Introduction to Electromagnetic Theory

Lecture topics

• Laws of magnetism and electricity
• Meaning of Maxwell’s equations
• Solution of Maxwell's equations

Electromagnetic radiation: wave model

• James Clerk Maxwell (1831-1879) – Scottish mathematician and physicist
• Wave model of EM energy
  • Unified existing laws of electricity and magnetism (Newton, Faraday, Kelvin, Ampère)
  • Oscillating electric field produces a magnetic field (and vice versa) – propagates an EM wave
  • Can be described by 4 differential equations
  • Derived speed of EM wave in a vacuum
  • ‘Speed of light’ measured by Fizeau and Foucault between 1849 and 1862
Electromagnetic radiation

- EM wave is:
  - Electric field (E) perpendicular to magnetic field (M)
  - Travels at velocity, $c \approx 3\times10^8$ m s$^{-1}$, in a vacuum

Dot (scalar) product

If $\mathbf{A}$ is perpendicular to $\mathbf{B}$, the dot product of $\mathbf{A}$ and $\mathbf{B}$ is zero
Cross (vector) product

\[ \mathbf{a} \times \mathbf{b} = [(a_2 b_3 - a_3 b_2), (a_3 b_1 - a_1 b_3), (a_1 b_2 - a_2 b_1)] \]

\[ \mathbf{a} \times \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \sin \theta \hat{n} \]

If \( \mathbf{a} \) is parallel to \( \mathbf{b} \), the cross product of \( \mathbf{a} \) and \( \mathbf{b} \) is zero

Div, Grad, Curl

Types of 3D vector derivatives:

The **Del** operator:

\[ \vec{\nabla} = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \]

The **Gradient** of a scalar function \( f \) (vector):

\[ \vec{\nabla}f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \]

The gradient points in the direction of steepest ascent.
Div, Grad, Curl

The Divergence of a vector function (scalar):
\[ \nabla \cdot \mathbf{f} = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z} \]

The Divergence is nonzero if there are sources or sinks.

A 2D source with a large divergence:

Div, Grad, Curl

The Curl of a vector function \( \mathbf{f} \):
\[ \nabla \times \mathbf{f} = \left( \frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}, \frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}, \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y} \right) \]

Functions that tend to curl around have large curls.

http://mathinsight.org/curl_idea
Div, Grad, Curl

The Laplacian of a scalar function:

\[ \nabla^2 f = \nabla \cdot \nabla f = \frac{\partial f}{\partial x} \frac{\partial}{\partial x} + \frac{\partial f}{\partial y} \frac{\partial}{\partial y} + \frac{\partial f}{\partial z} \frac{\partial}{\partial z} \]

The Laplacian of a vector function is the same, but for each component of \( f \):

\[ \nabla^2 f = \left( \frac{\partial^2 f_x}{\partial x^2} + \frac{\partial^2 f_y}{\partial y^2} + \frac{\partial^2 f_z}{\partial z^2} \right) \left( \frac{\partial^2 f_y}{\partial x^2} + \frac{\partial^2 f_y}{\partial y^2} + \frac{\partial^2 f_z}{\partial z^2} \right) + \frac{\partial^2 f_z}{\partial x^2} + \frac{\partial^2 f_z}{\partial y^2} + \frac{\partial^2 f_z}{\partial z^2} \]

The Laplacian tells us the curvature of a vector function.

Maxwell’s Equations

- Four equations relating electric (\( E \)) and magnetic fields (\( B \)) – vector fields
  \[ \nabla \cdot E = \frac{\rho}{\varepsilon_0} \]
  \[ \nabla \cdot B = 0 \]
  \[ \nabla \times E = -\frac{\partial B}{\partial t} \]
  \[ \nabla \times B = \mu_0 J + \varepsilon_0 \mu_0 \frac{\partial E}{\partial t} \]

- \( \varepsilon_0 \) is electric permittivity of free space (or vacuum permittivity - a constant) – resistance to formation of an electric field in a vacuum
  \( \varepsilon_0 = 8.854188 \times 10^{-12} \text{ Farad m}^{-1} \)

- \( \mu_0 \) is magnetic permeability of free space (or magnetic constant - a constant) – ability of a vacuum to support formation of a magnetic field
  \( \mu_0 = 1.2566 \times 10^{-6} \text{ T m A}^{-1} \) (T = Tesla; SI derived unit of magnetic field)

Note: \( \nabla \cdot \) is ‘divergence’ operator and \( \nabla \times \) is ‘curl’ operator
Biot-Savart Law (1820)

- Jean-Baptiste Biot and Felix Savart (French physicist and chemist)
- The magnetic field $B$ at a point a distance $R$ from an infinitely long wire carrying current $I$ has magnitude:
  \[ B = \frac{\mu_0 I}{2\pi R} \]
- Where $\mu_0$ is the magnetic permeability of free space or the magnetic constant
- Constant of proportionality linking magnetic field and distance from a current
- Magnetic field strength decreases with distance from the wire
- $\mu_0 = 1.2566 \times 10^{-6}$ T m A$^{-1}$ (T = Tesla; SI derived unit of magnetic field)

Coulomb’s Law (1783)

- Charles Augustin de Coulomb (French physicist)
- The magnitude of the electrostatic force ($F$) between two point electric charges ($q_1$, $q_2$) is given by:
  \[ F = \frac{q_1 q_2}{4\pi \varepsilon_0 r^2} \]
- Where $\varepsilon_0$ is the electric permittivity or electric constant
- Like charges repel, opposite charges attract
- $\varepsilon_0 = 8.854 \times 10^{-12}$ Farad m$^{-1}$
Maxwell’s Equations (1)

\[ \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \]

- Gauss’ law for electricity: the electric flux out of any closed surface is proportional to the total charge enclosed within the surface; i.e. a charge will radiate a measurable field of influence around it.

- \( \mathbf{E} \) = electric field, \( \rho \) = net charge inside, \( \varepsilon_0 \) = vacuum permittivity (constant)

- Recall: divergence of a vector field is a measure of its tendency to converge on or repel from a point.

- Direction of an electric field is the direction of the force it would exert on a positive charge placed in the field

- If a region of space has more electrons than protons, the total charge is negative, and the direction of the electric field is negative (inwards), and vice versa.

Maxwell’s Equations (2)

\[ \nabla \cdot \mathbf{B} = 0 \]

- Gauss’ law for magnetism: the net magnetic flux out of any closed surface is zero (i.e. magnetic monopoles do not exist)

- \( \mathbf{B} \) = magnetic field; magnetic flux = \( \mathbf{B} \mathbf{A} \) (\( \mathbf{A} \) = area perpendicular to field \( \mathbf{B} \))

- Recall: divergence of a vector field is a measure of its tendency to converge on or repel from a point.

- Magnetic sources are dipole sources and magnetic field lines are loops – we cannot isolate N or S ‘monopoles’ (unlike electric sources or point charges – protons, electrons)

- Magnetic monopoles could theoretically exist, but have never been observed
Maxwell’s Equations (3)

\[ \nabla \times E = -\frac{\partial B}{\partial t} \]

- Faraday’s Law of Induction: the curl of the electric field (\( E \)) is equal to the negative of rate of change of the magnetic flux through the area enclosed by the loop.
- \( E \) = electric field; \( B \) = magnetic field
- Recall: curl of a vector field is a vector with magnitude equal to the maximum ‘circulation’ at each point and oriented perpendicularly to this plane of circulation for each point.
- Magnetic field weakens \( \Rightarrow \) curl of electric field is positive and vice versa.
- Hence, changing magnetic fields affect the curl (‘circulation’) of the electric field – basis of electric generators (moving magnet induces current in a conducting loop).

Maxwell’s Equations (4)

\[ \nabla \times B = \mu_0 J + \varepsilon_0 \mu_0 \frac{\partial E}{\partial t} \]

- Ampère’s Law: the curl of the magnetic field (\( B \)) is proportional to the electric current flowing through the loop AND to the rate of change of the electric field. \( \leftarrow \) added by Maxwell
- \( B \) = magnetic field; \( J \) = current density (current per unit area); \( E \) = electric field
- The curl of a magnetic field is basically a measure of its strength.
- First term on RHS: in the presence of an electric current (\( J \)), there is always a magnetic field around it; \( B \) is dependent on \( J \) (e.g., electromagnets).
- Second term on RHS: a changing electric field generates a magnetic field.
- Therefore, generation of a magnetic field does not require electric current, only a changing electric field. An oscillating electric field produces a variable magnetic field (as \( \delta E/\delta t \) changes).
Putting it all together….

- An oscillating electric field produces a variable magnetic field. A changing magnetic field produces an electric field….and so on.
- In ‘free space’ (vacuum) we can assume current density (J) and charge (ρ) are zero i.e. there are no electric currents or charges.
- Equations become:

\[
\nabla \cdot E = 0 \\
\nabla \cdot B = 0 \\
\n\nabla \times E = -\frac{\partial B}{\partial t} \\
\n\nabla \times B = \varepsilon_0 \mu_0 \frac{\partial E}{\partial t}
\]

Solving Maxwell’s Equations

Take curl of:

\[
\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \\
\n\nabla \times [\nabla \times \vec{E}] = \nabla \times [-\frac{\partial \vec{B}}{\partial t}]
\]

Change the order of differentiation on the RHS:

\[
\nabla \times [\nabla \times \vec{E}] = -\frac{\partial}{\partial t} [\nabla \times \vec{B}]
\]
Solving Maxwell’s Equations (cont’d)

But (Equation 4):
\[ \vec{\nabla} \times \vec{B} = \mu \varepsilon \frac{\partial \vec{E}}{\partial t} \]

Substituting for \( \vec{\nabla} \times \vec{B} \), we have:
\[ \vec{\nabla} \times [\vec{\nabla} \times \vec{E}] = -\frac{\partial}{\partial t}[\vec{\nabla} \times \vec{B}] \Rightarrow \vec{\nabla} \times [\vec{\nabla} \times \vec{E}] = -\frac{\partial}{\partial t} [\mu \varepsilon \frac{\partial \vec{E}}{\partial t}] \]

Or:
\[ \vec{\nabla} \times [\vec{\nabla} \times \vec{E}] = -\mu \varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} \quad \text{assuming that } \mu \text{ and } \varepsilon \text{ are constant in time.} \]

Solving Maxwell’s Equations (cont’d)

Identity:
\[ \vec{\nabla} \times [\vec{\nabla} \times \vec{f}] = \vec{\nabla} (\vec{\nabla} \cdot \vec{f}) - \nabla^2 \vec{f} \]

Using the identity, \[ \vec{\nabla} \times [\vec{\nabla} \times \vec{E}] = -\mu \varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} \]

becomes:
\[ \vec{\nabla} (\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} = -\mu \varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} \]

Assuming zero charge density (free space; Equation 1):
\[ \vec{\nabla} \cdot \vec{E} = 0 \]

and we’re left with:
\[ \nabla^2 \vec{E} = \mu \varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} \]
Solving Maxwell’s Equations (cont’d)

\[ \nabla^2 \vec{E} = \mu \varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} \quad \nabla^2 \vec{B} = \mu \varepsilon \frac{\partial^2 \vec{B}}{\partial t^2} \]

The same result is obtained for the magnetic field B.
These are forms of the 3D wave equation, describing the propagation of a sinusoidal wave:

\[ \nabla^2 u = \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2} \]

Where \( v \) is a constant equal to the propagation speed of the wave

So for EM waves, \( v = \sqrt{\frac{1}{\mu \varepsilon}} \)

Solving Maxwell’s Equations (cont’d)

So for EM waves, \( v = \sqrt{\frac{1}{\mu \varepsilon}} \).

Units of \( \mu = \text{T} \cdot \text{m/A} \)
The Tesla (T) can be written as \( \text{kg} \cdot \text{A}^{-1} \cdot \text{s}^{-2} \)
So units of \( \mu \) are \( \text{kg} \cdot \text{m} \cdot \text{A}^{-2} \cdot \text{s}^{-2} \)

Units of \( \varepsilon = \text{Farad} \cdot \text{m}^{-1} \) or \( \text{A}^2 \cdot \text{s}^4 \cdot \text{kg}^{-1} \cdot \text{m}^{-3} \) in SI base units
So units of \( \mu \varepsilon \) are \( \text{m}^2 \cdot \text{s}^2 \)
Square root is \( \text{m}^{-1} \) s, reciprocal is \( \text{m} \cdot \text{s}^{-1} \) (i.e., velocity)
\( \varepsilon_0 = 8.854188 \times 10^{-12} \) and \( \mu_0 = 1.2566371 \times 10^{-6} \)

Evaluating the expression gives \( 2.998 \times 10^8 \) m \( \cdot \) s\(^{-1} \)

Maxwell (1865) recognized this as the (known) speed of light – confirming that light was in fact an EM wave.
EM waves carry energy – how much?

e.g., from the Sun to the vinyl seat cover in your parked car….

The energy flow of an electromagnetic wave is described by the **Poynting vector**:

\[ \vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} \]

The intensity (I) of a time-harmonic electromagnetic wave whose electric field amplitude is \( E_0 \), measured normal to the direction of propagation, is the average over one complete cycle of the wave:

\[ I = \frac{P}{A} = \frac{S_{\text{avg}}}{c} = \frac{1}{2\varepsilon_0} \frac{E_0^2}{c} = \frac{c\varepsilon_0}{2} \frac{E_0^2}{2} \, \text{WATTS/M}^2 \]

P = Power; A = Area; c = speed of light

**Key point:** intensity is proportional to the **square** of the amplitude of the EM wave

**NB.** \( \text{Intensity} = \text{Flux density} (F) = \text{Irradiance (incident)} = \text{Radiant Exitance (emerging)} \)

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**Summary**

- Maxwell unified existing laws of electricity and magnetism
- Revealed self-sustaining properties of magnetic and electric fields
- Solution of Maxwell’s equations is the three-dimensional wave equation for a wave traveling at the speed of light
- Proved that light is an electromagnetic wave
- EM waves carry energy through empty space and all remote sensing techniques exploit the modulation of this energy
Summary

- EM wave propagation: