Monte Carlo Methods

I, at any rate, am convinced that He does not throw dice.

Albert Einstein
Pseudo Random Numbers: 1/3

- Random numbers are numbers occur in a “random” way.
- If they are generated by an algorithm, they are not actually very random. Hence, they are usually referred to as *pseudo random* numbers.
- In Fortran 90, two subroutines help generate random numbers: `RANDOM_SEED()` and `RANDOM_NUMBER()`. 
- The generated random numbers are *uniform* because the probability to get each of these numbers is equal.
Pseudo Random Numbers: 2/3

- `RANDOM_SEED()` must be called, with or without actual arguments, before any use of `RANDOM_NUMBER()` or before you wish to “re-seed” the random number sequence.

- `RANDOM_NUMBER(x)` takes a `REAL` actual argument, which is a variable or an array element. The generated random number is returned with this argument.

- The generated random number is in \([0,1)\). Scaling and translation may be needed.
Simulate the throwing of two dice \( n \) times.

Array \( \text{count}() \) of 12 elements is initialized to 0, and \( p \) and \( q \) are the “random” numbers representing throwing two dice.

What does \( \text{INT}(6x) + 1 \) mean?

```
CALL RANDOM_SEED()
DO i = 1, n
   CALL RANDOM_NUMBER(x)
p = INT(6*x) + 1
   CALL RANDOM_NUMBER(x)
q = INT(6*x) + 1
   count(p+q) = count(p+q) + 1
END DO
WRITE(*,*) (count(i), i=1, 12)
```
Monte Carlo Methods

Monte Carlo techniques have their origin in WW2. Scientists found out problems in neutron diffusion were intractable by conventional methods and a probabilistic approach was developed.

Then, it was found that this probabilistic approach could be used to solve deterministic problems. In particular, it is useful in evaluating integrals of multiple dimensions.
Computing $\pi$: 1/3

- The unit circle (i.e., radius = 1) has an area of $\pi$.
- Consider the area in the first quadrant as shown below. Its area is $\pi/4 \approx 0.785398\ldots$
- If we generate $n$ pairs of random numbers $(x, y)$, representing $n$ points in the unit square, and count the pairs in the circle, say $k$, the area is approximately $k/n$. 

![Diagram of a unit circle and random points]
Computing $\pi$: $2/3$

- In the following, $n$ is the number of random number pairs to be generated, $\text{count}$ counts the number of pairs in the circle, and $r$ is the ratio.
- Hence, $r \approx \pi/4$ if enough number of $(x,y)$ pairs are generated.

```fortran
count = 0
CALL RANDOM_SEED
DO i = 1, n
    CALL RANDOM_NUMBER(x)
    CALL RANDOM_NUMBER(y)
    IF (x*x + y*y < 1.0)  count = count + 1
END DO
r = REAL(count)/n
```
The following shows some results.

Due to randomness, the results may be different if this program is run again.

<table>
<thead>
<tr>
<th>$n$</th>
<th>in circle</th>
<th>ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>9</td>
<td>0.9</td>
</tr>
<tr>
<td>100</td>
<td>72</td>
<td>0.72</td>
</tr>
<tr>
<td>1000</td>
<td>804</td>
<td>0.804</td>
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<td>0.7916</td>
</tr>
<tr>
<td>100000</td>
<td>78410</td>
<td>0.7841</td>
</tr>
<tr>
<td>1000000</td>
<td>785023</td>
<td>0.7850</td>
</tr>
</tbody>
</table>
Integration: 1/2

- The same idea can be applied to integration.
- Let us integrate $\frac{1}{1+x^2}$ on [0,1]. This function is bounded by the unit square.
- We may generate $n$ random number pairs and count the number of pairs $k$ in the area to be integrated. The ratio $k/n$ is approximately the integration.

$$\int_0^1 \frac{1}{1+x^2} \, dx = \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4}$$
Integration: 2/2

The following shows the results.

$$\text{ratio} \approx \frac{\pi}{4} = 0.785398...$$

<table>
<thead>
<tr>
<th>$n$</th>
<th>in area</th>
<th>ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>9</td>
<td>0.9</td>
</tr>
<tr>
<td>100</td>
<td>77</td>
<td>0.77</td>
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<tr>
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<td>781</td>
<td>0.781</td>
</tr>
<tr>
<td>10000</td>
<td>7940</td>
<td>0.794</td>
</tr>
<tr>
<td>100000</td>
<td>78646</td>
<td>0.786</td>
</tr>
<tr>
<td>1000000</td>
<td>784546</td>
<td>0.785</td>
</tr>
</tbody>
</table>

CALL RANDOM_SEED()
count = 0
DO i = 1, n
    CALL RANDOM_NUMBER(x)
    CALL RANDOM_NUMBER(y)
    fx = 1/(1 + x*x)
    IF (y <= fx) count = count + 1
END DO
r = REAL(count)/n
**Buffon Needle Problem: 1/4**

- Suppose the floor is divided into infinite number of parallel lines with a constant gap $G$.
- If we throw a needle of length $L$ to the floor randomly, what is the probability of the needle crossing a dividing line?
- This is the *Buffon needle problem*. The exact probability is \( \frac{2}{\pi} \times \frac{L}{G} \).
- If $L = G = 1$, the probability is $\frac{2}{\pi} \approx 0.63661 \ldots$
Buffon Needle Problem: 2/4

- We need two random numbers: $\theta$ for the angle between the needle and a dividing line, and $d$ the distance from one tip of the needle to the lower dividing line.
- If $d + L \times \sin(\theta)$ is less than 0 or larger than $G$, the needle crosses a dividing line.
Buffon Needle Problem: 3/4

- $\text{gap}$ and $\text{length}$ are gap and needle length.
- The generated random number is scaled by $\text{gap}$ and the angle by $2\pi$.

```fortran
count = 0
DO i = 1, n
    CALL RANDOM_NUMBER(x)
    distance = x*gap        ! distance in $[0, \text{gap})$
    CALL RANDOM_NUMBER(angle)
    angle = angle*2*PI      ! angle in $[0, 2\pi)$
    total = distance + length*sin(angle)
    IF (0 < total .AND. total < gap) count = count + 1
END DO
ratio = REAL(n-count)/n
```
Buffon Needle Problem: 4/4

The following has the simulated results with gap and needle length being 1.

<table>
<thead>
<tr>
<th>n</th>
<th>in area</th>
<th>ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>8</td>
<td>0.8</td>
</tr>
<tr>
<td>100</td>
<td>61</td>
<td>0.61</td>
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<tr>
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<td>0.63607</td>
</tr>
<tr>
<td>1000000</td>
<td>636847</td>
<td>0.63685</td>
</tr>
</tbody>
</table>

Exact value = $\frac{2}{\pi} \approx 0.63661\ldots$
The End