On the Shadow Geometries of $W(23,16)$

Assaf Goldberger[1] Yossi Strassler[2] Giora Dula[3]

[1]School of Mathematical Sciences
Tel-Aviv University

[2]Dan Yishay

[3]Department of Computer Science and Mathematics
Netanya College

26-30.08.2015 Algebraic Combinatorics and Applications Conference

Weighing matrices

Definition

A weighing matrix W of size n and weight k is a $\{0, 1, -1\}$ matrix that satisfies

$$WW^T = W^T W = kI_n.$$

We say that W is a $W(n, k)$ matrix.

Examples:

- Hadamard matrices are $W(n, n)$.
- Conference matrices are $W(n, n-1)$.
- Signed permutation matrices are $W(n, 1)$.

Examples and questions

- The following are $W(2,k), 1 \leq k \leq 2$

$$\left(I_2 \;\middle|\; \begin{matrix} 1 & 1 \\ 1 & -1 \end{matrix}\right)$$

- The following are $W(3,k), 1 \leq k \leq 3$

$$\left(I_3 \;\middle|\; \phi \;\middle|\; \phi\right)$$

- The following are $W(4,k), 1 \leq k \leq 4$

$$\left(\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \;\middle|\; \begin{matrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{matrix} \;\middle|\; \begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ -1 & -1 & 1 & 0 \end{matrix} \;\middle|\; \begin{matrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{matrix}\right)$$

- For which n and k $W(n,k) \neq \emptyset$ is an open question.
- Hadamard conjecture: $W(n,n) \neq \emptyset$ for every $n = 4k, k \in \mathbb{N}$.

Facts about weighing matrices

- Applications: Chemistry, Spectroscopy, Quantum Computing and Coding Theory.
- The main mathematical interest is to exhibit a concrete $W(n,k)$ or to prove that it does not exist.
- To date the smallest Hadamard matrix whose existence is unknown is $H(668)$.
- To date, the weighing matrix with smallest n whose existence is unknown is $W(23,16)$.
- In this note we present a concrete $W(23,16)$.

Fact

For odd n, a $W(n,k)$ exists $\implies$ k is a perfect square.

Facts about weighing matrices

- Applications: Chemistry, Spectroscopy, Quantum Computing and Coding Theory.
- The main mathematical interest is to exhibit a concrete $W(n,k)$ or to prove that it does not exist.
- To date the smallest Hadamard matrix whose existence is unknown is $H(668)$.
- To date, the weighing matrix with smallest n whose existence is unknown is $W(23,16)$.
- In this note we present a concrete $W(23,16)$.

Fact

For odd n, a $W(n,k)$ exists $\implies$ k is a perfect square.

Steps of $W(23,16)$ construction

- Let $W = (w_{i,j})$ be any weighing matrix. Let S, the associated shadow matrix, be defined by $S = (1 - w_{i,j}^2)$.
- Then

$$SS^T \equiv S^T S \equiv nJ_n + kI_n \mod 2.$$

where $J_n = (1)_{n \times n}$.

- Our method:
  1. Geometrizing (=Finding W mod 2, s.t. $WW^T \equiv W^T W \equiv kI_n \mod 2$) =(finding S).
  2. Coloring (= Signing $J_n - S$)

Shadow Geometry of $W(4,2)$

- $W = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & - & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & - \end{pmatrix} = \begin{pmatrix} H_2 & 0_2 \\ 0_2 & H_2 \end{pmatrix}$
- $S = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0_2 & J_2 \\ J_2 & 0_2 \end{pmatrix}$
- $SS^T = \begin{pmatrix} 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \end{pmatrix} = 2W^s.$

Shadow Geometry

- The shadow geometry of $W(23,16)$ was introduced in A. Goldberger, On the finite geometry of $W(23,16)$, http://arxiv.org/abs/1507.02063.
- Necessary conditions for the existence of the shadow geometry of $W(23,16)$ were given, alluding its non existence.
- It is a nice twist that those conditions serve to construct the shadow geometry.
- We used the shadow geometry to color it.

Axiomatic Shadow Geometries

Definition

A shadow geometry with parameters (n, k) is a finite set $\mathscr{P}$, $|\mathscr{P}| = n$ of elements called points together with a family $\mathscr{L}$, $|\mathscr{L}| = n$ of subsets of $\mathscr{P}$ called lines, such that

- L Each line contains $n - k$ points.
- P Each pair of distict lines intersects at $n \bmod 2$ points.
- LD Each point lies in $n - k$ lines.
- PD Each pair of distinct points lies in $n \bmod 2$ lines.

- Remark: Two members of $\mathscr{L}$ may have an equal underlying set.

Given $W(n,k)$, the conditions $WW^T = W^TW = kI_n$ imply that S defined above is an associated shadow geometry such that:

- **Point** Each column of S corresponds to a point.
- **Line** Each row of S corresponds to a line.
- **In** The j^{th} point lies on the i^{th} line if and only if $S_{i,j} = 1$.

Differently stated: S is the incidence matrix of the geometry.

Weighing matrices equivalence

- The following operations are well known to preserve weighing matrices.
  1. rows swap.
  2. columns swap.
  3. multiplying any row by -1.
  4. multiplying any column by -1.
- Swaps extend naturally to the associated shadow matrix S.
- One can use those equivalence operation to bring W and S to a normal form.
- Different normal forms have been used by different authors.

Top and base lines

- We normalize the associated shadow matrix S such that all the 1 digits of the top row live on the first $n-k$ columns. $(1, 1, \cdots 1_{n-k}, 0, 0, \cdots 0_k)$.
- We refer to the top row as the baseline.
- We will write ahead a set of equalities concerning the baseline.
- By symmetry, the same equalities hold with respect to any line.

Induced local geometry

- A choice of a baseline determines an associated local geometry with respect to this baseline.
- The local geometry is the part of the geometry that interacts with the baseline.
- The associated incidence matrix is a submatrix of S.
- Dualy choosing a basepoint there exists an associated dual local geometry with respect to this basepoint.
- Let $m \in \{0,1\}$ be so that $n \equiv m \mod 2$. Define $t = \lfloor \frac{n-k-m}{2} \rfloor$. The number of intersection points between any two lines may be $m + 2i, \forall 0 \leq i \leq t$.

Types

- Let z_{m+2i} denote the number of lines intersecting the baseline with $m+2i$ points.
- The following equations hold

$$\sum_{i=0}^{t} z_{m+2i} = n-1$$

$$\sum_{i=0}^{t}(m+2i)z_{m+2i} = (n-k)(n-k-1)$$

- There are finitely many $t+1$-tupples (z_i) that solve the equations.
- Any such $t+1$-tupple is called a a type for $W(n,k)$.

Reduction to projective geometries

Suppose given $q \in \mathbb{N}$, set $n = q^2 + q + 1$, $k = q^2$. Then $n - k = q + 1$, $m = 1$ and the above equalities become:

$$z_1 + z_3 + \cdots + z_{2[\frac{q}{2}]+1} = q^2 + q$$

$$z_1 + 3z_3 + \cdots + (2[\frac{q}{2}] + 1)z_{2[\frac{q}{2}]+1} = (q + 1)q$$

Subracting the equation gives:

$$2z_3 + \cdots + 2[\frac{q}{2}]z_{2[\frac{q}{2}]+1} = 0$$

Which implies that $z_3 = z_5 = z_{2[\frac{q}{2}]+1} = 0$, $z_1 = q^2 + q$ so that the shadow geometry becomes the well known projective geometry.

Non existence of weighing matrices

Suppose given $q \in \mathbb{Z}, k > q^2$. Then $n = q^2 + q + 1$ is an odd number and $n - k < q + 1$, $m = 1$ and the above equalities become:

$$z_1 + z_3 + \cdots + z_{2[\frac{q}{2}]+1} = q^2 + q$$

$$z_1 + 3z_3 + \cdots + (2[\frac{q}{2}] + 1)z_{2[\frac{q}{2}]+1} < (q + 1)q$$

Subracting the equation gives:

$$2z_3 + \cdots + 2[\frac{q}{2}]z_{2[\frac{q}{2}]+1} < 0$$

This implies that some of $z_3, z_5, \cdots$ must be negative, which is a contradiction, impying a well known result that there are no weighing matrices with such n and k.

Constructing candidates for local geometry matrices

- Given n, k and a corresponding type $(z_{m+2t}, z_{m-2+2t}, \cdots, z_m)$ there are matrices $LG_{n\times n-k}$ whose incidence relations correspond to the type.
- We will row normalize LG by descending order of the weights (i.e. decreasing z_i).
- Within a fixed z_i we normalize by increasing order of binary value.
- Any such matrix has to satisfy a parity condition discussed ahead.

The parity conditions

- Given a geometry matrix S every two distinct points (columns) lie on a n mod 2 number of mutual lines (rows). (parity conditions).
- This must hold for the submatrices LG discussed above.
- Any matrix $LG_{n\times n-k}$ needs to satisfiy $\binom{n-k}{2}$ parity condition.
- The parity conditions are equivalent to $LG^T LG = n \mod 2$ on the non diagonal terms.
- The parity conditions are necessary for LG to be a submatrix of a full geometry matrix S.

Reducing LG matrices

- For the case $n = 18$, $k = 14$ and the type $(z_4, z_2, z_0) = (0, 6, 11)$ there are 462 normalized LG matrices, but none of them satisfy all the parity conditions, as explained in the next slide.
- For the case $n = 18$, $k = 14$ and the type $(z_4, z_2, z_0) = (1, 4, 12)$ there are 126 normalized LG matrices, and 21 of them satisfy the parity conditions.
- For the case $n = 23$, $k = 16$ and the type $(z_7, z_5, z_3, z_1) = (3, 0, 1, 18)$ it can be shown that any LG matrix can not satisfy the parity conditions.

Explanation for the first type of $(n, k) = (18, 14)$

- There are $\binom{4}{2} = 6$ ways fill two digits in 4 places.
- Index the fillings by $\mathscr{F}_6 = \mathscr{F} = \{1, 2, 3, 4, 5, 6\}$.
- Those can be ordered arbirarily say in a non decreasing order.
- The type $(z_4, z_2, z_0) = (0, 6, 17)$ indicates that we need to fill in 6 positions elements from $\mathscr{F}_6$.
- After normalization the matrices LG correspond to non decreasing sequences of functions $\mathscr{F}_6 \to \mathscr{F}_6$.
- There are $\binom{11}{5} = 462$ such functions.

Explanation for the first type of $(n, k) = (18, 14)$ continued

- For example the baseline and the sequence $(1, 2, 3, 3, 4, 4)$ corresponds the matrix $LG_{18,4} = \binom{A_{7,4}}{0_{11,4}}$.
- $$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$
- $$LG^T LG = \begin{pmatrix} 3 & 1 & 1 & 3 \\ 1 & 4 & 3 & 2 \\ 1 & 3 & 4 & 2 \\ 3 & 2 & 2 & 5 \end{pmatrix} \neq 0 \mod 2.$$

The hook (reish) matrix

- Once a LG submatrix which satisfies the parity conditions has been established, it is of the following form $\begin{pmatrix} X_{n-k\times n-k} \\ Z_{k\times n-k} \end{pmatrix}$.
- The matrix X and the dual local geometry can be used to complete the data to the following hook type matrix.
$$\begin{pmatrix} X_{n-k\times n-k} & Y_{n-k\times k} \\ Z_{k\times n-k} & ??_{k\times k} \end{pmatrix}$$
- We remark that in principle the types of the local and the local dual geometries need not be the same.
- But for all the 7 geometries we happend to find for $(n,16), n=23,25,27,29$ it came out that X was symmetric, $Y=Z^T$ and the local and dual local types were the same.

Our reish matrix for $n = 23$ and $k = 16$

For the case $n = 23$ $k = 16$ we got the following reish matrix:

$$\begin{pmatrix} X_{7\times 7} & Y_{7\times 16} \\ Z_{16\times 7} & C_{16\times 16} \end{pmatrix}$$

With

$$X = \left(\begin{array}{ccc|cccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{array}\right) = \begin{pmatrix} J_3 & J_{3\times 4} \\ J_{4\times 3} & 0_4 \end{pmatrix}$$

and both the LG and the dual LG have the type $(z_7, z_5, z_3, z_1) = (2, 0, 4, 16)$.

Our reish matrix for $n = 23$ and $k = 16$, the matrix Y

The matrix $Y_{7\times 16}$ is composed of eight matrices

$$Y = \begin{pmatrix} 0_{3\times 4} & 0_{3\times 4} & 0_{3\times 4} & 0_{3\times 4} \\ K_1 & K_2 & K_3 & K_4 \end{pmatrix}$$

with $K1 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $K2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$K3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $K4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$

For us the matrix Z was equal to Y^T.

Our reish matrix for $n = 23$ and $k = 16$

Our total reish matrix with the core zero equals

$$reish = \left(\begin{array}{ccc|cccc|cccc|cccc|cccc|cccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
\hline
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)$$

The tiling of the core matrix

The core matrix $C_{16\times 16}$ has the following form:

$$C = \begin{pmatrix} t_{1,1} & t_{1,2} & t_{1,3} & t_{1,4} \\ t_{2,1} & t_{2,2} & t_{2,3} & t_{2,4} \\ t_{3,1} & t_{3,2} & t_{3,3} & t_{3,4} \\ t_{4,1} & t_{4,2} & t_{4,3} & t_{4,4} \end{pmatrix}$$

where each $t_{i,j}$ is a 4×4 matrix. Thus C is a 4×4 block matrix of 4×4 matrices (tiles) , and there is a tiling process (finding the tiles) needed to be completed.

Tiles

- The 1^{st} row of the core matrix must intersect the 4^{th} $(5^{th}, 6^{th}, 7^{th})$ row of the whole matrix by one or three points.
- This can be indicated on the matrix on page 24.
- The total weight of this row must be 6.
- 6 must be partitioned as $1 + 1 + 1 + 3$, (up to ordering).
- The same is true for all rows and columns in the core matrix.

Normalization within a tile

- Each row in each tile has either 1 or 3 digits so altogether there are $2i$ digits $\forall 2 \leq i \leq 6$.
- In the core matrix one can permute the rows 1-4 with any permutation, and similarly for columns 1-4.
- This permutations allow to normalize the top left tile, but not necessarily the tiles in the same row and column tile.
- There are 2^9 (non normalized) possible tiles.
- There are 7 normalized tiles.

The normalized tiles ordered by weight

$$T4N = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad T6_1N = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T6_2N = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad T8N = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

And the other normalized tiles are defined by $T10_iN = J - T6_iN, i = 1, 2$ and $T12N = J - T4N$.

Normalizing the tiling

- As each row has weight 6, then each layer of tiles has total weight 24.
- After reordering the tiles this allows only few sequences of tiles in each layer.
- $(T12N, T4N, T4N, T4N), (T10_iN, T6_jN, T4N, T4N)$
- $(T8N, T8N, T4N, T4N), (T8N, T6_iN, T6_jN, T4N)$
- $(T6_iN, T6_jN, T6_kN, T6_lN)$
- The above list carries 14 ordered sequences.

Normalizing the core reish

- By reordering the tiles we may bring the heaviest tile to the top left position.
- We may further normalize the heaviest tile to its normal form.
- In the reish of the heaviest tile we may not normalize the other tiles to their normal form, unless they are T4.
- If we did not normalize the T4, we can normalize the 3×3 core and then the 2×2 core.

The second layer of the inner core

- Choosing the sequence $T12N$, $T4N$, $T4N$, $T4N$ and plugging it in the reish of the core gives the matrix

$$\left(\begin{array}{cc|c|ccc} J_3 & J_{3\times 4} & 0_{3\times 4} & 0_{3\times 4} & 0_{3\times 4} & 0_{3\times 4} \\ J_{4\times 3} & 0_{4\times 4} & K1 & K2 & K3 & K4 \\ \hline 0_{4\times 3} & K1^T & T12N & T4N & T4N & T4N \\ \cline{3-6} 0_{4\times 3} & K2^T & T4N & ? & ? & ? \\ 0_{4\times 3} & K3^T & T4N & ? & ? & ? \\ 0_{4\times 3} & K4^T & T4N & ? & ? & ? \end{array}\right)$$

The second layer of the inner core, continued

- Each of the 4 lines in the second layer should intersect each of the 4 lines in the first layer by an odd number of points.
- There are 16 orthogonality conditions on 2^{27} fillings. If they were in general position we would be left with 2^{11} fillings.
- Unfortunately there are 2^{18} solutions for the second layer.
- There are $\binom{4}{2}$ parity conditions and also the condition that the 3 digits in each tile can not occur in the same line. This reduces the solutions to only 1224 cases.
- Each solution is a second layer completing the first layer.

The third layer of the inner core

- Any solution for the second layer is by symmetry also a solution for the third layer.
- A double loop on the solutions for the second later is run, and each pair of solutions has to satisfy orthogonality and the $(1, 1, 1, 3)$ conditions.
- This leaves only 1008 solutions for the second and third layers.

The fourth layer of the inner core

- For any of the 1008 solution above we try the fourth layer.
- Each solution has to satisfy orthogonality, the $(1, 1, 1, 3)$ and parity conditions, both horizontally and vertically.
- This leaves only 576 solutions for all the core matrix.
- On any full matrix check the equation SS^T has odd elements. This leaves 144 'kosher' geometries.

Our first normalized geometry for $(n, k) = (23, 16)$

- Some of the geometry matrices found can be normalized to the form

$$\left(\begin{array}{cc|c|ccc}
J_3 & J_{3\times 4} & 0_{3\times 4} & 0_{3\times 4} & 0_{3\times 4} & 0_{3\times 4} \\
J_{4\times 3} & 0_{4\times 4} & K1 & K2 & K3 & K4 \\
\hline
0_{4\times 3} & K1^T & T12N & T4N & T4N & T4N \\
\cline{3-6}
0_{4\times 3} & K2^T & T4N & T12 & T4 & T4 \\
0_{4\times 3} & K3^T & T4N & T4 & T12 & T4 \\
0_{4\times 3} & K4^T & T4N & T4 & T4 & T12
\end{array}\right)$$

- Observe the that the reish of the core is symmetric.
- The 3×3 inner core is not symmetric.

The first geometry matrix for $n = 23$ $k = 16$ specifically

1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0
1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
0	0	0	1	0	0	0	0	1	1	1	1	0	0	0	1	0	0	0	1	0	0	0
0	0	0	1	0	0	0	1	0	1	1	0	1	0	0	0	1	0	0	0	1	0	0
0	0	0	1	0	0	0	1	1	0	1	0	0	1	0	0	0	1	0	0	0	1	0
0	0	0	1	0	0	0	1	1	1	0	0	0	0	1	0	0	0	1	0	0	0	1
0	0	0	0	1	0	0	1	0	0	0	1	1	1	0	0	1	0	0	0	0	1	0
0	0	0	0	1	0	0	0	1	0	0	1	1	0	1	1	0	0	0	0	0	0	1
0	0	0	0	1	0	0	0	0	1	0	1	0	1	1	0	0	0	1	1	0	0	0
0	0	0	0	1	0	0	0	0	0	1	0	1	1	1	0	0	1	0	0	1	0	0
0	0	0	0	0	1	0	1	0	0	0	0	0	1	0	1	1	1	0	0	1	0	0
0	0	0	0	0	1	0	0	1	0	0	0	0	0	1	1	1	0	1	1	0	0	0
0	0	0	0	0	1	0	0	0	1	0	1	0	0	0	1	0	1	1	0	0	0	1
0	0	0	0	0	1	0	0	0	0	1	0	1	0	0	0	1	1	1	0	0	1	0
0	0	0	0	0	0	1	1	0	0	0	0	1	0	0	0	0	1	0	1	1	1	0
0	0	0	0	0	0	1	0	1	0	0	1	0	0	0	0	0	0	1	1	1	0	1
0	0	0	0	0	0	1	0	0	1	0	0	0	0	1	1	0	0	0	1	0	1	1
0	0	0	0	0	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	1	1

Our specific W in $W(23, 16)$

$$\begin{pmatrix}
0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&-&-&-&-&-&-&-&-\\
0&0&0&0&0&0&0&1&1&1&1&-&-&-&-&1&1&1&1&-&-&-&-\\
0&0&0&1&1&1&1&0&0&0&0&-&1&1&-&-&1&-&1&-&1&1&-\\
0&0&0&-&-&1&1&1&-&-&1&0&0&0&0&-&1&-&1&1&-&-&1\\
0&0&0&1&-&1&-&1&-&1&-&-&1&-&1&0&0&0&0&-&1&-&1\\
0&0&0&-&1&1&-&-&1&1&-&1&-&-&1&-&1&-&1&0&0&0&0\\
1&1&1&0&1&-&-&1&0&0&0&0&1&-&-&0&-&-&1&0&-&1&1\\
1&1&1&0&-&1&1&0&1&0&0&-&0&-&1&1&0&-&-&1&0&1&-\\
1&1&1&0&-&-&1&0&0&1&0&1&-&0&-&-&1&0&-&-&1&0&1\\
1&1&1&0&1&1&-&0&0&0&1&-&-&1&0&-&-&1&0&1&1&-&0\\
1&1&-&1&0&-&1&0&-&1&-&0&0&0&1&-&0&1&1&1&-&0&-\\
1&1&-&-&0&1&-&1&0&-&-&0&0&1&0&0&1&1&-&-&-&1&0\\
1&1&-&-&0&-&-&-&-&0&1&0&1&0&0&1&1&-&0&0&1&-&-\\
1&1&-&1&0&1&1&-&1&-&0&1&0&0&0&1&-&0&1&-&0&-&1\\
1&-&-&-&1&0&1&0&1&1&-&-&1&0&-&0&0&0&-&1&0&-&1\\
1&-&-&1&-&0&-&-&0&1&1&-&-&1&0&0&0&-&0&0&-&1&1\\
1&-&-&-&1&0&1&1&-&0&1&0&-&-&1&0&-&0&0&-&1&1&0\\
1&-&-&1&-&0&-&1&1&-&0&1&0&-&-&-&0&0&0&1&1&0&-\\
1&-&1&-&-&1&0&0&-&1&-&1&0&1&-&1&-&0&1&0&0&0&-\\
1&-&1&1&1&-&0&1&0&-&-&0&-&1&1&1&1&-&0&0&0&-&0\\
1&-&1&1&1&1&0&-&-&0&1&1&1&-&0&0&1&1&-&0&-&0&0\\
1&-&1&-&-&-&0&-&1&-&0&-&1&0&1&-&0&1&1&-&0&0&0
\end{pmatrix}$$

http://www.emba.uvm.edu/ jdinitz/hcd/W2316.txt

Our second normalized geometry for $(n,k) = (23,16)$

- The rest of the geometry matrices found can be normalized to the form

$$\left(\begin{array}{cc|c|ccc}
J_3 & J_{3\times 4} & 0_{3\times 4} & 0_{3\times 4} & 0_{3\times 4} & 0_{3\times 4} \\
J_{4\times 3} & 0_{4\times 4} & K1 & K2 & K3 & K4 \\
\hline
0_{4\times 3} & K1^T & T12N & T4N & T4N & T4N \\
\cline{3-6}
0_{4\times 3} & K2^T & T4N & T8 & T8 & T4 \\
0_{4\times 3} & K3^T & T4N & T8 & T4 & T8 \\
0_{4\times 3} & K4^T & T4N & T4 & T8 & T8
\end{array}\right)$$

- Observe the that the reish of the core is symmetric.
- The 3×3 inner core is not symmetric.
- All the geometries containing a $T12$ could be normalized to one of the two forms above.

Our second normalized geometry for $(n, k) = (23, 16)$, specifically

The other geometry matrices found can be normalized to the form

$$\left(\begin{array}{ccc|cccc|cccc|cccc|cccc|cccc}
1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
\hline
1&1&1&0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0\\
1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0\\
1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1\\
\hline
0&0&0&1&0&0&0&0&1&1&1&1&0&0&0&1&0&0&0&1&0&0&0\\
0&0&0&1&0&0&0&1&0&1&1&0&1&0&0&0&1&0&0&0&1&0&0\\
0&0&0&1&0&0&0&1&1&0&1&0&0&1&0&0&0&1&0&0&0&1&0\\
0&0&0&1&0&0&0&1&1&1&0&0&0&0&1&0&0&0&1&0&0&0&1\\
\hline
0&0&0&0&1&0&0&1&0&0&0&1&1&1&0&0&1&0&0&0&0&1&0\\
0&0&0&0&1&0&0&0&1&0&0&0&0&1&0&0&1&1&1&0&0&0&1\\
0&0&0&0&1&0&0&0&0&1&0&1&0&1&1&0&0&0&1&1&0&0&0\\
0&0&0&0&1&0&0&0&0&0&1&1&0&0&0&1&1&0&1&0&1&0&0\\
\hline
0&0&0&0&0&1&0&1&0&0&0&0&1&0&0&0&0&1&0&1&1&1&0\\
0&0&0&0&0&1&0&0&1&0&0&0&1&1&1&0&0&0&1&0&0&1&0\\
0&0&0&0&0&1&0&0&0&1&0&0&0&0&1&1&0&0&0&1&0&1&1\\
0&0&0&0&0&1&0&0&0&0&1&1&1&0&1&0&1&0&0&1&0&0&0\\
\hline
0&0&0&0&0&0&1&1&0&0&0&0&0&1&0&1&1&1&0&0&1&0&0\\
0&0&0&0&0&0&1&0&1&0&0&0&0&0&1&0&0&1&0&0&1&1&1\\
0&0&0&0&0&0&1&0&0&1&0&1&0&0&0&1&0&1&1&0&0&0&1\\
0&0&0&0&0&0&1&0&0&0&1&0&1&0&0&1&0&0&0&1&1&0&1
\end{array}\right)$$

the type and tiling for $n = 25$ and $k = 16$

- The geometry we found had the type $(z_9, z_7, z_5, z_3, z_1) = (4, 0, 4, 0, 16)$ for both the local and the dual local geometry.
- Again the core of 16×16 is divided to 4×4 tiles each of which is 4×4 matrix.
- Again each line and point intersect each tile with 1 or 3 digits.
- now $n - k = 9$ so we need to present 8 digits and this is possible only as $3 + 3 + 1 + 1$.
- This time the sum of the digits along one layer and column layer equals 32.

Our geometry for $n = 25$, $k = 16$

the reish matrix is of the form

$$\left(\begin{array}{cc|cccc} J_5 & J_{5\times4} & 0_{5\times4} & 0_{5\times4} & 0_{5\times4} & 0_{5\times4} \\ J_{4\times5} & 0_{4\times4} & K1 & K2 & K3 & K4 \\ \hline 0_{4\times5} & K1^T & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} \\ 0_{4\times5} & K2^T & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} \\ 0_{4\times5} & K3^T & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} \\ 0_{4\times5} & K4^T & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} \end{array}\right)$$

and is symmeteric.

Observe the tiling of the core matrix

$$\begin{pmatrix} T12N & T12 & T4N & T4N \\ T12 & T4 & T4 & T12N \\ T4N & T4 & T12 & T12 \\ T4N & T12N & T12 & T4 \end{pmatrix}$$

the geometry of 25, 16 specifically

1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0
1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
0	0	0	0	0	1	0	0	0	0	1	1	1	1	1	1	0	1	0	0	0	1	0	0	0
0	0	0	0	0	1	0	0	0	1	0	1	1	1	1	0	1	0	1	0	0	0	1	0	0
0	0	0	0	0	1	0	0	0	1	1	0	1	1	0	1	1	0	0	1	0	0	0	1	0
0	0	0	0	0	1	0	0	0	1	1	1	0	0	1	1	1	0	0	0	1	0	0	0	1
0	0	0	0	0	0	1	0	0	1	1	1	0	0	1	0	0	0	1	0	0	0	1	1	1
0	0	0	0	0	0	1	0	0	1	1	0	1	1	0	0	0	1	0	0	0	1	0	1	1
0	0	0	0	0	0	1	0	0	1	0	1	1	0	0	0	1	0	0	0	1	1	1	0	1
0	0	0	0	0	0	1	0	0	0	1	1	1	0	0	1	0	0	0	1	0	1	1	1	0
0	0	0	0	0	0	0	1	0	1	0	0	0	0	1	0	0	1	1	1	0	1	0	1	1
0	0	0	0	0	0	0	1	0	0	1	0	0	1	0	0	0	1	1	0	1	0	1	1	1
0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	1	1	0	1	1	1	1	1	0
0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	1	0	0	1	1	1	1	1	0	1
0	0	0	0	0	0	0	0	1	1	0	0	0	0	1	1	1	1	0	1	1	0	0	1	0
0	0	0	0	0	0	0	0	1	0	1	0	0	1	0	1	1	0	1	1	1	0	0	0	1
0	0	0	0	0	0	0	0	1	0	0	1	0	1	1	0	1	1	1	1	0	1	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	1	1	1	1	0	1	1	0	1	0	1	0	0

Observe that this matrix is symmetric!!!

Our geometry for $n = 27$, $k = 16$

Our reish for the matrix is:

$$\left(\begin{array}{cc|cccc} J_7 & J_{7\times 4} & 0_{7\times 4} & 0_{7\times 4} & 0_{7\times 4} & 0_{7\times 4} \\ J_{4\times 7} & 0_{4\times 4} & K1 & K2 & K3 & K4 \\ \hline 0_{4\times 7} & K1^T & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} \\ 0_{4\times 7} & K2^T & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} \\ 0_{4\times 7} & K3^T & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} \\ 0_{4\times 7} & K4^T & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} \end{array}\right)$$

and it corresponds to the type

$(z_{11}, z_9, z_7, z_5, z_3, z_1) = (6, 0, 4, 0, 0, 16)$

A property of tiles

- If T is a tile then J-T is a tile and both have 1 or 3 digits in each row and column
- Therefore

$$(J - T1)(J - T2)^T = JJ^T - JT2^T - T1J^T + T1T2^T.$$

- It follows that $(J - T1)(J - T2)^T$ and $T1T2^T$ have the corresponding terms equal mod 2.

The core of our $(n, k) = (27, 16)$ geometry

- As $n - k = 11$, the 10 core digits should be partitioned as $3 + 3 + 3 + 1$.
- replacing each core tile T in $(n, k) = (23, 16)$ with $J - T$ changes $1 + 1 + 1 + 3$ to $3 + 3 + 3 + 1$.
- This change replaces $T1T2^T$ by $(J - T1)(J - T2)^T$ which has the same parity in all terms.
- This gives the following core

$$\begin{pmatrix} T4N & T12N & T12N & T12N \\ T12N & T4 & T12 & T12 \\ T12N & T12 & T4 & T12 \\ T12N & T12 & T12 & T4 \end{pmatrix}$$

- This gives a full Shaddow geometry.

Our geometry for n=29,k=16

- Our reish for the matrix is:

$$\left(\begin{array}{cc|cccc} J_9 & J_{9\times 4} & 0_{9\times 4} & 0_{9\times 4} & 0_{9\times 4} & 0_{9\times 4} \\ J_{4\times 9} & 0_{4\times 4} & K1 & K2 & K3 & K4 \\ \hline 0_{4\times 9} & K1^T & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} \\ 0_{4\times 9} & K2^T & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} \\ 0_{4\times 9} & K3^T & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} \\ 0_{4\times 9} & K4^T & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} \end{array}\right).$$

- It corresponds to the type $(z_{13}, z_{11}, z_9, z_7, z_5, z_3, z_1) = (8, 0, 4, 0, 0, 0, 16)$.
- The core is obtained from that of $n = 21, k = 16$ by adjoinning each tile T to becomes $J - T$.
- This gives a full geometry.

The geometry for $(n, k) = (21, 16)$

- The reish matrix is

$$\left(\begin{array}{cc|cccc} J_1 & J_{1\times 4} & 0_{1\times 4} & 0_{1\times 4} & 0_{1\times 4} & 0_{1\times 4} \\ J_{4\times 1} & 0_{4\times 4} & K1 & K2 & K3 & K4 \\ \hline 0_{4\times 1} & K1^T & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} \\ 0_{4\times 1} & K2^T & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} \\ 0_{4\times 1} & K3^T & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} \\ 0_{4\times 1} & K4^T & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} \end{array}\right)$$

- Each line and point intersect each tile in a single digit,
- It holds that $1 + 1 + 1 + 1 = n - k - 1 = 21 - 16 - 1$.
- As in any projective geometry of order $q \in \mathbb{Z}$, it holds that $z_1 = q^2 + q = 20$.

The reish of the core

- The reish of the inner core can be normalized so that it will include only I_4 matrices.
- After this normalization S becomes

$$\left(\begin{array}{cc|cccc} J_1 & J_{1\times 4} & 0_{1\times 4} & 0_{1\times 4} & 0_{1\times 4} & 0_{1\times 4} \\ J_{4\times 1} & 0_{4\times 4} & K1 & K2 & K3 & K4 \\ \hline 0_{4\times 1} & K1^T & I_4 & I_4 & I_4 & I_4 \\ 0_{4\times 1} & K2^T & I_4 & ? & ? & ? \\ 0_{4\times 1} & K3^T & I_4 & ? & ? & ? \\ 0_{4\times 1} & K4^T & I_4 & ? & ? & ? \end{array}\right)$$

- All the matrices denoted with question marks are permutation matrices.
- The well known projective geometry $\mathbb{P}^2(\mathbb{F}_4)$ has a matrix of this form.

The inner core

- The 4 permutation matrices on each of rows and columns of the core must sum up to J.
- Each row 10-21 determines a permutation $\sigma = (i_1, i_2, i_3.i_4)$ where i_j is the position of the digit 1 in the j^{th} tile.
- Thus each layer gives a 4×4 latin square.

Multiple Latin Squares

- Denote $\mathscr{F} = \mathscr{F}_4 = \{1, 2, 3, 4\}$.
- The latin square defined by rows 10-13 can be presented as a function $ls : \mathscr{F}^2 \to \mathscr{F}$ each of which partial functions are 1-1.
- Similarly rows 14-17 and 18-21 give latin squares.
- All those latin squares can be put together in a function $3ls : \mathscr{F}^2 \to \mathscr{F}^3$ such that each coordinate projection of $3ls$ into $\mathscr{F}$ is a latin square.

Mutually orthogonal Latin Squares

- The above $3ls$ is a subgraph of $\mathscr{F}^5$ with the property that any projection into coordinates $\mathscr{F}^2$ is 1-1 and onto.
- This object is known as a triple of Mutually Orthogonal Latin Squares (MOLS).
- For a given number $q \in \mathbb{Z}$, there exists at most $q-1$-upple of MOLS.
- For a given number $q \in \mathbb{Z}$, (there exists a $q-1$-upple of $q \times q$ MOLS) $\iff$ (There exists a planar projective geometry of order q).
- It is conjectured that the above condition holds $\iff$ q is a prime power.