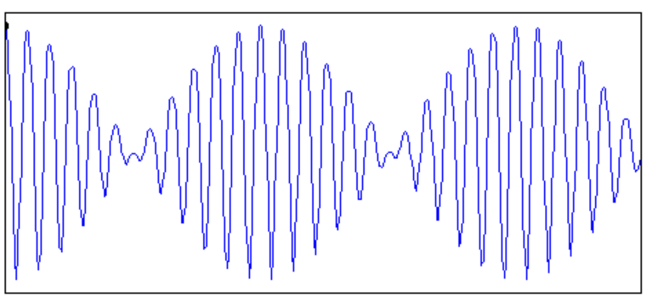


Shorter (blue) wavelengths refracted more than long (red) wavelengths.

Refractive index of blue light > red light.

10

Dispersion: phase/group velocity depends on frequency



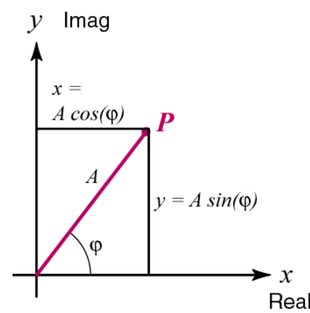
Black dot moves at group velocity. Red dot moves at phase velocity.

This is *anomalous dispersion* (refractive index *increases with increasing* λ)

11

Complex numbers

Consider a point, $P = (x,y)$, on a 2D Cartesian grid.



Let the x-coordinate be the real part and the y-coordinate the imaginary part of a complex number.

So, instead of using an ordered pair, (x,y), we write:

$$P = x + i\,y$$
$$= A\cos(\varphi) + i\,A\sin(\varphi)$$

where $i = \sqrt{(-1)}$ …or sometimes j = √(-1)

12

Euler's Formula

Links the trigonometric functions and the complex exponential function

$$\exp(i\varphi) = \cos(\varphi) + i\sin(\varphi)$$

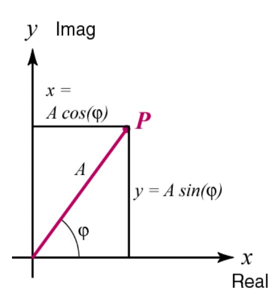
so the point, $P = A\cos(\varphi) + i\,A\sin(\varphi)$, can also be written:

$$P = A\exp(i\varphi) = A\,e^{i\varphi}$$

where

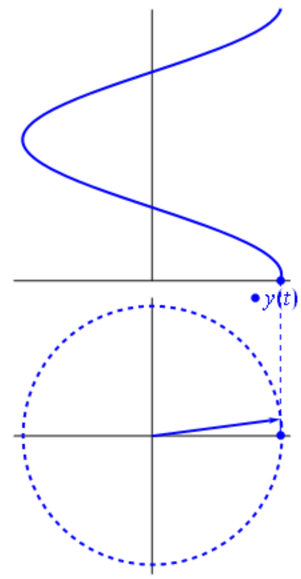
A = Amplitude

φ = Phase



13

Waves as rotating vectors



14

Waves using complex numbers

$$E = E_0 \cos k(x - ct); \quad \phi = k(x - ct)$$

The argument of the cosine function represents the *phase* of the wave, ϕ, or the fraction of a complete cycle of the wave.

Using complex numbers, we can write the harmonic wave equation as:

$$E = E_0 e^{ik(x-ct)} = E_0 e^{i(kx-\omega t)}$$

i.e., $E = E_0 \cos(\varphi) + i\, E_0 \sin(\varphi)$, where the 'real' part of the expression actually represents the wave.

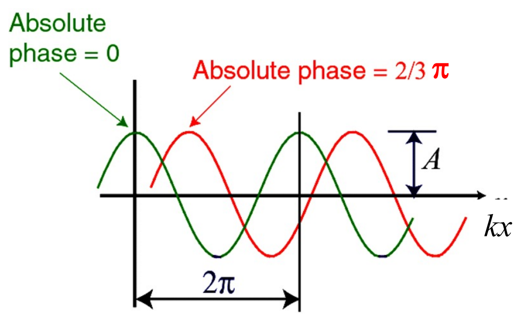
We also need to specify the displacement E at x = 0 and t = 0, i.e., the 'initial' displacement.

15

Amplitude and Absolute phase

$$E(x,t) = E_0 \cos[(k\,x - \omega\,t) - \theta]$$

E_0 = Amplitude
θ = Absolute phase (or initial, constant phase) at x = 0, t = 0



16

Waves using complex numbers

So the electric field of an EM wave can be written:

$$E(x,t) = E_0 \cos(kx - \omega t - \theta)$$

Since $\exp(i\varphi) = \cos(\varphi) + i\sin(\varphi)$, $E(x,t)$ can also be written:

$$E(x,t) = \mathrm{Re}\,\{\,E_0 \exp[i(kx - \omega t - \theta)]\,\}$$

Recall that the energy transferred by a wave (*flux density*) is proportional to the square of the amplitude, i.e., E_0^2. Only the interaction of the wave with matter can alter the energy of the propagating wave.

Remote sensing exploits this modulation of energy.

20

Waves using complex amplitudes

We can let the amplitude be complex:

$$E(x,t) = E_0 \exp[i(kx - \omega t - \theta)]$$

$$E(x,t) = \left[E_0 \exp(-i\theta)\right] \exp[i(kx - \omega t)]$$

Where the constant stuff is separated from the rapidly changing stuff.

The resulting "complex amplitude": $\left[E_0 \exp(-i\theta)\right]$
is constant in this case (as E_0 and θ are constant), which implies that the medium in which the wave is propagating is *nonabsorbing.*

What happens to the wave amplitude upon interaction with matter?

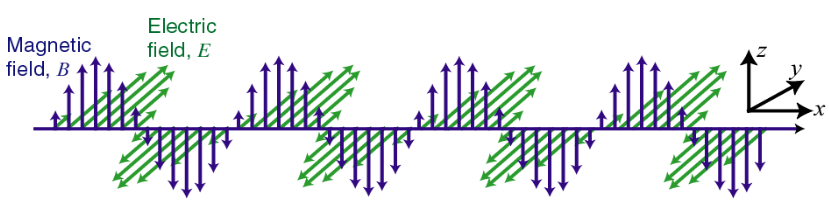
21

Vector fields

We also need to account for the fact that light is a 3D vector field.

A 3D vector field assigns a 3D vector (i.e., an arrow having both direction and length) to each point in 3D space.

A light wave has both electric and magnetic 3D vector fields:



And it can propagate in any direction, and point in any direction in space.

23

The 3D wave equation for the electric field and its solution

A light wave can propagate in any direction in space. So we must allow the space derivative to be 3D:

$$\vec{\nabla}^2 E - \mu\varepsilon \frac{\partial^2 E}{\partial t^2} = 0$$

whose solution is: $E(x,y,z,t) = \vec{E}_0 \exp(-\vec{k}''\cdot\vec{x})\exp([i(\vec{k}'\cdot\vec{x} - \omega t)]$

Where $\vec{E}_0$ is a constant, complex vector

And $\vec{k} = \vec{k}' + i\vec{k}''$ is a complex *wave vector* – the length of this vector is inversely proportional to the wavelength of the wave. Its magnitude is the angular wavenumber, k = 2π/λ.

$\vec{x} = (x,y,z)$ is a position vector

24

The 3D wave equation for the electric field and its solution

$$E(x,y,z,t) = \vec{E}_0 \exp(-\vec{k}''\cdot\vec{x})\exp([i(\vec{k}'\cdot\vec{x} - \omega t)]$$

The vector $\vec{k}'$ is normal to planes of constant phase (and hence indicates the direction of propagation of wave crests)

The vector $\vec{k}''$ is normal to planes of constant amplitude. Note that these are not necessarily parallel.

The amplitude of the wave at location $\vec{x}$ is now: $\vec{E}_0 \exp(-\vec{k}''\cdot\vec{x})$

So if $\vec{k}''$ is zero, then the medium is nonabsorbing, since the amplitude is constant.

25

EM propagation in homogeneous materials

The speed of an EM wave in free space is given by: $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = \frac{\omega}{k}$

ε_0 = permittivity of free space, μ_0 = magnetic permeability of free space

To describe EM propagation in other media, two properties of the medium are important, its *electric permittivity* ε and *magnetic permeability* μ. These are also complex parameters.

$\varepsilon = \varepsilon_0(1+ \chi) + \mathbf{i}\, \sigma/\omega$ = *complex permittivity*

σ = *electric conductivity*
χ = *electric susceptibility* (to polarization under the influence of an external field)

Note that ε and μ also depend on frequency (ω).

26

EM propagation in homogeneous materials

In a non-vacuum, the wave must still satisfy Maxwell's Equations:

$$v = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{\omega}{k}$$

We can now define the *complex index of refraction,* N, as the ratio of the wave velocity in free space to the velocity in the medium:

$$N = \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} = \frac{c}{v} \quad (n_i = 0) \;\; or \quad N = n_r + i\,n_i$$

If the imaginary part of N is zero, the material is *nonabsorbing,* and v is the phase velocity of the wave in the medium. For most physical media, N > 1 (i.e., the speed of light is reduced relative to a vacuum).

NB. N is a property of a particular medium and also a function of ω

27

EM propagation in homogeneous materials

Relationships between the wave vector and the refractive index (these are derived from Maxwell's Equations):

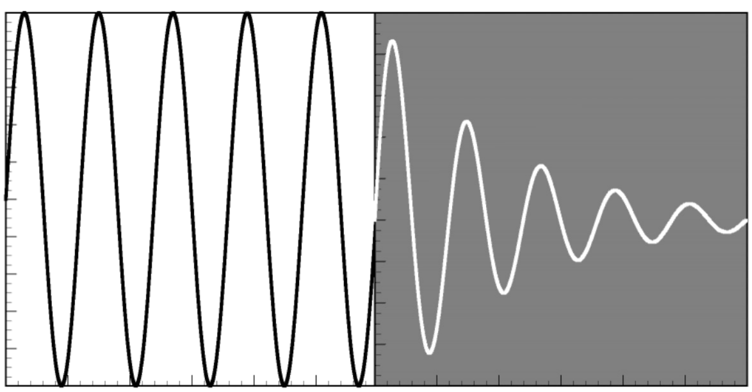
$$k' = \frac{\omega n_r}{c} = \frac{2\pi}{\lambda}$$ Real part of wave vector

$$k'' = \frac{\omega n_i}{c} = \frac{2\pi \nu n_i}{c}$$ Imaginary part of wave vector

These are the so-called 'dispersion relations' relating wavelength, frequency, velocity and refractive index.

28

Absorption of EM radiation



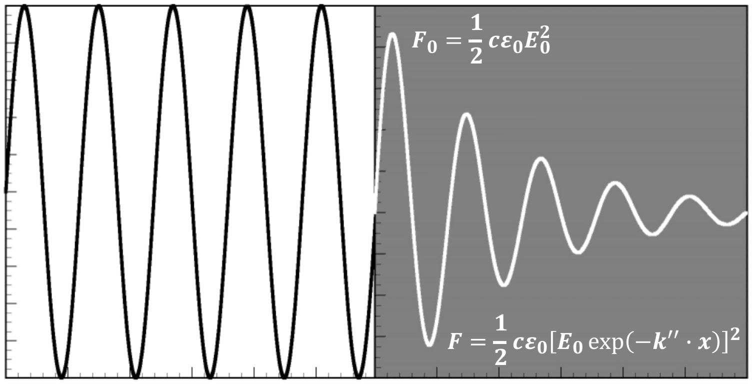
Recall the expression for the flux density of an EM wave (Poynting vector):

$$F = \frac{1}{2} c \varepsilon_0 E^2$$

When absorption occurs, the flux density of the absorbed frequencies is reduced.

29

Absorption of EM radiation



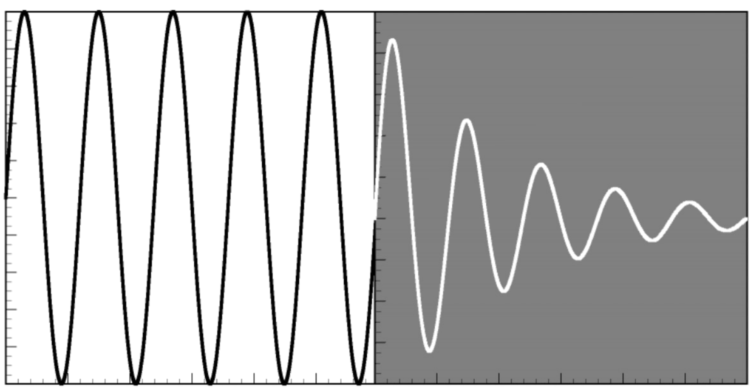
The scalar amplitude of an EM wave at location $\vec{x}$ is: $\vec{E}_0 \exp(-\vec{k}'' \cdot \vec{x})$

From the expression for the flux density we have:

$$F = F_0 \left[\exp(-\vec{k}'' \cdot \vec{x}) \right]^2 = F_0 \exp(-2\vec{k}'' \cdot \vec{x})$$

30

Absorption of EM radiation



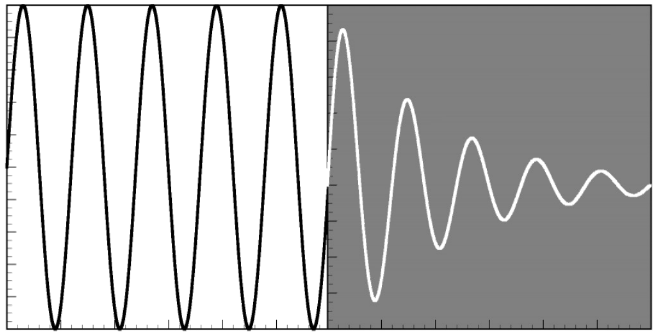
Now substitute the expression for k'': $k'' = \frac{\omega n_i}{c} = \frac{2\pi \nu n_i}{c}$

And we have: $F = F_0 \exp(-\frac{4\pi \nu n_i}{c} x)$

For a plane wave propagating in the x-direction.

31

Absorption coefficient and skin depth



$$F = F_0 \exp(-\frac{4\pi \nu n_i}{c} x) = F_0 e^{-\beta_a x}$$

Where β_a is known as the **absorption coefficient:** $\frac{4\pi \nu n_i}{c} = \frac{4\pi n_i}{\lambda}$

The quantity $1/\beta_a$ gives the distance required for the wave's energy to be attenuated to e^{-1} or ~37% of its original value, or the absorption/skin depth. It's a function of frequency/wavelength.

32

Absorption coefficient and skin depth

Within a certain material, an EM wave with λ = 1 μm is attenuated to 10% of its original intensity after propagating 1 cm. Determine the imaginary part of the refractive index n_i.

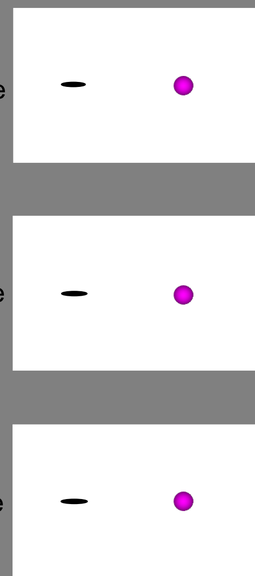
33

The Forced Oscillator: The relative phase of the oscillator motion with respect to the input force depends on the frequencies.

Let the oscillator's resonant frequency be ω_0, and the forcing frequency be ω.

We could think of the forcing function as a light electric field and the oscillator as a nucleus of an atom in a molecule.

	Force / Oscillator	
Below resonance $\omega << \omega_0$		Weak vibration. In phase.
On resonance $\omega = \omega_0$		Strong vibration. 90° out of phase.
Above resonance $\omega >> \omega_0$		Weak vibration. 180° out of phase.



Courtesy Prof. Rick Trebino, Georgia Tech

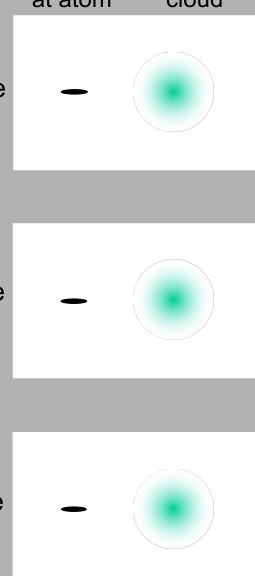
34

The relative phase of an electron cloud's motion with respect to input light depends on the frequency.

Let the atom's resonant frequency be ω_0, and the light frequency be ω.

The electron charge is negative, so there's a 180° phase shift in all cases (compared to the previous slide's plots).

	Electric field at atom / Electron cloud	
Below resonance $\omega << \omega_0$		Weak vibration. 180° out of phase.
On resonance $\omega = \omega_0$		Strong vibration. -90° out of phase.
Above resonance $\omega >> \omega_0$		Weak vibration. In phase.

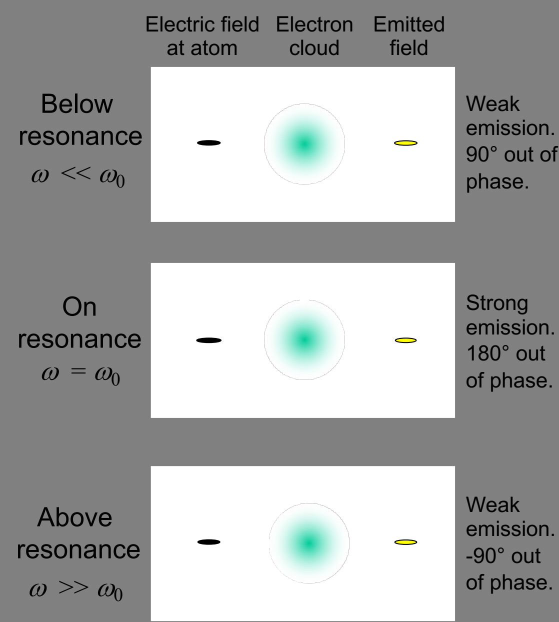


Courtesy Prof. Rick Trebino, Georgia Tech

35

The relative phase of emitted light with respect to the input light depends on the frequency.

The emitted light is 90° phase-shifted with respect to the atom's motion.



Courtesy Prof. Rick Trebino, Georgia Tech

36

Refractive index (n) – the dispersion equation

$$n = 1 + \frac{q_e^2}{2\varepsilon_0 m} \sum_k \frac{N_k}{\omega_k^2 - \omega^2 + i\gamma_k \omega}$$

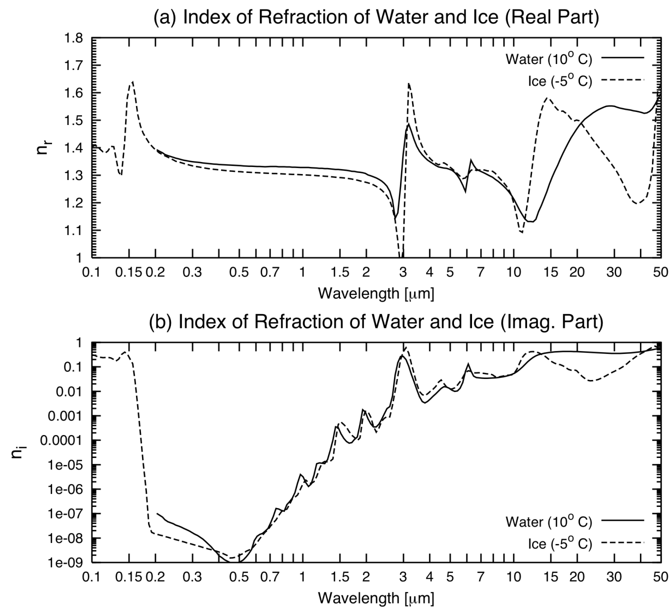
(Feynman, 1963)

Lorenz Harmonic oscillator model
q_e = charge on an electron
ε_0 = electric constant
m = mass of an electron
N_k = number of charges (oscillators) of type *k* per unit volume
ω = angular frequency of the EM radiation
ω_k = resonant frequency of an electron bound in an atom
γ = 'damping coefficient' for oscillator *k* (oscillation cannot be permanent)

What is the refractive index of visible light in air?
What happens as the frequency of EM radiation increases at constant ω_k?
What happens if the resonant frequency is in the visible range?
What happens if $\omega > \omega_k$? e.g., shine x-rays on glass, or radio waves on free electrons.

37

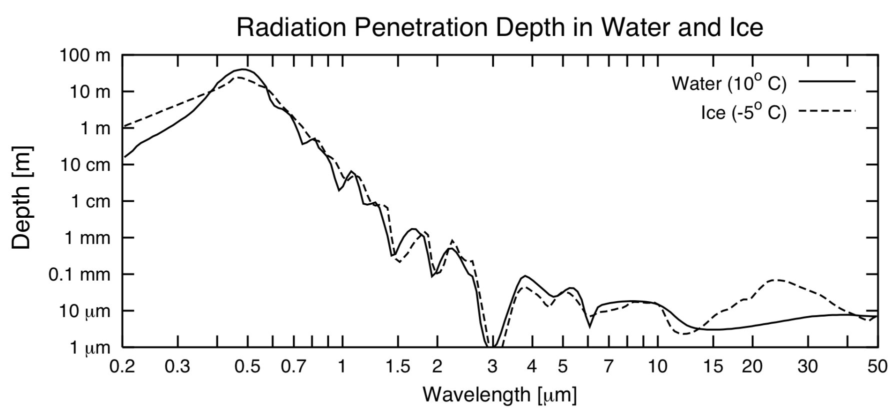
Refractive index (n) of water and ice



38

Penetration depth of water and ice

(also called absorption depth or skin depth)



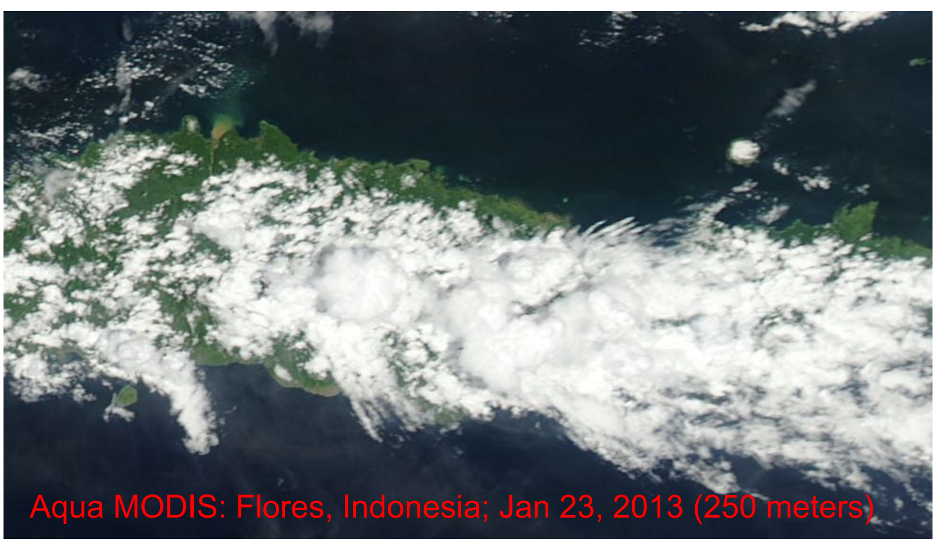
39

Refractive index and color

Aqua MODIS: Flores, Indonesia; Feb 2, 2013 (250 meters)

40

Color of suspended sediment



Aqua MODIS: Flores, Indonesia; Jan 23, 2013 (250 meters)

41

Color of suspended sediment

42

Complex refractive index of volcanic ash

Table 2. Refractive Indices of Volcanic Aerosol Models

Wavelength, nm	El Chichon Ash, April 6, 1982 [*Patterson et al.*, 1983]	WMO Volcanic Ash [*WMO*, 1986]	St. Helens Volcanic Ash [*Patterson*, 1981]	Fuego Volcanic Ash [*Patterson et al.*, 1983]	Sulfate Aerosol [*Bhartia et al.*, 1993]
312.5	1.5–0.003*i*	1.5–0.0093*i*	1.5–0.018*i*	1.5–0.0734*i*	1.466–0.0*i*
317.5	1.5–0.0027*i*	1.5–0.0091*i*	1.5–0.012*i*	1.5–0.0709*i*	1.464–0.0*i*
331.25	1.5–0.0025*i*	1.5–0.0083*i*	1.5–0.010*i*	1.5–0.0644*i*	1.461–0.0*i*
339.68	1.5–0.0023*i*	1.5–0.0080*i*	1.5–0.0080*i*	1.5–0.0610*i*	1.458–0.0*i*
360.0	1.5–0.0020*i*	1.5–0.0080*i*	1.5–0.0050*i*	1.5–0.0581*i*	1.452–0.0*i*
380.0	1.5–0.0018*i*	1.5–0.0080*i*	1.5–0.0040*i*	1.5–0.0460*i*	1.446–0.0*i*
500.0	1.5–0.0015*i*	1.5–0.0080*i*	1.5–0.0030*i*	1.5–0.020*i*	1.432–0.0*i*

- The complex refractive index indicates the relative importance of scattering (real part) and absorption (imaginary part) in a medium

From: Krotkov et al. (1997), Ultraviolet optical model of volcanic clouds for remote sensing of ash and sulfur dioxide. *J. Geophys. Res.*, 102 (D18), 21891-21904.

43