1. (15 pts) The following equations model the pressure and temperature dynamics in an insulated pressure vessel:

\[
\frac{d}{dt}\left(\frac{P}{T}\right) = \frac{1}{T} \frac{dP}{dt} - \frac{P}{T^2} \frac{dT}{dt} = \alpha_1 \sqrt{P_1 - P - \alpha_2 u} \sqrt{P_2}
\]

\[
\frac{dP}{dt} = \alpha_1 \left(\sqrt{P_1 - P} \cdot (T_1 - T_{\text{ref}}) - \alpha_2 u \sqrt{P_2} \cdot (T - T_{\text{ref}})\right)
\]

where the design parameters were determined to be:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\alpha_1)</td>
<td>3.5 (\text{K}^{-1}\text{sec}^{-1})</td>
</tr>
<tr>
<td>(\alpha_2)</td>
<td>4 (\text{K}^{-1}\text{sec}^{-1})</td>
</tr>
<tr>
<td>(T_{\text{ref}})</td>
<td>298 K</td>
</tr>
</tbody>
</table>

Find the steady state values of pressure and temperature of the gas in the vessel, i.e. \(P_{ss}\) and \(T_{ss}\) corresponding to a set of fixed values of external disturbances, \(P_1\), \(P_2\), \(T_1\), and manipulated variable, \(u\), given by

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(P_1 = P_{1,ss})</td>
<td>3 atm</td>
</tr>
<tr>
<td>(P_2 = P_{2,ss})</td>
<td>1 atm</td>
</tr>
<tr>
<td>(T_1 = T_{1,ss})</td>
<td>373 K</td>
</tr>
<tr>
<td>(u = u_{ss})</td>
<td>0.5</td>
</tr>
</tbody>
</table>
2. The dynamics of the concentration of a reactant in a given reactor was found to behave according to the following model:

$$\frac{d^2 C}{dt^2} + 2 \cdot C \frac{dC}{dt} + 3 \cdot u \cdot C_{in} = 1.5 \cdot C^2$$

where C is the concentration in the reactor while $C_{in}$ is the concentration in the feed and u is the fraction opening of the valve in the feed line.

a) (20 pts) Around the steady state conditions: $C_{ss}=0.975$, $u_{ss}=0.5$, $C_{in,ss}=0.95$, $(dC/dt)_{ss}=0$, find the values of $\alpha$, $\beta$ and $\eta$ such that the following equation is a linearized approximation of the given process around the steady state:

$$\frac{d^2 C}{dt^2} + 1.9 \cdot \frac{dC}{dt} + 2.925 \cdot C = \alpha \cdot (C_{in} - 0.95) + \beta \cdot (u - 0.5) + \eta$$

b) (15 pts) With $u=u(t)$ and $C_{in}=C_{in}(t)$, i.e. no feedback control, determine the damping coefficient of the linearized equation.

3. (20 pts) The process model for a process is given by the following set of simultaneous system:

$$\begin{align*}
\frac{dx_1}{dt} &= -2 \cdot x_1 + x_2 - u \\
\frac{dx_2}{dt} &= 8 \cdot x_1 - x_2 + 2 \cdot u
\end{align*}$$

Using a proportional control law given by,

$$u = k_c \cdot ( x_{1, set} - x_1 )$$

Determine values of $k_c$ that would stabilize the process.
4. For the dynamics of flow rate given by the equation,

\[ \frac{dF}{dt} = 2u - F \]

it was decided to use a PI controller given by

\[ u = \frac{1}{\tau_I} \left( e + \frac{1}{\tau_I} \int e \, dt \right) \]

where \( F_{set} \) is the constant set point.

a) (15 pts) Show that the feedback controlled dynamics is given by

\[ \left( \frac{\tau_I}{2k_c} \right) \frac{d^2 F}{dt^2} + \left( \frac{2k_c + 1}{2k_c} \right) \frac{dF}{dt} + F = F_{set} \]

b) (15 pts) Determine the values of \( k_c \) and \( \tau_I \) such that the resulting behavior has an overshoot of 1/6 and frequency of 2 rad/sec.

5. (Bonus: 5 pts) Suppose you are given the following set of differential equations,

\[ 2C^2 \frac{d^2 C}{dt^2} + 4 \left( \frac{dC}{dt} \right)^2 + C = \exp(-1.5t) \]

after introducing an auxiliary variable, \( y = dC/dt \), obtain the recursion equations using the Euler approximation method, i.e. what are the functions \( f_k \) and \( g_k \) below,

\[ C_{k+1} = C_k + \Delta t \left( f_k \right) \]
\[ y_{k+1} = y_k + \Delta t \left( g_k \right) \]