SPHERICAL EMBEDDINGS OF STRONGLY REGULAR GRAPHS

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PLAN

Two-distance tight frames

Spherical two-distance 2-designs

Strongly regular graphs

A finite collection of vectors $S = \{x_i, 1 \leq i \leq N\} \subset \mathbb{R}^n$ is called a finite frame for the Euclidean space $\mathbb{R}^n$ if there are constants $0 < A \leq B < \infty$ such that for all $x \in \mathbb{R}^n$

$$A||x||^2 \leq \sum_{i=1}^{N} \langle x, x_i \rangle^2 \leq B||x||^2. \tag{1}$$

If $A = B$, then S is called an A-tight frame.

An equivalent condition for A-tight frames is $Ax = \sum_{i=1}^{N} \langle x, x_i \rangle x_i$ for all $x \in \mathbb{R}^n$.

If in addition $\|x_i\| = 1$ for all i, then S is a unit-norm tight frame.

Theorem (Benedetto-Fickus, 2003)

If $N > n$ then

$$\sum_{i,j=1}^{N} \langle x_i, x_j \rangle^2 \geq \frac{N^2}{n} \tag{2}$$

with equality if and only if S is a tight frame.

A finite collection of unit vectors $S \subset \mathbb{R}^n$ is called a spherical two-distance set if there are two numbers a and b such that the inner products of distinct vectors from S are either a or b. If at the same time S is a finite unit-norm tight frame, we call it a two-distance tight frame.

If $a + b \neq 0$, the definition of a tight frame immediately shows that S must be regular, i.e. the distribution of inner products is the same for each vector x_i.

EQUIANGULAR TIGHT FRAMES

If the two inner products of a two-distance tight frame S satisfy the condition $a = -b$, then it is called an equiangular tight frame.

$$\boxed{\text{Equiangular tight frames}}$$
$$\updownarrow$$
$$\boxed{\text{Certain strongly regular graphs}}$$

See Waldron (Linear Alg. Appl., vol. 41, pp. 2228-2242, 2009).

For a natural number t, a finite set of vectors $S = \{x_i, 1 \leq i \leq N\} \subset \mathbb{S}^{n-1}$ is called a spherical t-design if for any polynomial f(x) of degree at most t

$$\frac{1}{|\mathbb{S}^{n-1}|} \int_{x \in \mathbb{S}^{n-1}} f(x) d\sigma(x) = \frac{1}{N} \sum_{i=1}^{n} f(x_i). \tag{3}$$

Examples:

- Icosahedron and dodecahedron are 5-designs
- 120-cell and 600-cell are 11-designs
- Root systems
- Minimal vectors of the Leech lattice form an 11-design

SPHERICAL 2-DESIGNS ARE TIGHT FRAMES

$S = \{x_i, 1 \leq i \leq N\} \subset \mathbb{S}^{n-1}$ is a spherical 2-design if and only if

$$\sum_{i,j=1}^{N} \langle x_i, x_j \rangle^2 = \frac{N^2}{n} \text{ and } \sum_{i=1}^{N} x_i = 0 \tag{4}$$

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$$\begin{tikzcd} \boxed{\text{Two-distance tight frames}} \\ \boxed{\text{Spherical two-distance 2-designs}} \arrow[u] \end{tikzcd}$$

A regular graph of degree k on v vertices is called strongly regular if every two adjacent vertices have λ common neighbors and every two non-adjacent vertices have μ common neighbors. We use the notation SRG(v, k, λ, μ) to denote such a graph.

Examples:

- Cycle of length 5
- Petersen graph
- Hoffman-Singleton graph
- Conference graphs
- $n \times n$ rook's graphs

Delsarte, Goethals, and Seidel obtained a spherical embedding of $\Gamma = \mathrm{SRG}(v, k, \lambda, \mu)$ by associating a basis of $\mathbb{R}^v$ with the vertices of Γ, projecting these vectors on an eigenspace of the adjacency matrix of Γ, and normalizing lengths of projections. They also showed that this embedding forms a two-distance 2-design.

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$$\begin{tikzcd}
\boxed{\text{Spherical two-distance 2-designs}} \\
\boxed{\text{Strongly regular graphs}} \arrow[u]
\end{tikzcd}$$

Proposition

If S is a regular 2-distance tight frame in $\mathbb{R}^n$, then S is either an n-dimensional spherical 2-design, or is similar to an $(n-1)$-dimensional spherical 2-design contained in a subsphere of radius $\sqrt{1 - 1/n}$.

Proof.

Let $s = \sum_{i=1}^{N} x_i$. The value $t := \langle x_i, s \rangle$ is the same for all i. Using an equivalent definition of tight frames, we get $\frac{N}{n} s = \sum_{i=1}^{N} t x_i = ts$. Hence either $s = 0$ or $t = \frac{N}{n}$. □

REGULAR TWO-DISTANCE TIGHT FRAMES ARE ALMOST 2-DESIGNS

Proposition

If S is a regular 2-distance tight frame in $\mathbb{R}^n$, then S is either an n-dimensional spherical 2-design, or is similar to an $(n-1)$-dimensional spherical 2-design contained in a subsphere of radius $\sqrt{1-1/n}$.

$$\begin{array}{c}\boxed{\text{Regular two-distance tight frames}}\\ \updownarrow \\ \boxed{\text{Spherical two-distance 2-designs}}\end{array}$$

Proposition

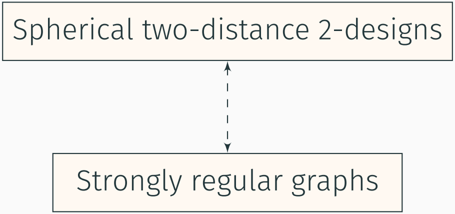
If S is a regular two-distance tight frame, then its associated graph Γ_1 (and Γ_2 as the complement of Γ_1) is a strongly regular graph.

Proof.

Use a theorem by Delsarte, Goethals, Seidel for 2-designs or just check the definition of tight frames carefully. $\square$

Proposition

If S is a regular two-distance tight frame, then its associated graph Γ_1 (and Γ_2 as the complement of Γ_1) is a strongly regular graph.



TWO-DISTANCE TIGHT FRAMES ARE DEFINED BY SRG'S

$$\begin{array}{c}\boxed{\text{Spherical two-distance 2-designs}}\\ \updownarrow\\ \boxed{\text{Strongly regular graphs}}\end{array}$$

Question

What two-distance spherical embeddings of SRG's form 2-designs?

For a given SRG(v, k, λ, μ) which is not a complete or empty graph, its adjacency matrix has three mutually orthogonal eigenspaces (subspaces) that correspond to three eigenvalues: the all-one vector $\mathbf{1}$ with eigenvalue k and subspaces E_1 and E_2.

Projecting an orthonormal basis of $\mathbb{R}^n$ on $\mathbf{1}$ and normalizing gives a trivial 1-dimensional embedding, where all inner products are 1.

Projections on E_1 or on E_2 after normalization give two-distance 2-designs.

Direct orthogonal sum of two spherical embeddings is a spherical embedding.

Proposition

For a given $\Gamma = \mathrm{SRG}(N, k, \lambda, \mu)$, any two-distance spherical embedding may be represented as a direct orthogonal sum of the trivial and Delsart-Goethals-Seidel embeddings.

Proof.

Since the Gram matrix is positive definite, the set of possible values of scalar products a and b associated to embeddings of Γ forms a triangle on (a, b)-plane with vertices corresponding to the trivial and two Delsarte-Goethals-Seidel embeddings. Therefore, any pair (a, b) may be obtained as a non-negative linear combination of scalar products from these embeddings. $\square$

Theorem

Any spherical two-distance 2-design with graph $\Gamma = \mathrm{SRG}(N, k, \lambda, \mu)$ for one of the distances is either one of two Delsarte-Goethals-Seidel embeddings, or a regular $(N-1)$-dimensional simplex.

Proof.

Use the previous proposition and the description of embeddings via eigenspaces of the adjacency matrix of Γ. $\square$

Theorem

Any spherical two-distance 2-design with graph $\Gamma = \mathrm{SRG}(N, k, \lambda, \mu)$ for one of the distances is either one of two Delsarte-Goethals-Seidel embeddings, or a regular $(N-1)$-dimensional simplex.

$$\begin{tikzcd} \boxed{\text{Spherical two-distance 2-designs}} \arrow[d, leftrightarrow] \\ \boxed{\text{Strongly regular graphs}} \end{tikzcd}$$

Theorem

Let S be a regular two-distance tight frame in $\mathbb{R}^n$. Then S forms a spherical two-distance 2-design or a shifted 2-design. In either case S can be obtained as a spherical embedding of a strongly regular graph. Under spherical embedding, every strongly regular graph gives rise to three different two-distance 2-designs and therefore, to six different two-distance tight frames, two of which are regular simplices.

SRG(N,k,λ,μ)	2-design (n,N,a,b) shifted 2-design (n,N,a,b)
$(10,6,3,4)$	$(4,10,\frac{1}{6},-\frac{2}{3})$; $(5,10,\frac{1}{3},-\frac{1}{3})$; $(5,10,\frac{1}{3},-\frac{1}{3})$; $(6,10,\frac{4}{9},-\frac{1}{9})$
$(15,8,4,4)$	$(5,15,\frac{1}{4},-\frac{1}{2})$; $(9,15,\frac{1}{6},-\frac{1}{4})$; $(6,15,\frac{3}{8},-\frac{1}{4})$; $(10,15,\frac{1}{4},-\frac{1}{8})$
$(16,10,6,6)$	$(5,16,\frac{1}{5},-\frac{3}{5})$; $(10,16,\frac{1}{5},-\frac{1}{5})$; $(6,16,\frac{1}{3},-\frac{1}{3})$; $(11,16,\frac{3}{11},-\frac{1}{11})$

THANK YOU!