Obtaining the Roots of a Cubic Equations

Given a cubic equation,

\[ z^3 + A \cdot z^2 + B \cdot z + C = 0 \]

Let

\[ z = x - \frac{A}{3} \]

then

\[ x^3 + \left( \frac{-1}{3} \cdot A^2 + B \right) \cdot x + \frac{2}{27} \cdot A^3 + C - \frac{1}{3} \cdot B \cdot A = 0 \]

or

\[ x^3 + p \cdot x = q \]

where,

\[ p = \left( \frac{-1}{3} \cdot A^2 + B \right) \]

\[ q = -\left( \frac{2}{27} \cdot A^3 + C - \frac{1}{3} \cdot B \cdot A \right) \]

Further, let

\[ x = y - \frac{p}{3y} \]

then we obtain a 6th order polynomial equation in \( y \) given by

\[ y^6 - q \cdot y^3 - \frac{1}{27} \cdot p^3 = 0 \]

whose roots are:

\[ y = \left[ \frac{q}{2} + \sqrt[3]{\left( \frac{q}{2} \right)^2 + \left( \frac{p}{3} \right)^3} \right] \]
Now let the discriminant $\Delta$ be the term inside the square root above, i.e.

$$\Delta = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3$$

then we will have two cases that will depend on whether the discriminant is positive or negative.

**Case 1:** $\Delta > 0$ Then we will have one real root and a complex conjugate pair

The first root is given by

$$z_1 = \text{sign}(h) \left( |h| \right)^{\frac{1}{3}} - \frac{A}{3}$$

where,

$$h = \frac{q}{2} + \sqrt{\Delta}$$

The other roots can then be obtained by using the values of the first root:

$$z_2 = \frac{-\left( A + z_1 \right) + \sqrt{\left( A + z_1 \right)^2 - 4\cdot\left( B + z_1\cdot A + z_1 \right)}}{2}$$

$$z_3 = \frac{-\left( A + z_1 \right) - \sqrt{\left( A + z_1 \right)^2 - 4\cdot\left( B + z_1\cdot A + z_1 \right)}}{2}$$

**Case 2:** $\Delta < 0$ There will be three real roots.

The first root will be obtained as follows (whose proof is given below):

$$z_1 = 2\left( \left( -\frac{p}{3} \right) \cdot \cos \left( \frac{\text{atan} \left( \frac{\sqrt{-\Delta}}{q} \right)}{3} \right) \right) - \frac{A}{3}$$
And the two remaining roots can be determined by the following equations:

\[
\begin{align*}
    z_2 &= -(A + z_I) + \frac{\sqrt{(A + z_I)^2 - 4B + z_I(A + z_I)}}{2} \\
    z_3 &= -(A + z_I) - \frac{\sqrt{(A + z_I)^2 - 4B + z_I(A + z_I)}}{2}
\end{align*}
\]

Proof for formula to obtain the first root:

Let \( h = \frac{q}{2} + i \left( -\frac{q}{2} - \left( \frac{p}{3} \right)^3 \right) \)

whose magnitude and angle are given by

\[
|h| = \sqrt{\left( \frac{-p}{3} \right)^3}
\]

\[
\theta = \arg(h) = \tan^{-1} \left( \frac{\sqrt{-A}}{q/2} \right)
\]

allowing one to evaluate the cube root of the polar representation:

\[
y = h^{\frac{1}{3}} = \sqrt[3]{\frac{-p}{3}} e^{i \frac{\theta}{3}}
\]

from which we obtain

\[
x = \sqrt[3]{\frac{-p}{3}} e^{i \frac{\theta}{3}} + \sqrt[3]{\frac{-p}{3}} e^{-i \frac{\theta}{3}}
\]
\[ x = \sqrt{-\frac{p}{3}} \left( e^{i\theta} + e^{-i\theta} \right) = 2 \sqrt{-\frac{p}{3}} \cos \left( \frac{\theta}{3} \right) \]

or

\[ z_1 = x - \frac{A}{3} = 2 \sqrt{-\frac{p}{3}} \cos \left( \frac{\theta}{3} \right) - \frac{A}{3} \]