The part-frequency matrices of a partition

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A *partition* of an integer n is a sequence $\lambda = (\lambda_1, \ldots, \lambda_r)$ such that $\lambda_1 \geq \lambda_r \geq 1$ and $\lambda_1 + \cdots + \lambda_r = n$. The partitions of 4 are

$$4,\ 3+1,\ 2+2,\ 2+1+1,\ 1+1+1+1.$$

The generating function for the number of partitions of n is

$$\sum_{n=0}^{\infty} p(q) q^n = \frac{1}{(q;q)_\infty}$$

where we have used the notation

$$(a;q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \qquad (a;q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k).$$

Threads

One of the oldest themes in partition theory is the relationship between partitions in which no part is divisible by a given number m, and those in which parts must appear fewer than m times. These two sets are equinumerous, as shown either by the equality of their generating functions

$$\prod_{m \nmid k} \frac{1}{1-q^k} = \prod_{k=1}^{\infty} \frac{1-q^{mk}}{1-q^k} = \prod_{k=1}^{\infty} \left(1 + q^k + q^{2k} + \cdots + q^{(m-1)k}\right)$$

or by Glaisher's bijection which maps the two sets together:

if λ contains am^k ℓ times, $m \nmid a$, then write ℓm^k appearances of a. Reverse by writing the number of appearances of a in m-ary digits.

Threads

A newer theme in partition theory (a mere 100 years old) is the study of congruences for $p(n)$. Ramanujan's congruences are

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5} \\ p(7n+5) &\equiv 0 \pmod{7}. \\ p(11n+6) &\equiv 0 \pmod{11} \end{aligned}$$

In fact for any $\alpha, \beta, \gamma \geq 0$, there is a δ such that

$$p(5^{\alpha} 7^{\lfloor \frac{\beta+1}{2} \rfloor} 11^{\gamma} n + \delta) \equiv 0 \pmod{5^{\alpha} 7^{\beta} 11^{\gamma}}.$$

Threads

These were first proved for low powers by Hardy and Ramanujan, then for all powers of 5 and 7 by Watson, and finally for powers of 11 by A. O. L. Atkin. In the meantime Freeman Dyson had constructed the *rank* of a partition

$$rank(\lambda) = \text{largest part} - \text{number of parts}$$

and Atkin and Swinnerton-Dyer showed that the classes of partitions with rank a mod 5 or 7 neatly divided the partitions of $5n+4$ and $7n+6$ into equally sized classes. It doesn't work for 11, but in 1987 George Andrews and Frank Garvan successfully produced the *crank* which did the job.

Indeed, Ahlgren and Ono showed that there are congruences for $p(n)$ modulo every prime from 5 on, and Karl Mahlburg showed that the crank explains every one of these.

Tying them together

One goal of this talk is to show that there is a partition statistic, related to a very natural generalization of Glaisher's bijection, which *also* realizes all of these congruences.

In order to prove this, we'll construct the *part-frequency matrices* of the title. We'll use them to give an identity or two and talk about some of the questions they inspire. A second goal is to demonstrate that these might be a useful tool for partition proofs.

Part-frequency matrices

The *modulus-m part-frequency matrices* of a partition λ are the infinite sequence of matrices M_j indexed by the j not divisible by m, in which if part jm^k appears in λ $a_{j,0}m^0 + a_{j,1}m^1 + a_{j,2}m^2 + \dots$ times, then row k of matrix M_j has entries $a_{j,\ell}$.

Example

If $m = 2$, the partition $(20, 5, 5, 4, 2, 2, 1, 1, 1, 1, 1)$ of 43 would be depicted

$$
\begin{array}{c|ccc}
1 & & & \\
\hline
 & 1 & 0 & 1 \\
 & 0 & 1 & 0 \\
 & 1 & 0 & 0
\end{array}
\quad
\begin{array}{c|ccc}
3 & & & \\
\hline
 & 0 & 0 & 0 \\
 & 0 & 0 & 0 \\
 & 0 & 0 & 0
\end{array}
\quad
\begin{array}{c|ccc}
5 & & & \\
\hline
 & 0 & 1 & 0 \\
 & 0 & 0 & 0 \\
 & 1 & 0 & 0
\end{array}
.
$$

Part-frequency matrices

Clearly Glaisher's bijection is just transposition of the matrices for partitions which have either only first-row entries nonzero, or those which have only first-column entries nonzero.

But it is easy to observe that *any transformation of the matrices which leaves the NW-SE antidiagonal sums unchanged will be a weight-preserving map on partitions*.

Orbits under rotation

Define $\phi_m(\lambda)$ by the following action on the modulus-m matrices M_j:

$$f(a_{k,i}) = a_{k-1,i+1} \text{ for } k > 0 \text{ and } f(a_{0,i}) = a_{i,0}.$$

That is, if jm^k appears $a_{i,k}m^i$ times in the m-ary expansion, rewrite this as $a_{i,k}m^{i-1}$ appearances of jm^{k+1}, and if jm^k appears $a_{0,k}m^0$ times, rewrite this as $a_{0,k}m^k$ appearances of j.

Glaisher's bijection can also be understood this way, as rotation applied to matrices with entries only in the first column.

Orbits under rotation

Example

In the 2-modular matrices for $(20, 5, 5, 4, 2, 2, 1, 1, 1, 1, 1)$ the first and second matrices are fixed; the third has six images

$$\begin{array}{c|ccc} 5 & & & \\ \hline & 0 & 1 & 0 \\ & 0 & 0 & 0 \\ & 1 & 0 & 0 \end{array} \qquad \begin{array}{c|ccc} 5 & & & \\ \hline & 0 & 0 & 1 \\ & 1 & 0 & 0 \\ & 0 & 0 & 0 \end{array} \qquad \begin{array}{c|ccc} 5 & & & \\ \hline & 0 & 1 & 0 \\ & 0 & 1 & 0 \\ & 0 & 0 & 0 \end{array}$$

$$\begin{array}{c|ccc} 5 & & & \\ \hline & 0 & 0 & 0 \\ & 1 & 0 & 0 \\ & 1 & 0 & 0 \end{array} \qquad \begin{array}{c|ccc} 5 & & & \\ \hline & 0 & 1 & 1 \\ & 0 & 0 & 0 \\ & 0 & 0 & 0 \end{array} \qquad \begin{array}{c|ccc} 5 & & & \\ \hline & 0 & 0 & 0 \\ & 1 & 1 & 0 \\ & 0 & 0 & 0 \end{array}$$

So under this action $(20, 5, 5, 4, 2, 2, 1, 1, 1, 1, 1)$ is part of an orbit of size 6.

Orbits under rotation

The fixed points of this action are matrices such as:

$$\begin{array}{c|ccc} 1 & & & \\ \hline & 1 & 0 & 1 \\ & 0 & 1 & 0 \\ & 1 & 0 & 0 \end{array}$$

in which entire antidiagonals are filled with the same entry. (Remember, if $m > 2$ the entry can be from 0 to $m - 1$.)

Orbits under rotation

Start by considering a subset of these, in which only the upper left corners may be filled:

$$\begin{array}{c|ccc} j & & & \\ \hline & 1 & 0 & 0 \\ & 0 & 0 & 0 \\ & 0 & 0 & 0 \end{array}$$

These are partitions into parts not divisible by m, appearing less than m times: the fixed points of Glaisher's bijection. Denote the number of these by $p_{m,m}(n)$. It's easy to see that their generating function is

$$\sum_{n=0}^{\infty} p_{m,m}(n) q^n = \prod_{\substack{k=1 \\ m \nmid k}}^{\infty} \left(1 + q^k + \cdots + q^{(m-1)k}\right) = \frac{(q^m; q^m)_\infty^2}{(q;q)_\infty (q^{m^2}; q^{m^2})_\infty}.$$

Orbits under rotation

$$\sum_{n=0}^{\infty} p_{m,m}(n)q^n = \prod_{\substack{k=1\\ m\nmid k}}^{\infty} \left(1 + q^k + \cdots + q^{(m-1)k}\right) = \frac{(q^m;q^m)_\infty^2}{(q;q)_\infty (q^{m^2};q^{m^2})_\infty}.$$

From this it's easy to see that we can write down a recurrence for $p_{m,m}(n)$ in terms of values of the partition function $p(n-\ell m)$, where the coefficients will be the terms of

$$\frac{(q^m;q^m)_\infty^2}{(q^{m^2};q^{m^2})_\infty}.$$

So if $p(An+B) \equiv 0 \pmod{C}$ for some (A,B,C), $A|m$, and all $n \geq 0$, then $p_{m,m}(n)$ will share this congruence, because all terms in the recurrence will share it.

Orbits under rotation

And now what are the generating functions for the orbit classes? We simply take the partitions enumerated by $p_{m,m}$ and add orbits that satisfy the requirements; for instance, to get an orbit of the required size. For instance, if $m = 5$, any upper left entries plus antidiagonals of the form

Example

$$\begin{array}{c|ccc} 1 & & & \\ \hline & ? & 1 & 0 \\ & 1 & 0 & 0 \\ & 0 & 0 & 0 \end{array} \qquad \begin{array}{c|ccc} 2 & & & \\ \hline & ? & 0 & 0 \\ & 0 & 0 & 0 \\ & 0 & 0 & 0 \end{array} \qquad \begin{array}{c|ccc} 3 & & & \\ \hline & ? & 4 & 0 \\ & 0 & 0 & 0 \\ & 0 & 0 & 0 \end{array}$$

will yield an orbit of size 2, and the number of these for n is just the number of upper-left fillings of $n - 70$.

Orbits under rotation

So the number of partitions of n with a given orbit class size can be written as a recurrence in the $p_{m,m}(n - \ell m)$ and, once again, share any congruence that these have.

The orbit class sizes themselves can be written with a similar recurrence, where we use equivalence classes of rotated antidiagonals.

Thus, the orbit class sizes for modulus m give a statistic that realizes any partition congruence $p(An + B) \equiv 0 \pmod{C}$ by dividing the partitions of n into classes of size divisible by C.

$\square$

Here are the orbit classes for $m = 5$, $n \equiv 5 \pmod 4$:

$5n+4$	1	2	3	6
4	5	0	0	0
9	20	5	0	0
14	75	30	0	0
19	220	135	0	0
24	605	485	0	0
29	1480	1535	5	0
34	3470	4375	20	5
39	7620	11580	75	30

1. Here we have a connection between partition congruences, and an elementary map that predates them. We've used nothing deep in the argument so far. Can we improve these results with more insight?
2. We used the existence of a partition congruence to guarantee the congruence for the orbit size classes. Can we reverse the argument, perhaps by finding a rank or m-fold map on the $p_{m,m}(n)$?
3. Study of the $p_{m,m}$ partitions immediately suggests itself. More information about these might yield more information about the orbits.

Generalizing an identity of Andrews, Dixit and Yee

In a recent paper, George Andrews, Atul Dixit and Ae Ja Yee consider identities related to the third order mock theta functions. Most involve partitions into odd or distinct parts with restrictions on the parts appearing. One such identity is (Theorem 3.4 there)

Theorem

$$\sum_{n=1}^{\infty} \frac{q^n}{(q^{n+1};q)_n(q^{2n+1};q^2)_\infty} = -1 + (-q;q)_\infty$$

which, interpreted combinatorially, yields their Theorem 3.5:

Corollary

The number of partitions of positive n with unique smallest part in which each even part does not exceed twice the smallest part equals the number of partitions of n into distinct parts.

Partitions with unique smallest part c in which each even part does not exceed twice the smallest part have a very specific form of 2-modular matrices M_j:

if an odd $j > c$, then no even multiple of j can appear, so only the first row of M_j can have nonzero entries;

if an odd $j < c$ is smaller than the smallest part, there is exactly one power of 2 such that $c < 2^k j < 2c$.

So only one row of each M_j can be occupied, except that both the smallest part and twice the smallest part may appear.

Generalizing an identity of Andews, Dixit and Yee

So write the matrices M_j and mark the allowed row. Say $\lambda = (15, 15, 15, 10, 10, 8, 6)$:

$$
\begin{array}{c|ccc}
1 & & & \\ \hline
 & 0 & 0 & 0 \\
 & 0 & 0 & 0 \\
 & 0 & 0 & 0 \\
* & 1 & 0 & 0
\end{array}
\quad
\begin{array}{c|ccc}
3 & & & \\ \hline
 & 0 & 0 & 0 \\
 & 1 & 0 & 0 \\
* & 0 & 0 & 0 \\
 & 0 & 0 & 0
\end{array}
\quad
\begin{array}{c|ccc}
5 & & & \\ \hline
 & 0 & 0 & 0 \\
* & 0 & 1 & 0 \\
 & 0 & 0 & 0 \\
 & 0 & 0 & 0
\end{array}
\quad \ldots \quad
\begin{array}{c|ccc}
15 & & & \\ \hline
* & 1 & 1 & 0 \\
 & 0 & 0 & 0 \\
 & 0 & 0 & 0 \\
 & 0 & 0 & 0
\end{array}
$$

Transpose the part of the matrices at and below the marked row.

$$
\begin{array}{c|ccc}
1 & & & \\ \hline
 & 0 & 0 & 0 \\
 & 0 & 0 & 0 \\
 & 0 & 0 & 0 \\
* & 1 & 0 & 0
\end{array}
\quad
\begin{array}{c|ccc}
3 & & & \\ \hline
 & 0 & 0 & 0 \\
 & 1 & 0 & 0 \\
* & 0 & 0 & 0 \\
 & 0 & 0 & 0
\end{array}
\quad
\begin{array}{c|ccc}
5 & & & \\ \hline
 & 0 & 0 & 0 \\
* & 0 & 0 & 0 \\
 & 1 & 0 & 0 \\
 & 0 & 0 & 0
\end{array}
\quad \ldots \quad
\begin{array}{c|ccc}
15 & & & \\ \hline
* & 1 & 0 & 0 \\
 & 1 & 0 & 0 \\
 & 0 & 0 & 0 \\
 & 0 & 0 & 0
\end{array}
$$

Generalizing an identity of Andews, Dixit and Yee

The entire operation works perfectly well for the m-modular matrices with analogous entries, and since it fixes the smallest part we have a refinement and generalization of the theorem: for any $n \geq 1$, $m \geq 2$, the terms with smallest part n yield

$$\frac{(q^{mn}; q^m)_\infty}{(q^{n+1}; q)_{(m-1)n}(q^{mn}; q)_\infty} = \frac{(q^{m(n+1)}; q^m)_\infty}{(q^{n+1}; q)_\infty}.$$

If you multiply by an extra factor of $\frac{q^n(1-q^{(m-1)n})}{1-q^n}$ and set $m = 2$, summing over all n obtains the previous theorem.

Concluding remarks

I hope I have demonstrated that the part-frequency matrices are

very *natural*,

connected to interesting partition objects, and

a *useful* technique for proving some kinds of theorems.

I would be thrilled if people found use for them elsewhere.

Thank You!