On the Hamilton Waterloo Problem for Complete Equipartite Graphs.

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08/28/2015

The Problem

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Originally v was supposed to be odd, but later the problem was extended to allow v even, and having it so that attendees would never sit next to their spouses.
In Graph Theory language this is equivalent to decomposing K_v (or $K_v - Q$, where Q is a 1-factor if v is even) into 2-factors, where K_v is the complete graph on v vertices and each 2-factor is isomorphic to a given 2-factor F.

The Problem

The Hamilton-Waterloo Problem is an extension of the Oberwolfach problem. In this versions dinners are at two different venues, Hamilton and Waterloo. The attendees will spend r nights in Hamilton, where the sizes of the tables are $m_1, m_2, \ldots, m_k$, and s nights in Waterloo, where the sizes of the tables are $n_1, n_2, \ldots, n_p$.

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In our language this means that we have two 2-factors, F_1 and F_2, and we are trying to decompose K_v (or $K_v - Q$) into r copies of F_1 and s copies of F_2.

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In Graph theoretical language, we want to decompose a complete equipartite graph, with m parts of size v each, into r copies of F_1 and s copies of F_2.
This type of problem has been studied in the Oberwolfach case.
Nevertheless, it has not been studied in the Hamilton-Waterloo case.

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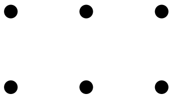
$$G = K_{(2:3)}$$



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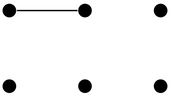

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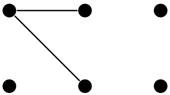

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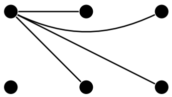

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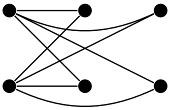

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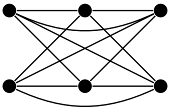

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Given a multipartite graph G with n parts, we will denote the parts $G_0, G_1, \ldots, G_{n-1}$.

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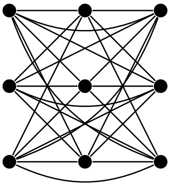

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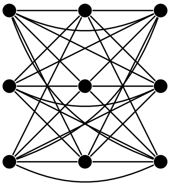

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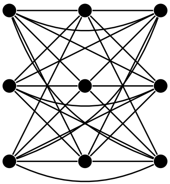


G_0 G_1 G_2

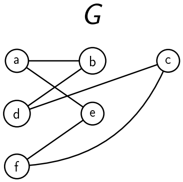
Product

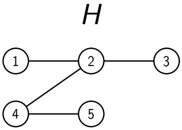
Definition

Let G and H be multipartite graphs with parts G_i and H_i respectively. Then we define the partite product of G and H, $G \otimes H$ as follows:

- $V(G \otimes H) = \{(g, h) | g \in G_i \text{ and } h \in H_i, \text{ for some } i\}$.
- $E(G \otimes H) = \{\{(g_1, h_1), (g_2, h_2)\} | \{g_1, g_2\} \in E(G) \text{ and } \{h_1, h_2\} \in E(H)\}$.

Example





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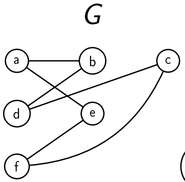
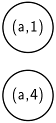
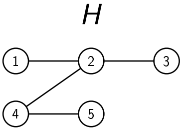


$G \otimes H$





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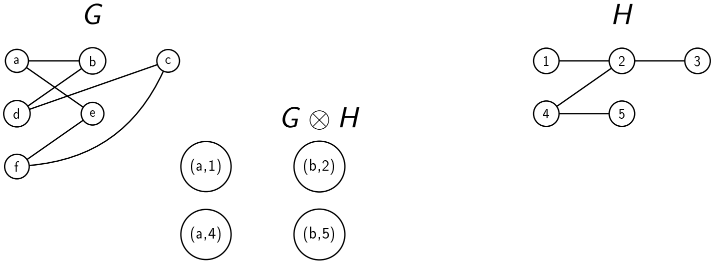

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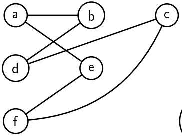

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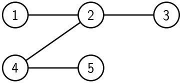

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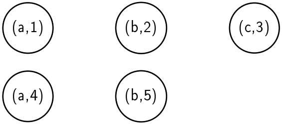

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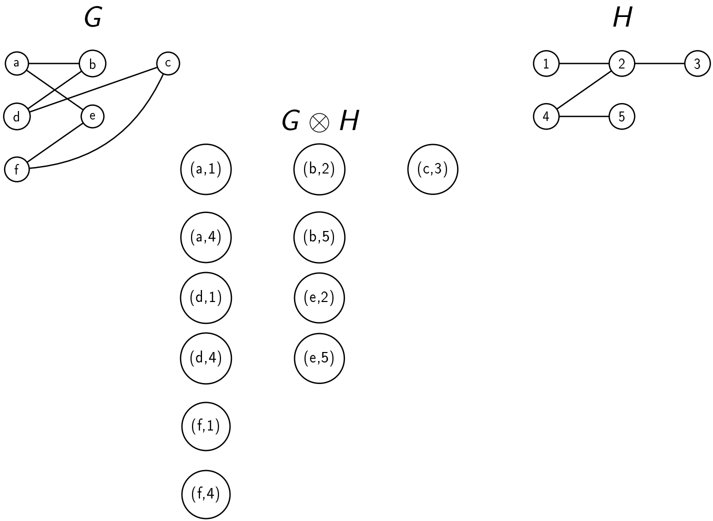

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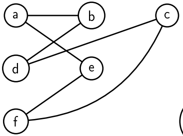

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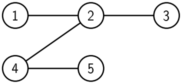

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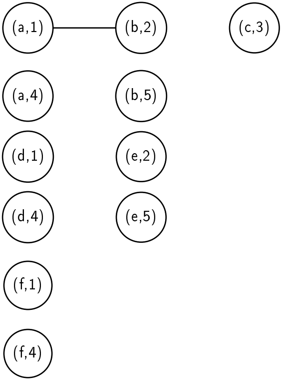

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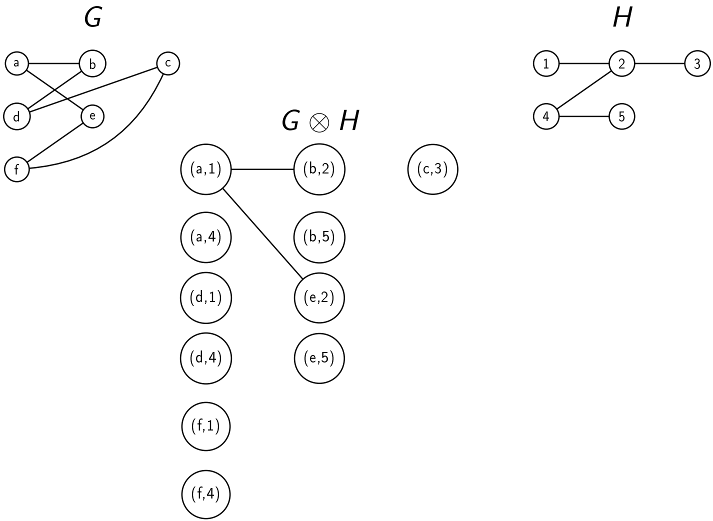

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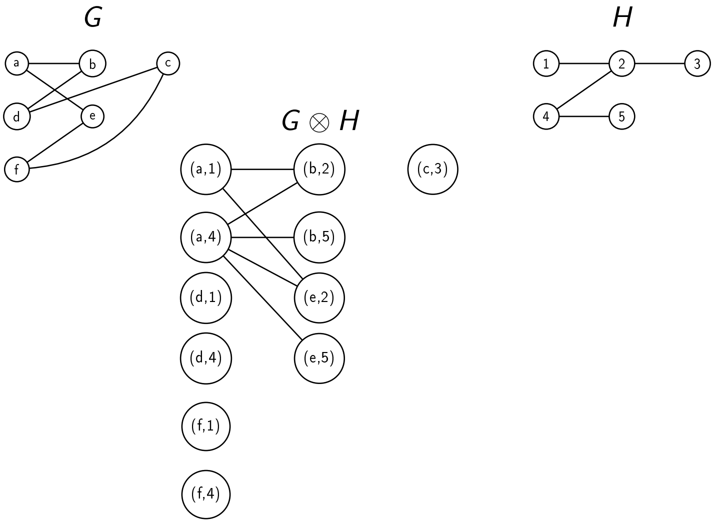

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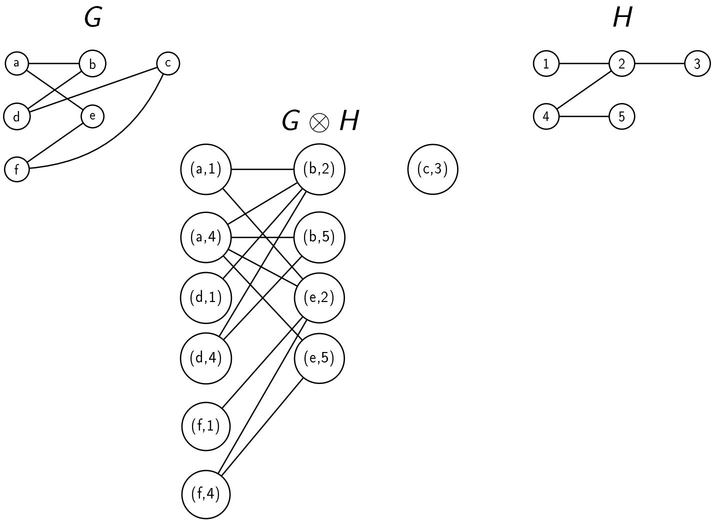

Definition Direct Sum

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If $V(G_1) = V(G_2)$ then $G = G_1 \oplus G_2$ is the Graph on the same set of vertices, having as edges the symmetric difference between the edges of G_1 and G_2. This is:

$$V(G) = V(G_1) = V(G_2)$$

$$E(G) = E(G_1) \cup E(G_2) \backslash (E(G_1) \cap E(G_2))$$

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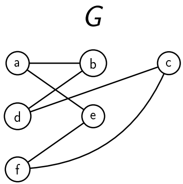
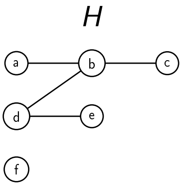
Remark

Notice that if there is a decomposition of a graph G into subgraphs $F_1, \ldots, F_s$, then

$$G = \bigoplus_{i=1}^{s} F_i$$

Example





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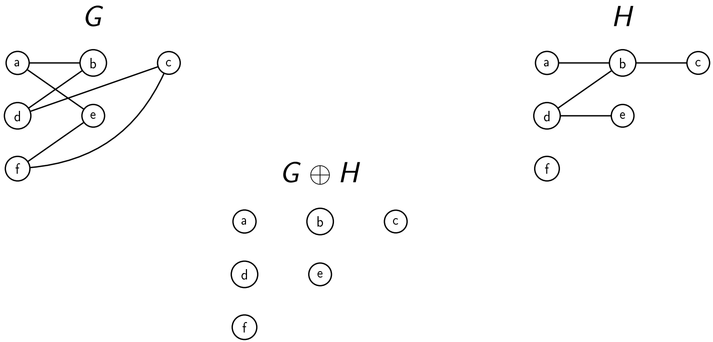

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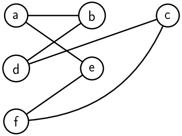

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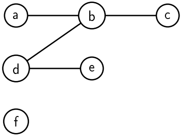
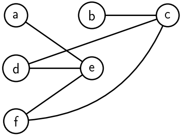


$G \oplus H$



Some Results on the Product and the Sum

Lemma

$K_{(x:n)} \otimes K_{(y:n)} = K_{(xy:n)}$

Lemma

Let $\mathfrak{G}_n$ be the set of n-partite graphs (where some vertices may be isolated). Then $(\mathfrak{G}_n, \oplus, \otimes)$ is a commutative ring with unity, where the 0 is the empty graph, and the 1 is $K_{(1:n)}$. More specifically:

- $G \oplus H = H \oplus G$.
- $G \oplus 0 = G$, *where 0 is the empty graph (a graph without any edges).*
- $G \otimes H = H \otimes G$.
- $G \otimes (H \oplus F) = (G \otimes H) \oplus (G \otimes F)$.
- $G \otimes K_{(1:n)} = G$.

Definition of $C_{(x:n)}$

Definition

The complete cyclic multipartite graph $C_{(x:n)}$ is the graph with n parts of size x, where two vertices $g \in G_i$ and $h \in G_j$ are neighbors if and only if $|i - j| = 1$, with this subtraction being done modulo n.

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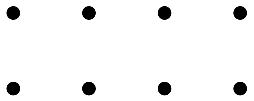
$$G = C_{(2:4)}$$



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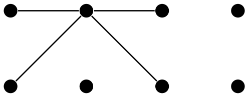

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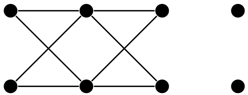

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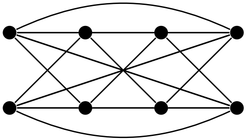
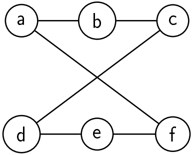
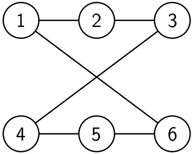


The Product of Cycles

G



G'



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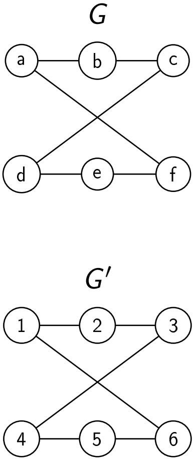

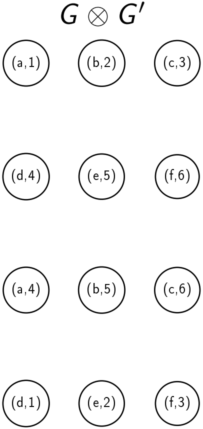

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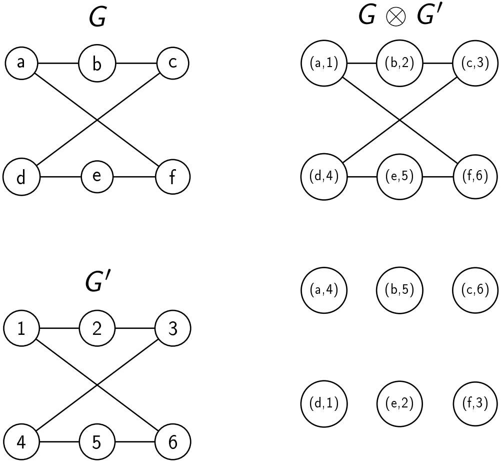

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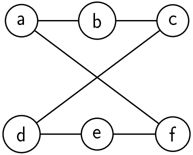

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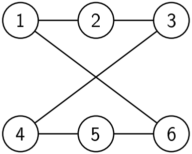
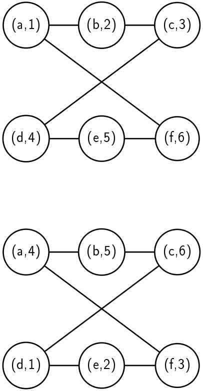


$G \otimes G'$



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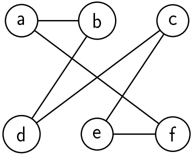
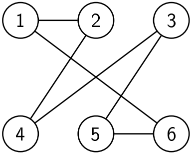


H'



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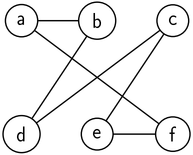

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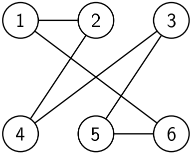

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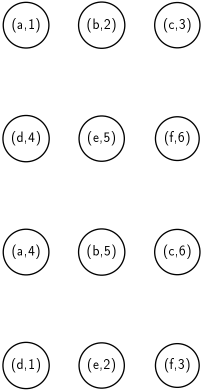

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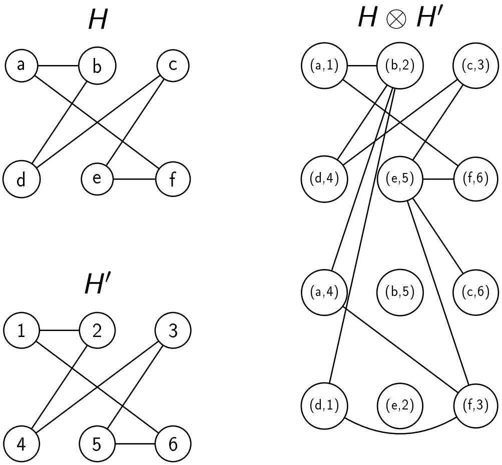

Definition n-balanced C_k-factor

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Let G be a subgraph of $C_{(x:n)}$. We will say that G is a $n-balanced$ C_k-factor if:

- G is a union of cycles of size k.
- If $v \in G_j$, then v has a neighbor in G_{j-1} and a neighbor in G_{j+1}.

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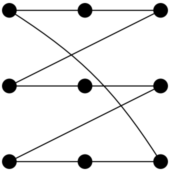

Products of Balanced Factors

Lemma

Let G and H be a n-balanced C_k-factor and a n-balanced C_m-factor, respectively. Then $G \otimes H$ is a n-balanced C_l-factor, where $l = \frac{km}{gcd(k,m)}$.

Lemma

The complete cyclic multipartite graph is the product of the complete multipartite graph by the cycle. This is: $K_{(x:n)} \otimes C_{(1:n)} = C_{(x:n)}$.

Construction Theorem

Lemma

Let m, n, x, y and v be positive integers. Suppose the following conditions are satisfied:

- *There exists a decomposition of K_m into C_n-factors.*
- *There exists a decomposition of $C_{(v:n)}$ into s_p C_{xn}-factors and r_p C_{yn}-factors.*

Let

$$s = \sum_{p=1}^{\frac{(m-1)}{2}} s_p \qquad \text{and} \qquad r = \sum_{p=1}^{\frac{(m-1)}{2}} r_p$$

Then there exists a decomposition of $K_{(v:m)}$ into s $C_{(xn)}$-factors and r C_{yn}-factors.

Needed Known Result + Basic Construction

**Theorem (Alspach, Haggkvist [1]
Alspach, Schellenberg, Stinson, Wagner[2])**

There exists a decomposition of K_m into C_n-factors if and only if $m \equiv 0 \pmod{n}$, $(m, n) \neq (6, 3)$ *and* $(m, n) \neq (12, 3)$.

Theorem (Not enough room in the slides to prove)

Let x, y and n be odd. Let $s \neq 1$. We have the following decompositions:

- $C_{(4x:n)}$ *can be decomposed into s k-balanced C_{2xn}-factors and r k-balanced C_n-factors.*

More Basic Constructions

Theorem (Not enough room in the slides to prove)

Let x, y and n be odd. Let $s, r \neq 1$. We have the following decompositions:

- $C_{(xy:n)}$ *can be decomposed into s k-balanced C_{xn}-factors and r k-balanced C_{yn}-factors.*
- $C_{(4x:n)}$ *can be decomposed into s k-balanced C_{xn}-factors and r k-balanced C_{2n}-factors.*
- $C_{(4xy:n)}$ *can be decomposed into s k-balanced C_{2xn}-factors and r k-balanced C_{yn}-factors.*

Using the Product Algebraically

Theorem

Let v be odd and x be an odd divisor of v. Then there is a decomposition of $C_{(v:n)}$ into s C_{xn}-factors and v C_n-factors, for any $s_p \neq 1$.

Proof.

Using the Product Algebraically

Theorem

Let v be odd and x be an odd divisor of v. Then there is a decomposition of $C_{(v:n)}$ into s C_{xn}-factors and v C_n-factors, for any $s_p \neq 1$.

Proof.

$$C_{(v:n)} = C_{(x:n)} \otimes C_{(\frac{v}{x}:n)}$$

$\square$

Using the Product Algebraically

Theorem

Let v be odd and x be an odd divisor of v. Then there is a decomposition of $C_{(v:n)}$ into s C_{xn}-factors and v C_n-factors, for any $s_p \neq 1$.

Proof.

$$C_{(v:n)} = C_{(x:n)} \otimes C_{(\frac{v}{x}:n)}$$

$s \not\equiv 1 \pmod{x}$

$$C_{(x:n)} \otimes C_{(\frac{v}{x}:n)} = C_{(x:n)} \otimes \left(\oplus_{i=1}^{\frac{v}{x}} H_{\frac{v}{x}}(i,i) \right) = \oplus_{i=1}^{\frac{v}{x}} \left(H_{\frac{v}{x}}(i,i) \otimes C_{(x:n)} \right)$$

Using the Product Algebraically

Theorem

Let v be odd and x be an odd divisor of v. Then there is a decomposition of $C_{(v:n)}$ into s C_{xn}-factors and v C_n-factors, for any $s_p \neq 1$.

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$$C_{(x:n)} \otimes C_{(\frac{v}{x}:n)} = C_{(x:n)} \otimes \left(\oplus_{i=1}^{\frac{v}{x}} H_{\frac{v}{x}}(i,i)\right) = \oplus_{i=1}^{\frac{v}{x}} \left(H_{\frac{v}{x}}(i,i) \otimes C_{(x:n)}\right)$$

$$\oplus_{i=1}^{\frac{v}{x}} \left(H_{\frac{v}{x}}(i,i) \otimes C_{(x:n)}\right) = \left(\oplus_{i=1}^{t} \left(H_{\frac{v}{x}}(i,i) \otimes C_{(x:n)}\right)\right)$$

$$\oplus \left(H_{\frac{v}{x}}(t+1,t+1) \otimes C_{(x:n)}\right) \oplus \left(\oplus_{i=t+2}^{\frac{v}{x}} \left(H_{\frac{v}{x}}(i,i) \otimes C_{(x:n)}\right)\right)$$

$\square$

Theorem

Let v be odd and x be an odd divisor of v. Then there is a decomposition of $C_{(v:n)}$ into s C_{xn}-factors and v C_n-factors, for any $s_p \neq 1$.

Proof.

$$C_{(v:n)} = C_{(x:n)} \otimes C_{(\frac{v}{x}:n)}$$

$$s \equiv 1 \pmod{x}$$

Theorem

Let v be odd and x be an odd divisor of v. Then there is a decomposition of $C_{(v:n)}$ into s C_{xn}-factors and v C_n-factors, for any $s_p \neq 1$.

Proof.

$$C_{(v:n)} = C_{(x:n)} \otimes C_{(\frac{v}{x}:n)}$$

$$s \equiv 1 \pmod{x}$$

$$\oplus_{i=1}^{\frac{v}{x}} \left(T_{\frac{v}{x}}(i) \otimes C_{(x:n)} \right) = \left(\oplus_{i=1}^{t-1} \left(T_{\frac{v}{x}}(i) \otimes C_{(x:n)} \right) \right)$$
$$\oplus \left(T_{\frac{v}{x}}(t) \otimes C_{(x:n)} \right) \oplus \left(T_{\frac{v}{x}}(t+1) \otimes C_{(x:n)} \right) \oplus \left(\oplus_{i=t+2}^{\frac{v}{x}} \left(T_{\frac{v}{x}}(i) \otimes C_{(x:n)} \right) \right)$$

□

The Path to Decomposing

- *Let* x, y, z, n, v *and* m *be integers, with* $n|m$*,* $xyz|v$ *and* $x, y, z \not\equiv 0 \pmod 4$*,* $\gcd(x, z) = \gcd(y, z) = 1$*.*

The Path to Decomposing

- *Let* x, y, z, n, v *and* m *be integers, with* $n|m$*,* $xyz|v$ *and* $x, y, z \not\equiv 0 \pmod 4$*,* $\gcd(x, z) = \gcd(y, z) = 1$*.*
- *Decompose* $K(v : m)$ *into copies of* $C_{(v:n)}$*.*

The Path to Decomposing

- *Let* x, y, z, n, v *and* m *be integers, with* $n|m$, $xyz|v$ *and* $x, y, z \not\equiv 0 \pmod 4$, $\gcd(x, z) = \gcd(y, z) = 1$.
- *Decompose* $K(v : m)$ *into copies of* $C_{(v:n)}$.
- *Write* $C_{(v:n)} = C_{(xy:n)} \otimes C_{(zw:n)}$.

The Path to Decomposing

- *Let* x, y, z, n, v *and* m *be integers, with* $n|m$*,* $xyz|v$ *and* $x, y, z \not\equiv 0 \pmod 4$*,* $\gcd(x, z) = \gcd(y, z) = 1$*.*
- *Decompose* $K(v : m)$ *into copies of* $C_{(v:n)}$*.*
- *Write* $C_{(v:n)} = C_{(xy:n)} \otimes C_{(zw:n)}$*.*
- *Decompose* $C_{(xy:n)}$ *into* C_{xn}*-factors and* C_{yn}*-factors.*

The Path to Decomposing

- *Let* x, y, z, n, v *and* m *be integers, with* $n|m$, $xyz|v$ *and* $x, y, z \not\equiv 0 \pmod 4$, $\gcd(x, z) = \gcd(y, z) = 1$.
- *Decompose* $K(v : m)$ *into copies of* $C_{(v:n)}$.
- *Write* $C_{(v:n)} = C_{(xy:n)} \otimes C_{(zw:n)}$.
- *Decompose* $C_{(xy:n)}$ *into* C_{xn}*-factors and* C_{yn}*-factors.*
- *Decompose* $C_{(zw:n)}$ *into* C_{zn}*-factors.*

The Path to Decomposing

- *Let* x, y, z, n, v *and* m *be integers, with* $n|m$*,* $xyz|v$ *and* $x, y, z \not\equiv 0 \pmod 4$*,* $\gcd(x, z) = \gcd(y, z) = 1$*.*
- *Decompose* $K(v : m)$ *into copies of* $C_{(v:n)}$*.*
- *Write* $C_{(v:n)} = C_{(xy:n)} \otimes C_{(zw:n)}$*.*
- *Decompose* $C_{(xy:n)}$ *into* C_{xn}*-factors and* C_{yn}*-factors.*
- *Decompose* $C_{(zw:n)}$ *into* C_{zn}*-factors.*
- *Multiply and add up.*

Main Theorem

Theorem

Let v, m and n be odd, such that $m \equiv 0 \pmod{n}$. Let s and r be such that $s, r \neq 1$ and $s + r = v\frac{m-1}{2}$. Let x, y, z and w be such that:

- $\gcd(x, z) = \gcd(y, z) = 1$,
- $w \notin \{2, 6\}$,
- *2 divides at most one of x, y and z,*
- $v = xyzw$ *if 2 divides none of x, y, z,*
- $v = 2xyzw$ *if 2 divides one of x, y, z.*

Then there is a decomposition of $K_{(v:m)}$ into s C_{xzn}-factors and r C_{yzn}-factors.

[1] B. Alspach and R. Haggkvist, Some observations on the Oberwolfach problem, *Journal of Graph Theory* **9** (1985), 177-187.

[2] B. Alspach, P. Schellenberg, D.R. Stinson, and D. Wagner, The Oberwolfach problem and factors of uniform length, *Journal of Combinatorial Theory, Ser. A* **52** (1989), 20-43.

Thank you!!