Do Blending and Offsetting Commute for Dupin Cyclides? ⋆

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Abstract

A common method for constructing blending Dupin cyclides for two cones having a common inscribed sphere of radius $r > 0$ involves three steps: (1) computing the $(-r)$-offsets of the cones so that they share a common vertex, (2) constructing a blending cyclide for the offset cones, and (3) computing the r-offset of the cyclide. Unfortunately, this process does not always work properly. Worse, for some half-cones cases, none of the blending cyclides can be constructed this way. This paper studies this problem and presents two major contributions. First, it is shown that the offset construction is correct for the case of $\epsilon \neq -r$, where ϵ is the signed offset value; otherwise, a procedure must be followed for properly selecting a pair of principal circles of the blending cyclide. Second, based on Shene's construction in "Blending two cones with Dupin cyclides", *CAGD*, Vol. 15 (1998), pp. 643–673, a new algorithm is available for constructing all possible blending cyclides for two half-cones. This paper also examines Allen and Dutta's theory of pure blends, which uses the offset construction. To help overcome the difficulties of Allen and Dutta's method, this paper suggests a new algorithm for constructing all possible pure blends. Thus, Shene's diagonal construction is better and more reliable than the offset construction.

Key words: Blending. Offset. Dupin Cyclide. Natural Quadrics. Common Inscribed Sphere

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1 Introduction

Two cones, $\mathcal{C}_1$ and $\mathcal{C}_2$, having a common inscribed sphere with center O and radius $r > 0$, can be blended with Dupin cyclides [1–5,7]. Let the set of all blending Dupin cyclides of $\mathcal{C}_1$ and $\mathcal{C}_2$ be denoted as $\mathcal{C}_1 \oplus \mathcal{C}_2$. Thus, $\oplus$ is a binary operator $\oplus : \langle \mathcal{C}_1, \mathcal{C}_2 \rangle \mapsto \mathcal{C}_1 \oplus \mathcal{C}_2$ sending a pair of cones having a common inscribed sphere to the set of their blending Dupin cyclides. For the purpose of blending, the offset of $\mathcal{C}_i$ can be defined uniquely with respect to the center of the common inscribed sphere. Let V_i and α_i be the vertex and cone angle of $\mathcal{C}_i$, respectively. Then, the ϵ-offset of $\mathcal{C}_i$, $\mathcal{C}_i^\epsilon$, is the cone with $\mathcal{C}_i$'s axis and vertex at $V_i - \frac{\epsilon}{\sin(\alpha_i)} \frac{\overrightarrow{V_iO}}{V_iO}$. Thus, offsetting is a unary operator sending a cone to its ϵ-offset. Moreover, the offset operator is invertible in the sense of $(\mathcal{C}^{-\epsilon})^\epsilon = \mathcal{C}$. The question in the title asks if the diagram in Figure 1 commutes.

$$
\begin{tikzcd}
\langle \mathcal{C}_1, \mathcal{C}_2 \rangle \arrow[r, "\oplus"] \arrow[d, "\epsilon"'] & \mathcal{C}_1 \oplus \mathcal{C}_2 \arrow[d, "\epsilon"] \\
\langle \mathcal{C}_1^\epsilon, \mathcal{C}_2^\epsilon \rangle \arrow[r, "\oplus"] & ?
\end{tikzcd}
$$

Fig. 1. Does this diagram commute (*i.e.*, $\mathcal{C}_1^\epsilon \oplus \mathcal{C}_2^\epsilon = (\mathcal{C}_1 \oplus \mathcal{C}_2)^\epsilon$)?

The answer to this question has an important impact. Since the offset operator is invertible, the question can be rephrased as: if $\mathcal{C}_1 \oplus \mathcal{C}_2 = (\mathcal{C}_1^\epsilon \oplus \mathcal{C}_2^\epsilon)^{-\epsilon}$ holds. Geometrically, this expression states that to construct a blending Dupin cyclide, one can take the following steps: (1) find an ϵ-offset of $\mathcal{C}_1$ and $\mathcal{C}_2$ with which a blending Dupin cyclide can easily be found, (2) construct a blending Dupin cyclide $\mathcal{Z}$ for $\mathcal{C}_1^\epsilon$ and $\mathcal{C}_2^\epsilon$ (*i.e.*, $\mathcal{Z} \in \mathcal{C}_1^\epsilon \oplus \mathcal{C}_2^\epsilon$), and (3) $\mathcal{Z}^{-\epsilon}$ is a desired blending cyclide for $\mathcal{C}_1$ and $\mathcal{C}_2$ (*i.e.*, $\mathcal{Z}^{-\epsilon} \in \mathcal{C}_1 \oplus \mathcal{C}_2$). This technique was suggested by Sabin and used by Boehm [3], Pratt [5], and Allen and Dutta [1], where ϵ is set to $-r$, making $\mathcal{C}_1^{-r}$ and $\mathcal{C}_2^{-r}$ to have a common vertex. Since this technique is frequently used to show the existence of a blending cyclide, it is a construction method, and will be referred to as the *offset construction* in this paper.

Pratt [5] pointed out that the r-offsets of some blending cyclides for $\mathcal{C}_1^{-r}$ and $\mathcal{C}_2^{-r}$ may not blend $\mathcal{C}_1$ and $\mathcal{C}_2$. This was further clarified in Shene [7], since in general there are two families of blending cyclides if the cones do not share a common vertex and there are four otherwise. As a result, if blending cyclides for $\mathcal{C}_1^{-r}$ and $\mathcal{C}_2^{-r}$ are not chosen properly, offsetting will not produce valid blending cyclides for $\mathcal{C}_1$ and $\mathcal{C}_2$. This paper revisits the offset construction and proves that the above diagram commutes if $\epsilon \neq -r$.

Recently, Allen and Dutta [1,2] used the offset construction to construct blending cyclides for half-cones. Since the offset construction may fail for full cones, it may also fail for half-cones. In fact, it will be shown that some blending cyclides for two half-cones cannot be constructed using the offset technique. To address this problem, this paper presents a modification to Shene's construction algorithm in [7]. This new method can construct all possible blending cyclides. Allen and Dutta [1] also looked at a special type of blends, called *pure blends*. This paper will show that pure blends can be obtained easily from the general theory for cones.

In what follows, Section 2 reviews the definition of offset for Dupin cyclides and establishes a number of fundamental properties to be used in later sections. Section 3 contains the main results of this paper. It starts with the definition of a *reference configuration* on which all offsets of the involved surfaces (*i.e.*, cones and blending cyclides) are based. This is followed by two important technical lemmas to be used for establishing the two major theorems of this paper: (1) the ϵ-offset of a blending cyclide for two cones blends the ϵ-offsets of the cones if the cones do not share a common vertex, and (2) the diagram in Figure 1 commutes if $\epsilon \neq -r$. These results also provide an interesting interpretation of blending cyclide construction using offset. Section 4 shows that the offset construction is incomplete for half-cones and suggests a new, complete algorithm. An algorithm is *complete* if it can construct *all* possible blending cyclides [7]. Section 5 examines Allen and Dutta's theory for "pure blends" which uses the offset construction, and provides a complete algorithm for constructing all possible pure blends. Finally, Section 6 has our conclusion.

Notation: This paper uses the same notation as in [7]. The construction using offset and the construction in [7] are referred to as the *offset* and *diagonal* construction methods, respectively. Moreover, this paper assumes the cones are in general positions and always have a common inscribed sphere so that a blending cyclide can be constructed. Special cases, such as identical cone axes and cubic cyclide blends, are ignored; however, the same line of reasoning also applies to these special cases with minimal modifications.

2 Offsets

Given two positive constants a and c ($c < a$), the Dupin cyclide $\mathcal{Z}(a, c, \mu)$ (Figure 2), whose longitudinal principal circles have centers at $(\pm a, 0, 0)$ and radii $\mu \mp c$ and latitudinal principal circles have centers at $(\pm c, 0, 0)$ and radii $a \mp \mu$, is defined by the following equation (Pratt [5]):

$$\mathcal{Z}(a, b, \mu) : \quad \left(x^2 + y^2 + z^2 - \mu^2 + b^2\right)^2 = 4(ax - c\mu)^2 + 4b^2y^2 \quad \text{where } b^2 = a^2 - c^2 \tag{1}$$

Here, μ is referred to as the *offset parameter*. Note that μ can be any positive, zero or negative value.

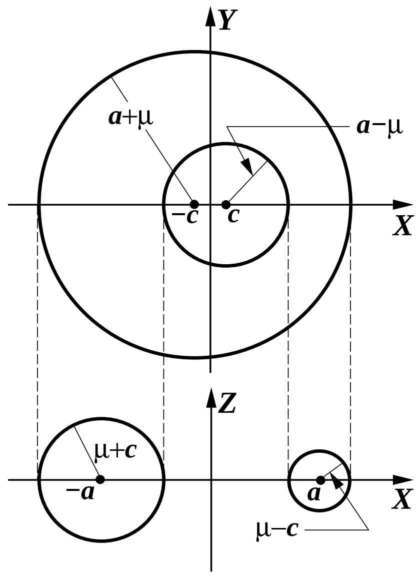


Fig. 2. Longitudinal and latitudinal principal circles of a Dupin cyclide

Lemma 1 (Fundamental Properties – The Quartic Case) *Let $\mathcal{Z}(a, c, \mu)$ be the Dupin cyclide defined by μ and constants a and c ($c < a$). Then, the following offset properties hold:*

(1) For any μ_1 and μ_2, $\mathcal{Z}(a, c, \mu_1)$ and $\mathcal{Z}(a, c, \mu_2)$ are offsets of each other. Moreover, $\mathcal{Z}(a, c, \mu_2)$ is the $\mu_2 - \mu_1$ offset of $\mathcal{Z}(a, c, \mu_1)$, and an offset of a cyclide is a cyclide. Hence, the meaning of the offset parameter μ is well-defined.
(2) The type of $\mathcal{Z}(a, c, \mu)$ is shown in the following table, where R, SH, DH, 1S *and* 2S *denote ring, singly horned, doubly horned, one-singularity spindle and two-singularity spindle, respectively.*

μ	$< -a$	$-a$	$(-a, -c)$	$-c$	$(-c, c)$	c	(c, a)	a	$> a$
Type	2S	1S	R	SH	DH	SH	R	1S	2S

(3) The offset relation is an equivalence relation.
(4) Two Dupin cyclides $\mathcal{Z}_1(a_1, c_1, \mu_1)$ and $\mathcal{Z}_2(a_2, c_2, \mu_2)$ are offsets of each other if and only if they have the same directrix conics, and if and only if $a_1 = a_2$ and $c_1 = c_2$.

PROOF. Recall that the four principal circles uniquely determine a Dupin cyclide. Thus, when the radii of the latitudinal principal circles change from $a - \mu_1$ and $a + \mu_1$ to $a - \mu_2$ and $a + \mu_2$, respectively, the generated Dupin cyclide is the envelope of a moving sphere with center on the same directrix ellipse of $\mathcal{Z}(a, b, \mu_1)$ and radius being $\mu_2 - \mu_1$ larger than that of the corresponding moving sphere of $\mathcal{Z}(a, b, \mu_2)$. Therefore, $\mathcal{Z}(a, b, \mu_2)$ is a $\mu_2 - \mu_1$ offset of $\mathcal{Z}(a, b, \mu_1)$, and Part (1) is established. Part (2) can be obtained by observing the relationship between the radii of the two principal circles of the same family and the distance between their centers. Part (3) is obvious.

Consider Part (4). By the definition of offsets of Dupin cyclides, if $\mathcal{Z}(a_1, b_1, \mu_1)$ and $\mathcal{Z}(a_2, b_2, \mu_2)$ are offsets of each other, they have the same directrix conics. If the directrix conics of $\mathcal{Z}(a_1, b_1, \mu_1)$ are the same as those of $\mathcal{Z}(a_2, b_2, \mu_2)$, then $a_1 = a_2$ and $c_1 = c_2$ because $a_1 = a_2$ (*resp.*, $c_1 = \sqrt{a_1^2 - b_1^2} = \sqrt{a_2^2 - b_2^2} = c_2$) is the focal length of the directrix ellipse (*resp.*, hyperbola). Finally, if $a_1 = a_2$ and $c_1 = c_2$, $\mathcal{Z}(a_1, b_1, \mu_1)$ and $\mathcal{Z}(a_2, b_2, \mu_2)$ are offsets of each other by Part (1). Thus, Part (4) is established. $\square$

3 Main Results

This section presents the main results of this paper. In what follows, Section 3.1 precisely defines the offsets of two cones and their blending cyclides so that the blending cyclides "shrink" as the cones "shrink"; Section 3.2 establishes two technical lemmas to be used in later sections; Section 3.3 deals with the non-singular case which states that blending and offsetting commute (Figure 1); Section 3.4 covers the singular case and shows that the offset construction must be used with care since some blending cyclides for two offset cones cannot be offset back for the given cones correctly; and, finally, Section 3.5 discusses an interesting interpretation of blending cyclides construction with the offset method.

3.1 Offsets of a Blending Configuration

The concept of offsets of blending cyclides for two cones have to be defined carefully because a blending cyclide may not "grow" when the cones it blends "grow". For example, if a ring cyclide blends two cones along its longitudinal circles and lies outside (*resp.*, inside) of the cones, this cyclide "shrinks" (*resp.*, "grows") when the cones "grow". Worse, it is difficult to define "shrinking/growing" with respect to a doubly horned cyclide because one part of it shrinks while the other part grows. To overcome this problem, all offsets are defined with respect to a reference configuration.

Suppose the cones $\mathcal{C}_1(V_1, \ell_1, \alpha_1)$ and $\mathcal{C}_2(V_2, \ell_2, \alpha_2)$ have a common inscribed sphere $\mathcal{S}$ with center $O = \ell_1 \cap \ell_2$ and radius $r > 0$. The ϵ-offsets of $\mathcal{C}_1$ and $\mathcal{C}_2$ are computed as follows. Since $\mathcal{S}$ lies in the interior of both cones, it will grow and shrink simultaneously with the cones. Therefore, the ϵ-offset of $\mathcal{C}_i$ is the cone with the same axis and cone angle, and a new vertex $V_i' = V_i - \frac{\epsilon}{\sin(\alpha_i)} \frac{\overrightarrow{V_iO}}{\overline{V_iO}}$. More precisely, the ϵ-offset of $\mathcal{C}_i(V_i, \ell_i, \alpha_i)$ is $\mathcal{C}_i(V_i - \frac{\epsilon}{\sin(\alpha_i)} \frac{\overrightarrow{V_iO}}{\overline{V_iO}}, \ell_i, \alpha_i)$.

To properly define the ϵ-offset of a blending cyclide $\mathcal{Z}$ for $\mathcal{C}_1$ and $\mathcal{C}_2$, a reference configuration is required. Let the two principal circles of $\mathcal{Z}$ on the axial plane, the plane containing the axes of cones, be Z_1 and Z_2. Consider an offset of $\mathcal{S}$ that is large enough to contain both Z_1 and Z_2. Let this sphere be the δ-offset of $\mathcal{S}$ for some $\delta > 0$ and be named as $\mathcal{S}_\infty$. Thus, we have $\mathcal{S}^\delta = \mathcal{S}_\infty$ and $\mathcal{S} = \mathcal{S}_\infty^{-\delta}$. It will be shown in Theorem 4 that a blending cyclide $\mathcal{Z}_\infty$ for $\mathcal{C}_{1\infty}$ and $\mathcal{C}_{2\infty}$ can be constructed such that the centers of the principal circles (on the axial plane) of $\mathcal{Z}_\infty$ are identical to that of Z_1 and Z_2. In fact, $\mathcal{Z}_\infty$ is an offset of $\mathcal{Z}$. We shall define $\mathcal{Z}_\infty$ to be the δ-offset of $\mathcal{Z}$ (Figure 3). Surfaces $\mathcal{S}_\infty = \mathcal{S}^\delta$, $\mathcal{C}_{1\infty} = \mathcal{C}_1^\delta$, $\mathcal{C}_{2\infty} = \mathcal{C}_2^\delta$ and $\mathcal{Z}_\infty = \mathcal{Z}^\delta$ are referred to as a *reference configuration* which is shown in light color in Figure 3. The offsets of cones and their blending cyclides are computed with respect to this configuration.

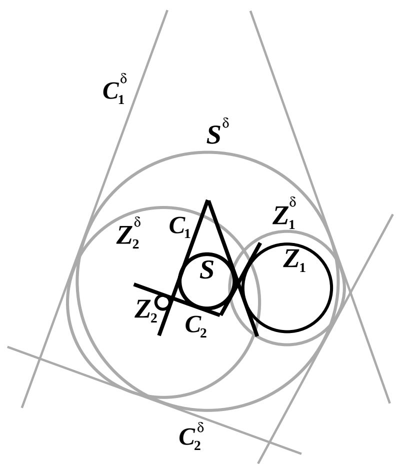


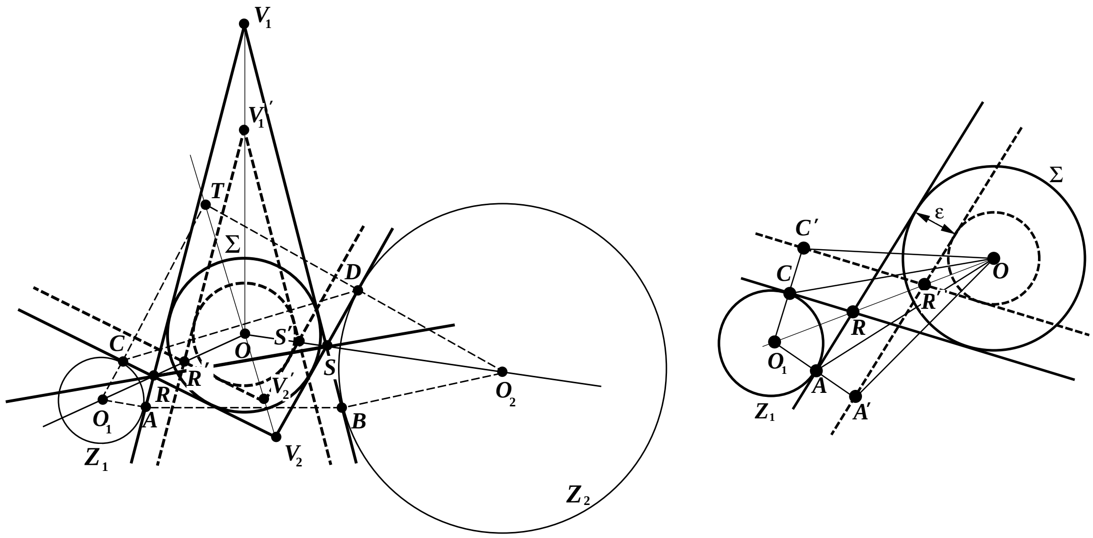
Fig. 3. A reference Configuration

By Part (2) of Lemma 1, we know that if $\mathcal{S}_\infty$ is sufficiently large, $\mathcal{Z}_\infty$ is a two-singularity spindle cyclide, which can be considered as a cyclide whose principal circles have very large radii. As a result, offsets of $\mathcal{Z}$ are obtained by "shrinking" $\mathcal{Z}_\infty$. The radii of the principal circles on the axial plane may become negative; however, this will not cause any problem because a negative radius only change the orientation of a circle without affecting its shape.

3.2 Two Technical Lemmas

Suppose two cones $\mathcal{C}_1(V_1, \ell_1, \alpha_1)$ and $\mathcal{C}_2(V_2, \ell_2, \alpha_2)$ have a common inscribed sphere with center $O = \ell_1 \cap \ell_2$ and radius $r > 0$. Let $\mathcal{H}$ be the *axial plane*. Let

R and S be a pair of opposite vertices of the intersection quadrilateral Q of the cones and $\mathcal{H}$ (*i.e.*, $Q = \mathcal{H} \cap (\mathcal{C}_1 \cup \mathcal{C}_2)$) and let Σ be the intersection circle of $\mathcal{H}$ and the common inscribed sphere (Figure 4). Thus, $\overleftrightarrow{RS}$ (*resp.*, Σ) is a diagonal (*resp.*, inscribed circle) of quadrilateral Q. If S is at infinity, $\overleftrightarrow{V_1S}$ and $\overleftrightarrow{V_2S}$ are parallel to each other. Let $\mathcal{Z}$ be a blending Dupin cyclide constructed at a point on $\overleftrightarrow{RS}$. Let $\mathcal{Z}$ and $\mathcal{H}$ intersect in two principal circles Z_1 and Z_2, with centers O_1 and O_2, such that Z_1 (*resp.*, Z_2) is tangent to $\overleftrightarrow{V_1R}$ and $\overleftrightarrow{V_2R}$ (*resp.*, $\overleftrightarrow{V_1S}$ and $\overleftrightarrow{V_2S}$) at A and C (*resp.*, B and D).



(a) (b)

Fig. 4. The basic configuration for the general case

Consider the ϵ-offsets of cones $\mathcal{C}_1$ and $\mathcal{C}$. On the axial plane $\mathcal{H}$, V_1, V_2, R and S move to V_1', V_2', R' and S', respectively (Figure 4(a)). Let the lines $\overleftrightarrow{O_1A}$, $\overleftrightarrow{O_1C}$, $\overleftrightarrow{O_2B}$ and $\overleftrightarrow{O_2D}$ meet lines $\overleftrightarrow{V_1'R'}$, $\overleftrightarrow{V_2'R'}$, $\overleftrightarrow{V_1'S'}$ and $\overleftrightarrow{V_2'S'}$ at A', C', B' and D', respectively (Figure 4(b)).

Lemma 2 *Given the above notation, the following propositions hold:*

*(1) Points O, O_1, R and R' are collinear and points O, O_2, S and S' are collinear (*i.e., *O_1 and O_2 lie on $\overleftrightarrow{OR}$ and $\overleftrightarrow{OS}$, respectively);*
(2) Lines $\overleftrightarrow{RS}$ and $\overleftrightarrow{R'S'}$ are parallel;
(3) Quadrangles V_1RV_2S and $V_1'R'V_2'S'$ are homothetic with respect to O;
(4) Points A', B', C' and D' are cocircular.

PROOF. To prove Part (1), let X and Y be two arbitrary points on the diagonal $\overleftrightarrow{RS}$ (Figure 5). The diagonal construction at X (*resp.*, Y) determines point A (*resp.*, E) on $\overleftrightarrow{V_1R}$ and point C (*resp.*, F) on $\overleftrightarrow{V_2R}$ such that there is a principal circle with center O_1 (*resp.*, G) tangent to $\overleftrightarrow{V_1R}$ and $\overleftrightarrow{V_2R}$ at A and C (*resp.*, E and F), respectively. Consider triangles $\triangle ACX$ and $\triangle EFY$. Since the lines joining the corresponding vertices (*i.e.*, $\overleftrightarrow{AE}$, $\overleftrightarrow{CF}$ and $\overleftrightarrow{XY}$) meet at point R, by Desargues' theorem, the intersection points of the corresponding sides (*i.e.*, $\overleftrightarrow{AC} \cap \overleftrightarrow{EF}$, $\overleftrightarrow{XA} \cap \overleftrightarrow{YE}$, and $\overleftrightarrow{XC} \cap \overleftrightarrow{YF}$) are collinear. Since $\overleftrightarrow{XA}$ and $\overleftrightarrow{XC}$ are parallel to $\overleftrightarrow{YE}$ and $\overleftrightarrow{YF}$ by construction, their intersection points lie on the line at infinity. As a result, the third intersection points $\overleftrightarrow{AC} \cap \overleftrightarrow{EF}$ must also be at infinity, and hence $\overleftrightarrow{AC}$ and $\overleftrightarrow{EF}$ are parallel to each other. Now consider $\triangle O_1AC$ and $\triangle GEF$. Since the corresponding sides are parallel to each other, their intersection points lie on the line at infinity, by Desargues' theorem, the lines joining their corresponding vertices meet at the same point. Therefore, O_1, G and R are collinear. Since O_1 and G are arbitrarily chosen and since the common inscribed circle is a degenerate case of a cyclide (*i.e.*, identical principal circles), its center O must also lie on $\overleftrightarrow{O_1R}$. Consequently, O, O_1 and R are collinear.

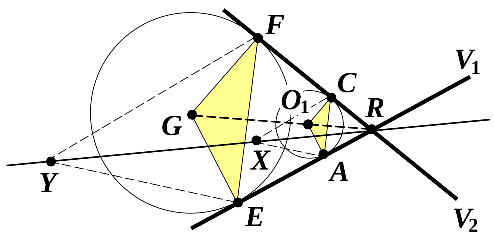


Fig. 5. O, O_1 and R are collinear

After offsetting, $\overleftrightarrow{V_1'R'}$ and $\overleftrightarrow{V_2'R'}$ are tangent to the circles with center O_1 and radius $r+\epsilon$ or $r-\epsilon$ depending on the position of O_1 (Figure 4). Therefore, O, O_1 and R' are collinear. As a result, O, O_1, R and R' are collinear. If S is at infinity, $\overleftrightarrow{V_1S}$ and $\overleftrightarrow{V_2S}$ are parallel and Z_2 and Σ have equal radii. Hence, O, O_1, S and S' are collinear.

Part (2) can also be proved with Desargues' theorem. In $\triangle V_1RS$ and $\triangle V_1'R'S'$ (Figure 4), since the lines joining the corresponding vertices (*i.e.*, $\overleftrightarrow{V_1V_1'}$, $\overleftrightarrow{RR'}$ and $\overleftrightarrow{SS'}$) meet at O, the intersection points of corresponding sides are collinear. Since the corresponding lines $\overleftrightarrow{V_1R}$ and $\overleftrightarrow{V_1'R'}$, and $\overleftrightarrow{V_1S}$ and $\overleftrightarrow{V_1'S'}$ meet at points at infinity, the third pair $\overleftrightarrow{RS}$ and $\overleftrightarrow{R'S'}$ must also meet at a point at infinity and, hence $\overleftrightarrow{RS}$ and $\overleftrightarrow{R'S'}$ are parallel.

Two figures are *homothetic* with respect to O if and only if for any point X on the first and its corresponding point $X' \in \overleftrightarrow{OX}$ on the second the ratio $\overline{OX}/\overline{OX'}$ is a constant. Thus, Part (3) is obvious since V_1 and V_1', V_2 and V_2',

R and R', and S and S' lie on lines though O, and the corresponding sides of V_1RV_2S and $V_1'R'V_2'S'$ are parallel.

Since circle Z_1 is tangent to $\overleftrightarrow{V_1R}$ and $\overleftrightarrow{V_2R}$ at A and C, $\overline{RA} = \overline{RC}$ (Figure 4(b)). Since $\mathcal{Z}$ is a blending cyclide, A, B, C and D lie on a circle with center O and radius $\overline{OA} = \overline{OC}$ [7]. Therefore, $\triangle ORA \cong \triangle ORC$. Consequently, $\angle OAR = \angle OCR$ and $\angle OAA' = 90° - \angle OAR = 90° - \angle OCR = \angle OCC'$. This implies $\triangle OAA' \cong \triangle OCC'$ since $\overline{AA'} = \overline{CC'} = |\epsilon|$ and $\overline{OA} = \overline{OC}$. Hence, $\overline{OA'} = \overline{OC'}$. Because of symmetry, $\overline{OA'} = \overline{OB'}$. As a result, the distances from O to A', B', C' and D' are all equal and these four points are cocircular. $\square$

This lemma has two immediate implications. First, the centers of the principal circles on the axial plane must be on $\overleftrightarrow{OR}$ and $\overleftrightarrow{OS}$ (Part (1)). Therefore, circles whose centers are in the regions not containing $\overleftrightarrow{OR}$ and $\overleftrightarrow{OS}$ cannot be used as principal circles. The diagonal construction never places principal circles in these areas. Second, as ϵ approaches $-r$, $\overleftrightarrow{RS}$ moves parallelly toward O and contains O when $\epsilon = -r$ (Part (2)).

Lemma 3 *Let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be two blending Dupin cyclides constructed from the same diagonal for cones $\mathcal{C}_1$ and $\mathcal{C}_2$. Then, the line of centers of the principal circles of $\mathcal{Z}_1$ is parallel to that of $\mathcal{Z}_2$.*

PROOF. See Figure 4(a). Let Z_1 and Z_2 be the principal circles, with centers O_1 and O_2 respectively, of blending Dupin cyclide $\mathcal{Z}_1$. Let Z_1 and Z_2 be tangent to cone $\mathcal{C}_2$ at C and D, respectively. Then, $\overleftrightarrow{O_1C}$ and $\overleftrightarrow{O_2D}$ meet at a point T on $\mathcal{C}_2$'s axis.

Let $\bar{O}_1$ and $\bar{O}_2$ be the centers of principal circles of $\mathcal{Z}_2$ and let $\bar{T}$ be constructed in a similar way as T. Thus, $\bar{T}$ lies on $\mathcal{C}_2$'s axis. Note that $\bar{O}_1$, $\bar{O}_2$ and $\bar{T}$ are not shown in Figure 4(a); however, this should not impose any problem. Consider $\triangle O_1O_2T$ and $\triangle\bar{O}_1\bar{O}_2\bar{T}$. Since $\bar{O}_1$ and $\bar{O}_2$ lie on $\overleftrightarrow{OO_1}$ and $\overleftrightarrow{OO_2}$ (Part (1) of Lemma 2), and since T and $\bar{T}$ lie on $\mathcal{C}_2$'s axis which passes through O, the center of the common inscribed sphere, we have $\overleftrightarrow{O_1\bar{O}_1}$, $\overleftrightarrow{O_2\bar{O}_2}$ and $\overleftrightarrow{T\bar{T}}$ meeting at O. By Desargues' theorem, the intersection points $\overleftrightarrow{O_1T} \cap \overleftrightarrow{\bar{O}_1\bar{T}}$, $\overleftrightarrow{O_2T} \cap \overleftrightarrow{\bar{O}_2\bar{T}}$ and $\overleftrightarrow{O_1O_2} \cap \overleftrightarrow{\bar{O}_1\bar{O}_2}$ are collinear. Since $\overleftrightarrow{O_1T}$ is parallel to $\overleftrightarrow{\bar{O}_1\bar{T}}$ and $\overleftrightarrow{O_2T}$ is parallel to $\overleftrightarrow{\bar{O}_2\bar{T}}$, $\overleftrightarrow{O_1O_2}$ and $\overleftrightarrow{\bar{O}_1\bar{O}_2}$ are also parallel to each other. $\square$

This section deals with the non-singular case, where $\epsilon \neq -r$ and r is the radius of the common inscribed sphere of the given cones. We shall first prove that the offset of a blending cyclide blends the offsets of the cones (Theorem 4). Then, this fact is used to prove that the diagram in Figure 1 commutes (Theorem 5).

Theorem 4 *Suppose cones $\mathcal{C}_1$ and $\mathcal{C}_2$ have a common inscribed sphere with radius r and distinct vertices. If cyclide $\mathcal{Z}$ blends cones $\mathcal{C}_1$ and $\mathcal{C}_2$, then the ϵ-offset of $\mathcal{Z}$ ($\epsilon \neq r$) blends the ϵ-offsets of $\mathcal{C}_1$ and $\mathcal{C}_2$.*

PROOF. Let $\mathcal{Z}$ blend $\mathcal{C}_1$ and $\mathcal{C}_2$. This proof consists of three steps. The first step constructs a blending cyclide $\bar{\mathcal{Z}}$ for $\mathcal{C}_1^\epsilon$ and $\mathcal{C}_2^\epsilon$ from $\mathcal{Z}$; the second step proves that $\bar{\mathcal{Z}}$ is an offset of $\mathcal{Z}$; and the third step shows that $\mathcal{Z}$ is indeed the ϵ-offset of $\bar{\mathcal{Z}}$. We shall use Figure 4 throughout this proof, where the solid and dashed lines represent the given surfaces and their offsets, respectively, R and S are two opposite diagonal vertices, and $\overleftrightarrow{RS}$ is a diagonal. Note that if $\mathcal{C}_1$ and $\mathcal{C}_2$ share a common vertex, R and S do not exist and this proof fails.

Step (1). Let Z_1 and Z_2 be the principal circles of $\mathcal{Z}$ on the axial plane. Let Z_1 (*resp.*, Z_2) be tangent to $\overleftrightarrow{V_1R}$ and $\overleftrightarrow{V_2R}$ (*resp.*, $\overleftrightarrow{V_1S}$ and $\overleftrightarrow{V_2S}$) at A and C (*resp.*, B and D). Let the lines $\overleftrightarrow{O_1A}$, $\overleftrightarrow{O_1C}$, $\overleftrightarrow{O_2B}$ and $\overleftrightarrow{O_2D}$ meet the ϵ-offsets of $\overleftrightarrow{V_1R}$, $\overleftrightarrow{V_2R}$, $\overleftrightarrow{V_1S}$ and $\overleftrightarrow{V_2S}$ at A', C', B' and D', respectively. Since it is not difficult to see $\overline{V_1'A'} = \overline{V_1'B'}$, there is a circle tangent to $\overleftrightarrow{V_1'R'}$ and $\overleftrightarrow{V_1'S'}$ at A' and B', respectively. Similarly, there is a circle tangent to $\overleftrightarrow{V_2'R'}$ and $\overleftrightarrow{V_2'S'}$ at C' and D', respectively. Since $\overline{R'A'} = \overline{R'C'}$ and $\overline{S'B'} = \overline{S'D'}$, there is a circle (with center O_1) tangent to $\overleftrightarrow{V_1'R'}$ and $\overleftrightarrow{V_2'R'}$ at A' and C', and a circle (with center O_2) tangent to $\overleftrightarrow{V_1'S'}$ and $\overleftrightarrow{V_2'S'}$ at B' and D'. By Part (4) of Lemma 2, A', B', C' and D' lie on a circle with center O. By the Specification Lemma for quartic cyclides in [7], there exists a Dupin cyclide $\bar{\mathcal{Z}}$ that blends the offset cones.

Step (2). We shall prove that $\bar{\mathcal{Z}}$ is an offset of $\mathcal{Z}$. Note that the principal circles of $\bar{\mathcal{Z}}$ and $\mathcal{Z}$ have the same centers O_1 and O_2, and are on the axial plane. If $\overleftrightarrow{O_1O_2}$ and the midpoint of $\overline{O_1O_2}$ are chosen to be the x-axis and the coordinate origin, we have $a_1 = a_2$ or $c_1 = c_2$ depending upon the type of the principal circles (*i.e.*, longitudinal or latitudinal), where a_1 and c_1 (*resp.*, a_2 and c_2) are the a and c parameters of $\mathcal{Z}$ (*resp.*, $\bar{\mathcal{Z}}$) in Equation (1) and Figure 2. In Figure 4(a), since the principal circles are longitudinal circles, we have $a_1 = a_2$. In what follows, we shall prove $c_1 = c_2$. The proof of $a_1 = a_2$ given $c_1 = c_2$ is similar.

The directrix conic on the axial plane is a hyperbola with foci O_1 and O_2. Since the sphere with center T and radius $\overline{TC} = \overline{TD}$ is tangent to the principal circles Z_1 and Z_2 of cyclide $\mathcal{Z}$, T lies on $\mathcal{Z}$'s directrix hyperbola. Similarly, since the sphere with center T and radius $\overline{TC'} = \overline{TD'}$ is tangent to the principal circles of $\bar{\mathcal{Z}}$, T also lies on $\bar{\mathcal{Z}}$'s directrix hyperbola. Since both directrix hyperbolas have the same foci and a common point T, they must be identical,[2] and both directrix hyperbolas intersect the x-axis (*i.e.*, $\overleftrightarrow{O_1O_2}$) at the same point (*i.e.*, $c_1 = c_2$). By Part (4) of Lemma 1, $\mathcal{Z}$ and $\bar{\mathcal{Z}}$ are offsets of each other.

Step (3). Finally, we shall show $\mathcal{Z} = \bar{\mathcal{Z}}^\epsilon$. Suppose the reference cones $\mathcal{C}_{1\infty}$ and $\mathcal{C}_{2\infty}$ are the δ-offsets of $\mathcal{C}_1$ and $\mathcal{C}_2$ (*i.e.*, $\mathcal{C}_i = \mathcal{C}_{i\infty}^{-\delta}$ and $\mathcal{C}_i^\epsilon = \mathcal{C}_{i\infty}^{-\delta+\epsilon}$). Following the construction in Step (1), we can construct a blending cyclide $\mathcal{Z}_\infty$ for $\mathcal{C}_{1\infty}$ and $\mathcal{C}_{2\infty}$ from $\mathcal{Z}$, $\mathcal{C}_1$ and $\mathcal{C}_2$. Since $\mathcal{C}_i^\epsilon$ is the $(-\delta+\epsilon)$-offset of $\mathcal{C}_{i\infty}$, $\mathcal{Z}$ is also the $(-\delta+\epsilon)$-offset of $\mathcal{Z}_\infty$ (*i.e.*, $\mathcal{Z} = \mathcal{Z}_\infty^{-\delta+\epsilon}$). Since $\mathcal{C}_i$ is the $(-\delta)$-offset of $\mathcal{C}_{i\infty}$, $\bar{\mathcal{Z}}$ is the $(-\delta)$-offset of $\mathcal{Z}_\infty$ (*i.e.*, $\bar{\mathcal{Z}} = \mathcal{Z}_\infty^{-\delta}$), and, as a result, $\mathcal{Z} = \mathcal{Z}_\infty^{-\delta+\epsilon} = (\mathcal{Z}_\infty^{-\delta})^\epsilon = \bar{\mathcal{Z}}^\epsilon$. $\square$

The main result of this paper is a direct and easy consequence of the above theorem.

Theorem 5 (The Non-Singular Case) *Suppose cones $\mathcal{C}_1$ and $\mathcal{C}_2$ have a common inscribed sphere with radius r and distinct vertices. If $\epsilon \neq -r$, then $\mathcal{C}_1^\epsilon \oplus \mathcal{C}_2^\epsilon = (\mathcal{C}_1 \oplus \mathcal{C}_2)^\epsilon$ holds and the main diagram commutes.*

PROOF. Let $\mathcal{Z}$ be a blending cyclide for $\mathcal{C}_1^\epsilon$ and $\mathcal{C}_2^\epsilon$. By Theorem 4, since $\mathcal{Z}$ blends $\mathcal{C}_1^\epsilon$ and $\mathcal{C}_2^\epsilon$, $\bar{\mathcal{Z}} = \mathcal{Z}^{-\epsilon}$ blends $\mathcal{C}_1 = (\mathcal{C}_1^\epsilon)^{-\epsilon}$ and $\mathcal{C}_2 = (\mathcal{C}_2^\epsilon)^{-\epsilon}$. Therefore, we have $\mathcal{Z} = \bar{\mathcal{Z}}^\epsilon$, and $\mathcal{Z}$ is the ϵ-offset of a blending cyclide for $\mathcal{C}_1$ and $\mathcal{C}_2$. Consequently, $\mathcal{C}_1^\epsilon \oplus \mathcal{C}_2^\epsilon \subset (\mathcal{C}_1 \oplus \mathcal{C}_2)^\epsilon$.

Let $\mathcal{Z}$ be the ϵ-offset of a blending cyclide for $\mathcal{C}_1$ and $\mathcal{C}_2$. Since $\mathcal{Z} \in (\mathcal{C}_1 \oplus \mathcal{C}_2)^\epsilon$, there exists a blending cyclide $\bar{\mathcal{Z}}$ for $\mathcal{C}_1$ and $\mathcal{C}_2$ such that $\mathcal{Z} = \bar{\mathcal{Z}}^\epsilon$ holds. Since $\bar{\mathcal{Z}}$ blends $\mathcal{C}_1$ and $\mathcal{C}_2$, by Theorem 4, $\bar{\mathcal{Z}}^\epsilon$ blends $\mathcal{C}_1^\epsilon$ and $\mathcal{C}_2^\epsilon$. Therefore, $\mathcal{Z}$ blends $\mathcal{C}_1^\epsilon$ and $\mathcal{C}_2^\epsilon$ and $\mathcal{Z} \in \mathcal{C}_1^\epsilon \oplus \mathcal{C}_2^\epsilon$. Consequently, $(\mathcal{C}_1 \oplus \mathcal{C}_2)^\epsilon \subset \mathcal{C}_1^\epsilon \oplus \mathcal{C}_2^\epsilon$. $\square$

3.4 The Singular Case

Since $\mathcal{C}_1 \oplus \mathcal{C}_2$ has two families of blending cyclides and $\mathcal{C}_1^{-r} \oplus \mathcal{C}_2^{-r}$ has four [7], $(\mathcal{C}_1 \oplus \mathcal{C}_2)^{-r}$ is a proper subset of $\mathcal{C}_1^{-r} \oplus \mathcal{C}_2^{-r}$. More precisely, there are blending

[2] A central conic with foci $(\pm f, 0)$ has a form of $x^2/c^2 \pm y^2/(f^2 - c^2) = 1$, which is parameterized by c. Thus, this one-parameter family has only one member that contains a given point.

cyclides for $\mathcal{C}_1^{-r}$ and $\mathcal{C}_2^{-r}$ whose r-offsets do not blend $\mathcal{C}_1$ and $\mathcal{C}_2$. Figure 6 is an example, where $\mathcal{C}_1^{-r}$ and $\mathcal{C}_2^{-r}$ are blended with a Dupin cyclide whose principal circles are shown in solid lines. As $\mathcal{C}_2^{-r}$ is offset back to $\mathcal{C}_2$, the two principal circles are also offset as shown in dashed lines. While the new circles still define a Dupin cyclide that is tangent to $\mathcal{C}_2$, it is not tangent to $\mathcal{C}_1$ and hence does not blend $\mathcal{C}_1$ and $\mathcal{C}_2$. Therefore, $(\mathcal{C}_1 \oplus \mathcal{C}_2)^{-r}$ is a proper subset of $\mathcal{C}_1^{-r} \oplus \mathcal{C}_2^{-r}$ and the offset construction in general does not work properly. Consequently, the main diagram does not commute when the offset value is $-r$.

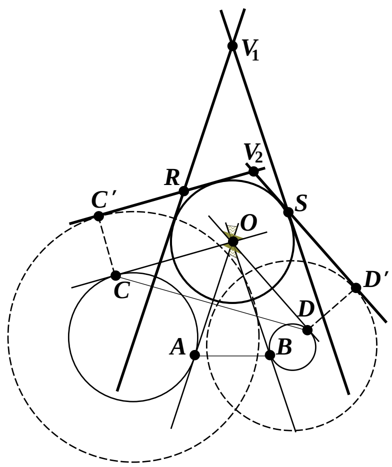


Fig. 6. An example showing $\mathcal{C}_1^{-r} \oplus \mathcal{C}_2^{-r} \neq (\mathcal{C}_1 \oplus \mathcal{C}_2)^{-r}$

Theorem 6 (The Singular Case) *Suppose cones $\mathcal{C}_1$ and $\mathcal{C}_2$ have a common inscribed sphere with radius r and distinct vertices. Then, $(\mathcal{C}_1 \oplus \mathcal{C}_2)^{-r} \subset \mathcal{C}_1^{-r} \oplus \mathcal{C}_2^{-r}$.*

The failure of the offset construction is due to the fact that the centers of the principal circles of the cyclide that blends $\mathcal{C}_1^{-r}$ and $\mathcal{C}_2^{-r}$ are not on $\overleftrightarrow{OR}$ and $\overleftrightarrow{OS}$ (Lemma 2). To construct a blending cyclide whose offset will also blend the given cones, $\overleftrightarrow{OR}$ and $\overleftrightarrow{OS}$ must be constructed first. Then, one must choose a principal circle with center on $\overleftrightarrow{OR}$ and construct the corresponding principal circle with center on $\overleftrightarrow{OS}$. While this construction is correct, it is not as clean as the diagonal construction and cannot distinguish the two possible families of blending cyclides.

Pratt [5] observed the same problem in the example. Allen and Dutta [1] used the offset construction for developing a theory of “pure blends” for half-cones; however, as will be shown in Section 4.1, this construction is incomplete. Section 4.2 suggests a remedy to this problem.

As shown in previous sections, the center O of the common inscribed sphere, lines $\overleftrightarrow{OR}$ and $\overleftrightarrow{OS}$, and the centers O_1 and O_2 of the principal circles remain fixed throughout the offset process. Moreover, quadrangles V_1RV_2S and $V_1'R'V_2'S'$ are homothetic to each other, and many results are proved using Desargues' theorem. These facts suggest a very interesting interpretation among offset configurations.

Let the axial plane be the xy-coordinate plane with origin at O. Using the intersection circle C of the axial plane and the common inscribed sphere as the base circle, one can construct a cone with cone angle 45° and vertex $V = (0, 0, -r)$ (*i.e.*, $\mathcal{C}(V, \overleftrightarrow{OV}, 45°)$), where r is the radius of the common inscribed sphere. The vertex V and the lines $\overleftrightarrow{V_1R}$, $\overleftrightarrow{V_1S}$, $\overleftrightarrow{V_2R}$ and $\overleftrightarrow{V_2S}$ define four planes that are tangent to cone $\mathcal{C}$. Let these four planes be $\mathcal{P}_{V_1R}$, $\mathcal{P}_{V_1S}$, $\mathcal{P}_{V_2R}$ and $\mathcal{P}_{V_2S}$, respectively. Since $\mathcal{P}_{V_1R}$ and $\mathcal{P}_{V_2R}$ both contain V and R, their intersection line $\overleftrightarrow{VR}$ lies on the plane determined by V, R and O. Therefore, the line through O_1 and perpendicular to the axial plane meets $\overleftrightarrow{VR}$ at a point V_{1Z}. It is not difficult to see that the cone with vertex V_{1Z} and base circle Z_1 has a cone angle 45°. Thus, we have a cone $\mathcal{C}_{1Z}(V_{1Z}, \overleftrightarrow{V_{1Z}O_1}, 45°)$ defined on Z_1. Similarly, there is a cone $\mathcal{C}_{2Z}(V_{2Z}, \overleftrightarrow{V_{2Z}O_2}, 45°)$ defined on Z_2. Note that $\mathcal{C}_{1Z}$ (*resp.*, $\mathcal{C}_{2Z}$) is tangent to planes $\mathcal{P}_{V_1R}$ and $\mathcal{P}_{V_2R}$ (*resp.*, $\mathcal{P}_{V_1S}$ and $\mathcal{P}_{V_2S}$).

Consider plane $\mathcal{P}_\epsilon$ whose equation is $z = \epsilon$ in the coordinate system established earlier. Let $V_1' = \mathcal{P}_\epsilon \cap \overleftrightarrow{VV_1}$, $V_2' = \mathcal{P}_\epsilon \cap \overleftrightarrow{VV_2}$, $R' = \mathcal{P}_\epsilon \cap \overleftrightarrow{VR}$ and $S' = \mathcal{P}_\epsilon \cap \overleftrightarrow{VS}$, and let $\overleftrightarrow{V_1'R'} = \mathcal{P}_\epsilon \cap \mathcal{P}_{V_1R}$, $\overleftrightarrow{V_1'S'} = \mathcal{P}_\epsilon \cap \mathcal{P}_{V_1S}$, $\overleftrightarrow{V_2'R'} = \mathcal{P}_\epsilon \cap \mathcal{P}_{V_2R}$ and $\overleftrightarrow{V_2'S'} = \mathcal{P}_\epsilon \cap \mathcal{P}_{V_2S}$. The intersection circle $C' = \mathcal{P}_\epsilon \cap \mathcal{C}$ is the ϵ-offset of C and tangent to $\overleftrightarrow{V_1'R'}$, $\overleftrightarrow{V_1'S'}$, $\overleftrightarrow{V_2'R'}$ and $\overleftrightarrow{V_2'S'}$. Plane $\mathcal{P}_\epsilon$ also intersects $\mathcal{C}_{1Z}$ and $\mathcal{C}_{2Z}$ in two circles Z_1' and Z_2', respectively. Thus, the vertical projections (onto the axial plane) of circles C', Z_1' and Z_2', and lines $\overleftrightarrow{V_1'R'}$, $\overleftrightarrow{V_1'S'}$, $\overleftrightarrow{V_2'R'}$ and $\overleftrightarrow{V_2'S'}$ constitute the ϵ-offset of the original configuration. More precisely, the ϵ-offset is no more than a set of level curves of the cones and planes. As ϵ approaches $-r$, diagonal $\overleftrightarrow{R'S'}$, the "offset" of $\overleftrightarrow{RS}$, moves parallelly toward O, and eventually passes through O when $\epsilon = -r$. Hence, all possible ϵ-offsets of the given cones and blending cyclides can be obtained from the intersections, or level curves, of a plane and the above constructed configuration.

4 A Theory for Half-Cones

A complete theory of blending with Dupin cyclides for axial natural quadric surfaces (*i.e.*, cylinders and cones) is presented in [4,6,7]. Recently, Allen and Dutta presented a theory for half-cones [1,2] using the offset construction. As discussed in Section 3.4, the offset construction has to be used with care because not all blending cyclides for two cones with a common vertex can be offset without problems. In fact, it is even worse for half-cones because some blending cyclides cannot be constructed this way. This section presents an analysis of the offset construction for half-cones. More precisely, we shall show that the offset construction is incomplete and none of the blending cyclides of a particular configuration can be constructed (Section 4.1). This is followed by a correct and complete construction algorithm (Section 4.2).

4.1 The Offset Construction is Incomplete

Suppose two half-cones $\vec{\mathcal{C}}_1(V_1, \vec{\ell}_1, \alpha_1)$ and $\vec{\mathcal{C}}_2(V_2, \vec{\ell}_2, \alpha_2)$ are blended with a cyclide. Let the radius and center of the common inscribed sphere of the containing cones be $r > 0$ and O. Let one of the two principal circles have center X and be tangent to $\overrightarrow{V_1R}$ and $\overrightarrow{V_2R}$ at A and C, respectively, where R is one of the four vertices of the intersection quadrilateral (Figure 7). If the angle between $\overrightarrow{OX}$ and $\overrightarrow{V_1R}$ is greater than 90°, the perpendicular foot from X to $\overrightarrow{V_1R}$ lies in $\overline{V_1E}$, where E is the perpendicular foot from O to $\overrightarrow{V_1R}$. Hence, the $(-r)$-offset of this principal circle cannot be tangent to the offset of $\overrightarrow{V_1R}$ (dashed rays in Figure 7). As a result, this principal circle cannot be obtained, using the offset construction, from any principal circle that is tangent to the offset of $\vec{\mathcal{C}}_1$ and $\vec{\mathcal{C}}_2$. On the other hand, if the angle between $\overrightarrow{OX}$ and $\overrightarrow{V_2R}$ is less than 90°, the perpendicular foot C from X to $\overrightarrow{V_2R}$ is not in $\overline{V_2F}$, where F is the perpendicular foot from O to $\overrightarrow{V_2R}$, and its extension will intersect the $(-r)$-offset of $\overrightarrow{V_2R}$. Consequently, the $(-r)$-offset of this principal circle is tangent to the $(-r)$-offset of $\overrightarrow{V_2R}$. This idea is summarized in the following proposition:

Proposition 7 *Let X be the center of a principal circle, O the center of the common inscribed sphere, and R a diagonal vertex such that X, O and R are collinear. Then, the $(-r)$-offset of this principal circle is tangent to the offset cones if and only if the angle between $\overrightarrow{OX}$ and $\overrightarrow{V_1R}$ and the angle between $\overrightarrow{OX}$ and $\overrightarrow{V_2R}$ are both acute.*

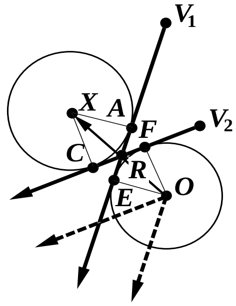
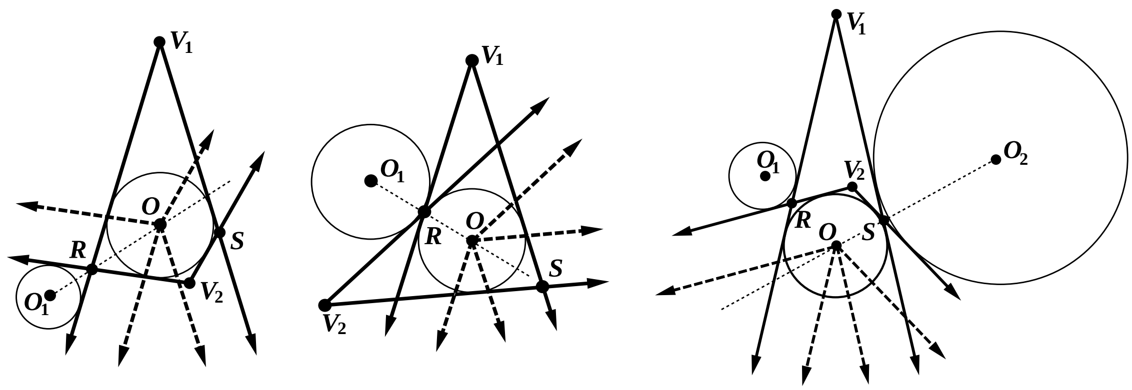


Fig. 7. Characterization of the constructibility of the offset construction



(a) (b) (c)

Fig. 8. An analysis of the constructibility of the offset construction

Now we can use the above proposition to analyze the incompleteness of the offset construction. Figure 8(a) shows a configuration of two half-cones. It is clear that $\overrightarrow{OR}$ makes acute angles with both $\overrightarrow{V_1R}$ and $\overrightarrow{V_2R}$. Thus, this principal circle can be constructed from its $(-r)$-offset which is tangent to the $(-r)$-offsets or $\overrightarrow{V_1R}$ and $\overrightarrow{V_2R}$. However, since $\overrightarrow{RO}$ makes non-acute angles with both $\overrightarrow{V_1R}$ and $\overrightarrow{V_2R}$, the $(-r)$-offsets of those principal circles whose centers are on $\overleftrightarrow{RS} - \overrightarrow{OR}$ are not tangent to the offset cones, and, as a result, principal circles with centers on $\overleftrightarrow{RS} - \overrightarrow{OR}$ cannot be constructed with the offset construction. Consequently, the offset construction does not construct *all* possible blending cyclides.

Figure 8(b) is another example. Since $\overrightarrow{OR}$ makes non-acute angles with both $\overrightarrow{V_1R}$ and $\overrightarrow{V_2R}$, the indicated principal circle with center O_1 cannot be constructed. Unfortunately, this configuration is commonly used in cyclide blending. Moreover, those blending cyclides that can be constructed (with centers on $\overleftrightarrow{OR} - \overrightarrow{OR}$) are singular.

Figure 8(c) shows an extreme case. Since $\overrightarrow{OS}$ makes an acute angle with $\overrightarrow{V_2S}$ and a non-acute angle with $\overrightarrow{V_1S}$, any principal circle whose center is on $\overrightarrow{OS}$

cannot be constructed. Since the opposite direction of $\overrightarrow{OS}$ makes an acute angle with $\overrightarrow{V_1S}$ and a non-acute angle with $\overrightarrow{V_2S}$, any principal circle whose center is on $\overleftrightarrow{OS} - \overrightarrow{OS}$ cannot be constructed either. Hence, none of the principal circles whose centers are on $\overleftrightarrow{OS}$ can be constructed, and consequently no blending cyclide can be constructed with the offset construction. That is, the offset construction fails completely for this case, and an existence proof based on the offset construction is likely to be incorrect.

4.2 A Correct Construction Algorithm

With a minor modification, the diagonal construction can be used for half-cones. Since a blending cyclide of two half-cones is also a blending cyclide of the two containing cones, to construct blending cyclides for half-cones, one only needs to properly add some restrictions to the algorithm for cones. Given two half-cones $\vec{\mathcal{C}}_1(V_1, \vec{\ell}_1, \alpha_1)$ and $\vec{\mathcal{C}}_2(V_2, \vec{\ell}_2, \alpha_2)$, we first extend them to two cones $\mathcal{C}_1(V_1, \ell_1, \alpha_1)$ and $\mathcal{C}_2(V_2, \ell_2, \alpha_2)$. These two cones must have a common inscribed sphere to have a blending cyclide. The axial plane intersects the cones, half-cones, and the common inscribed sphere in two pairs of intersecting lines with intersection points V_1 and V_2, two pairs of rays with base points V_1 and V_2, and a circle (Figure 9). The two pairs of intersection lines form a complete quadrilateral with three pairs of opposite vertices, V_1 and V_2, R and S, and R' and S'. Let the center of the common inscribed sphere be O. In what follows, we shall only consider the general case in which the axes of the cones do not coincide and none of R, S, R' and S' is at infinity.

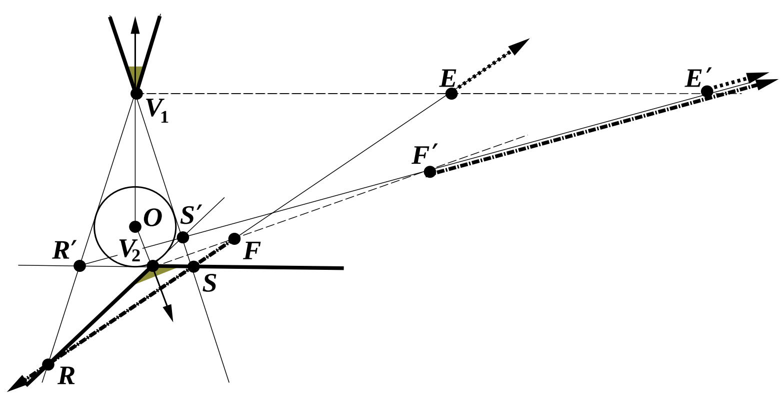


Fig. 9. A construction algorithm for half-cones

Since a blending cyclide is constructed by dropping perpendicular lines to the axes of the cones from a point on a diagonal, say $\overleftrightarrow{RS}$, if the constructed cyclide blends the half-cones, the line perpendicular to the axes of the cones must intersect the axis rays of the given half-cones. Construct a line through V_1 and perpendicular to ℓ_1, meeting diagonal $\overleftrightarrow{RS}$ at E (Figure 9), which subdivides $\overleftrightarrow{RS}$ into two rays in opposite directions with base point E. Note that only one of these two rays makes an acute angle with $\vec{\ell}_1$. It is clear that from any point on this ray one can construct a line perpendicular to $\vec{\ell}_1$. This idea can be incorporated into the diagonal construction to yield an algorithm for half-cones as follows:

- **Input:** two half-cones $\vec{\mathcal{C}}_1(V_1, \vec{\ell}_1, \alpha_1)$ and $\vec{\mathcal{C}}_2(V_2, \vec{\ell}_2, \alpha_2)$
- **Output:** a series of blending cyclides for $\vec{\mathcal{C}}_1$ and $\vec{\mathcal{C}}_2$ if they exist
- **Algorithm:**

(1) Extend the half-cones to full cones $\mathcal{C}_1(V_1, \ell_1, \alpha_1)$ and $\mathcal{C}_2(V_2, \ell_2, \alpha_2)$, and compute the two diagonals $\overleftrightarrow{RS}$ and $\overleftrightarrow{R'S'}$, where R and S, and R' and S' are two pairs of opposite vertices.

(2) Let the line through V_1 and perpendicular to ℓ_1 meet $\overleftrightarrow{RS}$ and $\overleftrightarrow{R'S'}$ at E and E', respectively.

(3) Let the line through V_2 and perpendicular to ℓ_2 meet $\overleftrightarrow{RS}$ and $\overleftrightarrow{R'S'}$ at F and F', respectively.

(4) If the angle between $\vec{\ell}_1$ (*resp.*, $\vec{\ell}_2$) and $\overrightarrow{RS}$ is acute, let $\vec{e}_E$ (*resp.*, $\vec{e}_F$) be the ray with base point E (*resp.*, F) in the direction of $\overrightarrow{RS}$. Otherwise, $\vec{e}_E$ (*resp.*, $\vec{e}_F$) is the ray with base point E (*resp*, F) in the direction of $\overrightarrow{SR}$.

(5) Do the same for E' and F', yielding rays $\vec{e}_{E'}$ and $\vec{e}_{F'}$, respectively.

(6) If $\vec{e}_E \cap \vec{e}_F$ is empty, no blending cyclide can be constructed from diagonal $\overleftrightarrow{RS}$. Otherwise, from any point in $\vec{e}_E \cap \vec{e}_F \subset \overleftrightarrow{RS}$ a blending cyclide for the half-cones $\vec{\mathcal{C}}_1$ and $\vec{\mathcal{C}}_2$ can be constructed.

(7) Do the same for $\vec{e}_{E'}$ and $\vec{e}_{F'}$ for diagonal $\overleftrightarrow{R'S'}$

Figure 9 illustrates two examples. Since $\vec{e}_E$ and $\vec{e}_F$ do not intersect, no blending cyclide can be constructed. On the other hand, since $\vec{e}_{E'}$ is a subset of $\vec{e}_{F'}$, a blending cyclide can be constructed from any point in $\vec{e}_{E'}$. The correctness and completeness proofs are easy and hence are omitted. A complete analysis of the types of the constructed cyclides can be found in [7]. In general, $\overleftrightarrow{RS}$ can be subdivided into five intervals with four points. Each of these division points corresponds to a singly horned or a one-singularity spindle cyclide. All points in the same interval correspond to cyclides of the same type, and of these five intervals two correspond to ring cyclides. In fact, points that are close to either R or S and are in the exterior of both half-cones correspond to ring cyclides.

5 On Allen-Dutta's Theory

This section re-examines Allen and Dutta's theory of "pure" cyclide blends for half-cones. We shall show that this type of pure blends is a small part of the general theory for cones and can be obtained quite easily (Section 5.1). Since the offset construction, which is also used by Allen and Dutta for constructing pure cyclide blends, is incomplete (Section 4.1), a modified algorithm that uses the diagonal method is given in Section 5.2.

5.1 Pure Cyclide Blends

In [1,2], Allen and Dutta defined a pure cyclide blend for two half-cones to be a non-singular (*i.e.*, ring) cyclide satisfying three conditions: (1) the intersection curve is nonempty and closed, (2) the cyclide is tangent to each half-cone along a latitudinal circle, and (3) the intersection curve must wrap around the axis of each half-cone being blended. Since a necessary and sufficient condition for two axial quadrics, with intersecting axes, to have a blending Dupin cyclide is that they intersect in planar curves [4,6,7], the intersection curve in conditions (1) and (3) must be an ellipse. In [7], it is also proved that if the line of vertices lies in the interior of both cones, any blending cyclide is tangent to the given cones along latitudinal circles. For the half-cones case, this is equivalent to the vertex of a half-cone being in the interior of the other. Consequently, we have the following:

Definition 8 (Pure Cyclide Blend) *A blending Dupin cyclide for two half-cones is a pure blend if and only if the half-cones intersect in an ellipse and the vertex of one half-cone lies in the interior of the other.*

Allen-Dutta's theory can be considered as a special case of the general theory of cyclide blending. By properly specializing the general results, a theory that is capable of constructing all possible pure cyclide blends can be obtained easily. In fact, an numeration of all possible relative positions of the half-cones and common inscribed sphere will suffice. In what follows, we assume that the containing cones of the given half-cones have a common inscribed sphere, and will base our discussion on the axial plane.

The intersection of the axial plane and the two full cones is a complete quadrilateral with two four-side regions, one bounded while the other unbounded. The intersection circle of the common inscribed sphere and the axial plane is an inscribed circle of the quadrilateral. This inscribed circle can be in the bounded or the unbounded area as shown in Figure 10 and Figure 11, respectively. The vertices of the given cones are opposite vertices of the quadrilateral. Thus, there are three possibilities for placing the vertices of the cones. Each cone has two half-cones, and once a pair of opposite vertices are fixed to be the cones' vertices, there are four half-cones. Therefore, there are $2 \times 3 \times 4 = 24$ different configurations of the given half-cones. Figure 10 (*resp.*, Figure 11) illustrates the configurations where the inscribed circles are in the bounded (*resp.*, unbounded) region. Note that due to symmetry, each pair of the second and third configurations in Figure 10(b), Figure 10(c), Figure 11(b), and Figure 11(c) are equivalent. Consequently, there are only 20 different configurations.

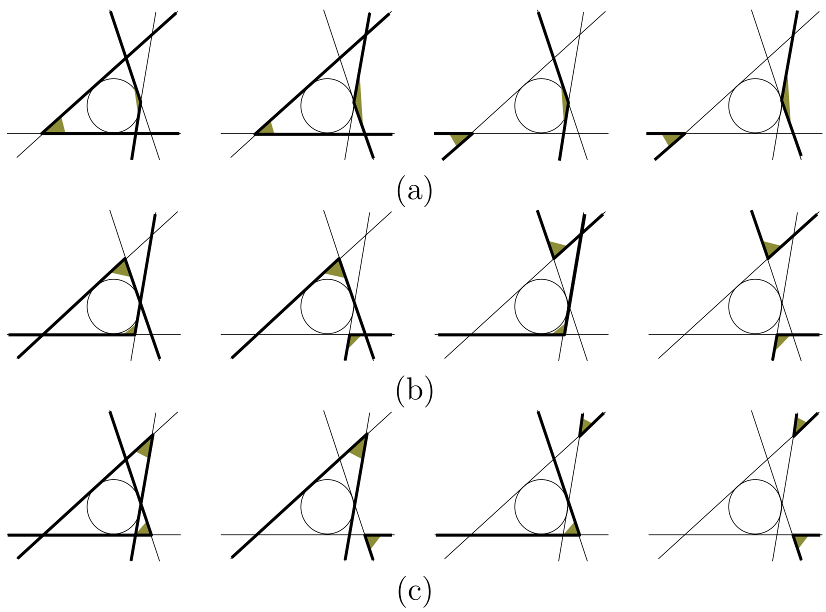


Fig. 10. The inscribed circle lies in the bounded region

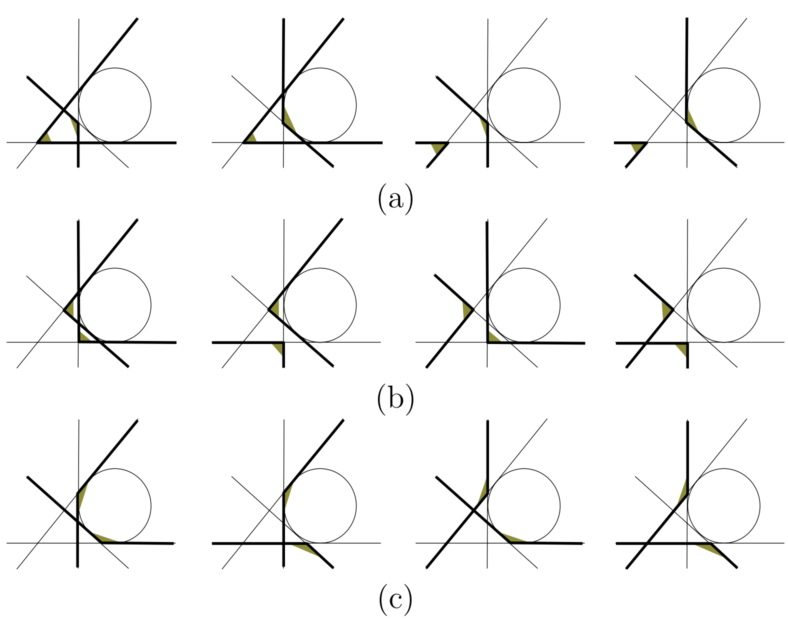


Fig. 11. The inscribed circle lies in the unbounded region

Of these 20 configurations, only the following seven have an ellipse in the intersection of the given half-cones: (1) the first and second of Figure 10(a), (2) the first of Figure 10(b), (3) the first of Figure 10(c), (4) the first and second of Figure 11(a), and (5) the first of Figure 11(b). Note that these figures are not required for determining if a particular intersection contains an ellipse, and the use of diagrams here is for the sake of simplicity. Shene and Johnstone [9] has a fast algorithm for determining the types of the intersection conics of two cones. Of these seven, only the configurations in (1), (2), and (4) satisfy the definition of a pure cyclide blending (Definition 8). Thus, Allen-Dutta's theory of pure cyclide blending only covers 25% of the general cases.

The existence of a common inscribed sphere for cones carries over to half-cones naturally. Of the above mentioned five configurations, only the first configurations of Figure 10(a) and Figure 10(b), and the second of Figure 11(a) have intersecting axis rays. The half-cones of these three configurations all contain the common inscribed sphere and the common inscribed sphere criterion in [1] is established.

Allen and Dutta [2] also discussed a common inscribed sphere criterion for blending half-cones. Note that, in this case, blending cyclides are not restricted to be "pure". To have a common inscribed sphere, the axis rays of the half-cones must intersect. The first configurations in Figure 10, the second configuration in Figure 11(a), and the first configurations of Figure 11(b) and (c) satisfy this requirement. Therefore, each of these six configurations has a common inscribed sphere. Note that there are only two cases (*i.e.*, the third configurations in Figure 10(a) and Figure 11(a)) in which one half-cone contains the other, and in both cases the axis rays do not intersect. As a result, the second condition of Allen-Dutta's result is redundant (Theorem 5.1 of [2]). Note also that this common inscribed sphere criterion only accounts for 30% of the general cases. Hence, the power of common inscribed sphere for half-cones is quit limited. Moreover, the existence of a pure blend cannot be tested using the common inscribed sphere criterion (*e.g.*, the second configuration of Figure 10(a) and the first configuration of Figure 11(a)).

5.2 The Construction of Pure Cyclide Blends

As shown in Section 4.1, the offset construction does not deliver all possible blending cyclides. Since Section 4.2 has already presented a construction algorithm for correctly constructing blending cyclides using the diagonal construction, what remains is to determine the given half-cones intersect in an ellipse. This is easy as shown below.

Suppose two half-cones intersect in an ellipse on the diagonal $\overleftrightarrow{RS}$. It is easy

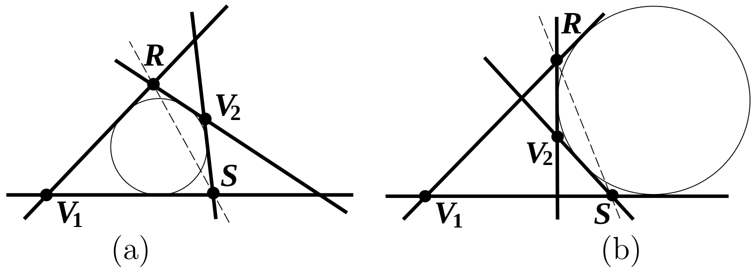


Fig. 12. The two possible positions of the inscribed circle relations

to verify that if the inscribed circle lies in the bounded region (Figure 12(a)), $\overline{V_1R} - \overline{V_1S} = \overline{V_2R} - \overline{V_2S}$ holds; otherwise, $\overline{V_1R} - \overline{V_1S} = -(\overline{V_2R} - \overline{V_2S})$ holds (Figure 12(b)). Conversely, if one of these two relations holds, it can be shown that the quadrilateral has an inscribed circle and that the half-cones have planar intersection [6,8]. Hence, by selecting a point on $\overleftrightarrow{RS}$, using the algorithm presented in Section 4.2 one can construct a blending cyclide.

To construct a "pure" blending cyclide, one has to address two requirements: (1) one cone contains the vertex of the other, and (2) the cyclide must be of the ring type. Condition (1) is easy to test. However, condition (2) may not be satisfied at all as discussed in Section 4.1 and in particular in Figure 8, which is the second configuration of Figure 10(a). Based on this finding, Allen and Dutta's algorithm can be replaced with the following simplified version:

input: Two half-cones $\vec{\mathcal{C}}_1(V_1, \vec{\ell}_1, \alpha_1)$ and $\vec{\mathcal{C}}_2(V_2, \vec{\ell}_2, \alpha_2)$;
Output: A series of pure blending cyclides;
Algorithm:

```
if the axis rays are not coplanar then
    there is no blending cyclide
else if none of the cones contains the vertex of the other then
    there is no pure blend (Definition 8)
else
    begin
        compute a pair of finite opposite vertices R and S;
        if R or S does not exist then
            the half-cones do not intersect in an ellipse
        else if abs(V1R − V1S) ≠ abs(V2R − V2S) then
            the half-cones do not intersect in an ellipse
        else
            apply the algorithm in Section 4.2
    end
```

6 Conclusion

This paper has successfully established the fact that the offset of a blending cyclide for two cones also blends the offsets of the latter if the cones do not share a common vertex. Otherwise, this result holds only if a blending cyclide for the offset cones is chosen carefully. This paper includes an interesting interpretation from which all possible offset blending configurations can be obtained as level curves of a simple configuration involving three cones and four planes. We also re-examine the use of the offset construction for half-cones. It is shown that the offset construction is incomplete in general, and none of the possible blending cyclides can be constructed in a particular case. As a final contribution, this paper looks at Allen and Dutta's theory of constructing "pure blends" for half-cones using the offset construction. Our findings include: (1) the results for "pure blends" only account for a small part of the general theory for cones and can be obtained easily and quickly, and (2) Allen and Dutta's construction is incomplete since it shares the same difficulties of the offset construction. To address this problem, this paper also suggests a correct and complete construction algorithm. Based on these evidences, one can conclude that the diagonal construction in [7] is better and more reliable than the offset construction.

References

[1] Seth Allen and Debasish Dutta, Cyclides in Pure Blending I and II, *Computer Aided Geometric Design*, Vol. 14 (1997), pp. 51–102.

[2] Seth Allen and Debasish Dutta, Results on Nonsingular, Cyclide Transition Surfaces, *Computer Aided Geometric Design*, Vol. 15 (1998), pp. 127–145.

[3] Wolfgang Boehm, On Cyclides in Geometric Modeling, *Computer Aided Geometric Design*, Vol. 7 (1990), pp. 243–255.

[4] John K. Johnstone and Ching-Kuang Shene, Dupin Cyclides as Blending Surfaces for Cones, in *The Mathematics of Surfaces V*, edited by R. B. Fisher, Clarendon Press, Oxford, England, 1994, pp. 3–29.

[5] Michael J. Pratt, Cyclides in Computer Aided Geometric Design, *Computer Aided Geometric Design*, Vol. 7 (1990), pp. 221–242.

[6] Ching-Kuang Shene, Planar Intersection and Blending of Natural Quadrics, Ph.D. thesis, Department of Computer Science, The Johns Hopkins University, Baltimore, Maryland, August 1992.

[7] Ching-Kuang Shene, Blending Two Cones with Dupin Cyclides, *Computer Aided Geometric Design*. Vol. 15 (1998), No. 7, pp. 643-673.

[8] Ching-Kuang Shene and John K. Johnstone, When Does a Quadrilateral Have an Inscribed Circle? Technical Report JHU-91/22, Department of Computer Science, The Johns Hopkins University, December 1991.

[9] Ching-Kuang Shene and John K. Johnstone, On the Lower Degree Intersections of Two Natural Quadrics, *ACM Transactions on Graphics*, Vol. 13 (1994), No. 4 (October), pp. 400–424.