1. (30 pts) A second order bioreactor process for the biomass concentration, $x_1$, and the substrate concentration, $x_2$, is described by the following equations:

$$\frac{dx_1}{dt} = (\mu - D)x_1$$

$$\frac{dx_2}{dt} = D(x_{2f} - x_2) - \frac{\mu x_1}{Y}$$

where,

$$\mu = \frac{\mu_{\text{max}} x_2}{k_m + x_2}$$

$x_1(0) = 1$

$x_2(0) = 1$

By treating the dilution rate, $D$, as the manipulated variable and the substrate feed concentration, $x_{2f}$, as the disturbance to the process, obtain a linearized set of equations for $x_1$ and $x_2$ around the steady states corresponding to $D_{\text{ss}} = 0.3$ and $x_{2f,ss} = 4.0$, with $x_{1,ss} > 0$. The process and design parameters are given as:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_{\text{max}}$</td>
<td>0.53</td>
</tr>
<tr>
<td>$k_m$</td>
<td>0.12</td>
</tr>
<tr>
<td>$Y$</td>
<td>0.45</td>
</tr>
</tbody>
</table>

2. A linearized model for a CSTR is given by the following set of equations:

$$\frac{dT}{dt} = -4T + 6C + 2u$$

$$\frac{dC}{dt} = 4T - 2C$$

a) (20 pts) Evaluate the eigenvalues and determine the stability of process if $u = 5 e^{-2t}$.

b) (20 pts) Now apply a proportional feedback control law given by

$$u = 3 + k_c(T^{\text{set}} - T)$$

where $k_c$ is the proportional control gain and $T^{\text{set}}$ is a constant set point. What range of $k_c$ values will guarantee the stability of the process under feedback control?
3. (30 pts) The temperature of a process was determined to be a first order process given by the following equation:

\[ 5 \frac{dT}{dt} + T = 2u \quad ; \quad T(0) = 10 \]

where \( u \) is the manipulated variable. With the implementation of a proportional control law given by:

\[ u = 5 + k_c (T^{set} - T) \]

the feedback controlled response is shown in Figure 1.

![Figure 1. Temperature response under feedback control.](image)

Using the plot in Figure 1, determine the value of \( k_c \) and \( T^{set} \) used.

(Hint: You can measure the new time constant of the feedback controlled system from Figure 1 and use it to determine \( k_c \). Then from the steady state value of \( T \) and \( k_c \) value just obtained, solve for \( T^{set} \).)

4. (Bonus: 10 pts) One wishes to simulate the process described by

\[ \frac{dP}{dt} = -2\sqrt{P - 0.5} \]

Write the equations for \( \delta_1, \delta_2, \delta_3 \) and \( \delta_4 \) required for the implementation of the 4th order Runge-Kutta method such that the update equation for \( P_{k+1} \) is given by

\[ P_{k+1} = P_k + \frac{1}{6}(\delta_1 + 2\delta_2 + 2\delta_3 + \delta_4) \]