Non-existence of strongly regular graphs with feasible block graph parameters of quasi-symmetric designs

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Abstract

A quasi-symmetric design(QSD) is a (v, k, λ) design with two intersection numbers x, y, where $0 \leq x < y < k$. The block graph of QSD is a strongly regular graph(SRG). It is known that there are SRGs which are not block graphs of QSDs. We derive necessary conditions on the parameters of a SRG to be feasible as the block graph of a QSD. We apply these condtions to rule out many infinite families such SRGs.

Outline of talk

- Preliminary concepts
- Preliminary results needed
- Main method
- Some results obtained

Strongly regular graph (SRG)

- A regular graph Γ is an SRG with parameters (n, a, c, d)
- n $=$#vertices of Γ, a $=$valency of Γ
- c $=$ # of vertices adjacent to two adjacent vertices
- d $=$ # vertices adjacent to two non-adjacent vertices

Γ is assumed to be non-null, non-complete, connected

Quasi-symmetric design (QSD)

- A QSD is a 2-(v, k, λ) design $D = (X, \beta)$ with two block intersection numbers x, y, where $x < y$

Block graph of QSD

- The block graph Γ of a QSD $D = (X, \beta)$ has vertices the b blocks of D, where two distinct blocks B, B' are adjacent iff $|B \cap B'| = y$
- The block graph Γ is an SRG with parameters (b, a, c, d)
- result due to S.S. Shrikhande & Bhagwandas (1965) and Goethals & Seide(1970)

Is SRG the block graph of a QSD?

- Goethals and Seidel proved that the SRG lattice graph $L_2(n)$ is not the block graph of a QSD
- Deciding which SRGs are block graphs of QSDs appears to be a difficult open problem Coster & Haemers DCC (1995)

Some previous non-existence results

- S.S. Shrikhande & Jain (1962),
- S.S. Shrikhande, Raghavarao,& Tharthare(1963)
- considered duals of PBIBDs ($=$ SRGs) & using Hasse-Minkowski Theory
- Haemers (1992), Coster & Haemers (1995) used quadratic forms theory

Preliiminary results need

Lemma (1)

Let **D** *be a QSD, with the standard parameter set* $(v, b, r, k, \lambda; x, y)$. *Then the following relations hold:*

1. $vr = bk$ *and* $\lambda(v-1) = r(k-1)$.
2. $k(r-1)(x+y-1) - xy(b-1) = k(k-1)(\lambda-1)$.
3. $r(-r+kr+\lambda) = bk\lambda$.
4. $y - x$ *divides* $k - x$ *and* $r - \lambda$.

Preliminary results need

Lemma (2)

1. Let Γ be a connected SRG (b, a, c, d), Then Γ has three distinct eigenvalues, $\theta_0 = a$ with multiplicity 1, θ_1 with multiplicity f, and θ_2 with multiplicity g, where θ_1, θ_2 $(\theta_1 > \theta_2)$ are the roots of the quadratic equation

$$\rho^2 - (c - d)\rho - (a - d) = 0,$$

statement of lemma contd.

2. the multiplicities f and g are positive integers given by

$$f, g = \frac{1}{2}\left(b - 1 \pm \frac{(b-1)(c-d) + 2a}{\sqrt{(c-d)^2 + 4(a-d)}}\right).$$

and

$$c = a + \theta_1 + \theta_2 + \theta_1\theta_2, d = a + \theta_1\theta_2.$$

Krein Conditions

Let Γ be a connected SRG (b, a, c, d), with three distinct eigenvalues, $\theta_0 = a$, θ_1, θ_2 $(\theta_1 > \theta_2)$. Then

1. $(\theta_1 + 1)(a + \theta_1 + 2\theta_1\theta_2) \leq (a + \theta_1)(\theta_2 + 1)^2$;
2. $(\theta_2 + 1)(a + \theta_2 + 2\theta_1\theta_2) \leq (a + \theta_2)(\theta_1 + 1)^2$.

Lemma (3)

Let **D** *be a* $(v, b, r, k, \lambda; x, y)$ *QSD. Form the block graph* Γ *of* **D**. *Assume* Γ *is connected. Then,* Γ *is a SRG with parameters* (b, a, c, d), *where the eigenvalues of* Γ *are given by*
$a = \theta_0 = \frac{k(r-1)+(1-b)x}{y-x}, \theta_1 = \frac{r-\lambda-k+x}{y-x}$ *and*
$\theta_2 = \frac{-(k-x)}{y-x}$.

Lemma (4)

- *The eigenvalues* $\theta_0, \theta_1, \theta_2$ *are integers, with* $\theta_0 > 0, \theta_1 \geq 0$, *and* $\theta_2 < 0$;
- $a = \frac{k(r-1)+(1-b)x}{y-x}$ (1);
- $c = \frac{(x-k+r-\lambda)(x-k)}{(y-x)^2} + \frac{x-k}{y-x} + \frac{x-k+r-\lambda}{y-x} + \frac{k(r-1)+(1-b)x}{y-x}$ (2);
- $d = \frac{k(r-1)+(1-b)x}{y-x} + \frac{(x-k)(-k+r+x-\lambda)}{(y-x)^2}$ (3)

Lemma [4] contd.

- $\frac{r-\lambda}{y-x} \geq \frac{k-x}{y-x}$,
- $y - x$ divides both $r - \lambda$ and $k - x$, so we take $y = z + x, k = mz + x$ and $r = nz + \lambda$, for positive integers m and n, assuming $m \leq n$.
- If $\lambda > 1$, then $\lambda \geq x + 1$.
- The block graph of $\mathbf{D}$ and block graph of $\overline{\mathbf{D}}$, the complement of the design $\mathbf{D}$, are isomorphic.

Remarks on SRGs arisng in paper

1. The block graph of a *Steiner graph* is a SRG.
2. Given $m-2$ mutually orthogonal Latin squares of order n, the vertices of a *Latin square graph* $LS_m(n)$ are the n^2 cells; two vertices are adjacent if and only if they lie in the same row or column or they have same entry in one of the Latin squares. This graph is a SRG, denoted by $L_m(n)$.
3. A *Negative Latin square graph* $NL_m(n)$, is a SRG obtained by replacing m and n by their negatives in the parameters of $LS_m(n)$.

Main tool

Theorem (5)

Let $\mathbf{D}$ *be a* $(v, b, r, k, \lambda; x, y)$ *QSD and* Γ *the* (b, a, c, d) *strongly regular block graph of* $\mathbf{D}$*. Let* $y = z + x, k = mz + x$ *and* $r = nz + \lambda$*, for positive integers* m *and* n*, assuming* $m \leq n$*. Then,*

1. $n = \dfrac{m^2 - 2m + a - c}{m - 1}, c - d = n - 2m$ *and* $a - d = m(n - m)$,

2. $m = \dfrac{1}{2}\left(d - c + \sqrt{(d - c)^2 + 4(a - d)}\right)$,

Theorem 5 cont.

3. $z = \frac{(-a+c-d+m+b\,m)\,(b-s)\,s}{b\,(c-d+2\,m)\,(-a-m+b\,m)}$
for some positive integer s

4. $0 \le b^2 - 4q$,
where $q = \frac{b\,(c-d+2\,m)\,(-a-m+b\,m)}{\gcd(b\,(c-d+2\,m)\,(-a-m+b\,m), -a+c-d+m+b\,m)}$.

Proof: From (3) of Lemma 4, get $m^2 - (d - c)m - (a - d) = 0$ and note that m is a positive root of this quadratic.
From (1) of Lemma 4, get $\lambda = \frac{b\,x+a\,z+m\,z-n\,x\,z-m\,n\,z^2}{x+m\,z}$. Observe that $(b\,x+a\,z+m\,z-n\,x\,z-m\,n\,z^2)-(a+m-b\,m)\,z = (x+m\,z)\,(b-n\,z)$. Hence $x+m\,z$ divides $(-a-m+b\,m)\,z$. Taking $(-a-m+b\,m)\,z = s\,(x+m\,z)$ for positive integer s, we get $x = \frac{-(a+m-b\,m+m\,s)\,z}{s}$. Substitute these values with $n = c - d + 2\,m$ in (3) of Lemma to get desired expression for z.

From expression of z, q divides $(b-s)s$. Hence $(b-s)s = pq$ for some positive integer p. The discriminant of this quadratic in s $=b^2 - 4pq$ is non-negative. Observe that
$0 \leq b^2 - 4pq \leq b^2 - 4q$.
Thus, $0 \leq b^2 - 4q$,

Algorithm

Let Γ be a (b, a, c, d) strongly regular graph. To find feasible parameters of a QSD whose block graph is Γ, the following steps are followed.
(1) m is obtained using (2) of the Theorem 1 and then n by (1).
(2) If $b^2 - 4q < 0$, then there is no QSD, whose block graph parameters are (b, a, c, d).
(3) If $b^2 - 4q \geq 0$, then for each integer p, $1 \leq p \leq b^2/4q$, we take integer $s = \frac{b+\sqrt{b^2-4\,p\,q}}{2}$,

$$x = \frac{-(a+m-b\,m+m\,s)\,z}{s},\ z = \frac{(-a+c-d+m+b\,m)\,(b-s)\,s}{b\,(c-d+2\,m)\,(-a-m+b\,m)}.$$

(4) Other feasible parameters of design can be obtained from Lemma , satisfying all known necessary conditions.

Designs associated with different values of s satisfying expression (3) of Theorem 1, are complements of each others.

Theorem (6)

The following SRGs (b, a, c, d) are not block graphs of QSDs.

1(a) $(t^3, (t-1)(t+2), t-2, t+2),\ t \geq 2$

1(b) $(t^3, (t-1)^2(t+1), t^3-2t^2-t+4, (t-2)(t-1)(t+1)),\ t \geq 2$

2(a) $(t^2(t+2), t(t+1), t, t),\ t \geq 2$

2(b) $((t^2(t+2), (t-1)(t+1)^2, t^3-t-2), (t-1)t(t+1)),\ t \geq 2$

theorem contd.

3 $((t+1)(t^2+1), t^3, (t-1)t^2, (t-1)t^2),\ t \geq 3$

4(a) $((t+1)(t^3+1), t(t^2+1), (t-1), t^2+1),\ t \geq 2$

4(b) $((t+1)(t^3+1), t^4, (t-1)t(t^2+1), (t-1)t^3),$ $t \geq 2$

theorem contd.

5 $((t^2+1)(t^3+1), t^5, t(t-1)(t^3+t^2-1), t^3(t-1)(t+1)),\ t \geq 2$

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17 $((t^2+1)(t^3+1), t^5, (t-1)t(t^3+t^2-1), (t-1)t^3(t+1)),\ t \geq 2$

sketch of proof of Theorem 6

We use (4) of Main Tool (Theorem 3) to rule out the possibility of QSD whose block graph parameters are (b, a, c, d) given in theorem by observing that $\Delta = b^2 - 4p\,q < 0$

Observe that $z = -\frac{s\left(t^2+2t-1\right)\left(s-t^3\right)}{2(t-1)t^3(t+1)(t+2)}$. Consider two cases, t even and t odd.
If $t = 2e$ then $z = \frac{\left(4e^2+4e-1\right)\left(8e^3-s\right)s}{32e^3(e+1)(2e-1)(2e+1)}$. As $32e^3(e+1)(2e-1)(2e+1)$ and $\left(4e^2+4e-1\right)$ are relatively prime, $\left(8e^3 - s\right)s = 32e^3(e+1)(2e-1)(2e+1)p$ for some positive integer p. Observe that $\Delta = -64e^3\left((8p-1)e^3 + 8pe^2 - 2pe - 2p\right)$, the discriminant of this quadratic in s is negative.

If $t = 2e + 1$ then $z = \frac{\left(2e^2+4e+1\right)\left(8e^3+12e^2+6e-s+1\right)s}{4e(e+1)(2e+1)^3(2e+3)}$.

$\left(8e^3 + 12e^2 + 6e - s + 1\right) s =$
$4e(e + 1)(2e + 1)^3(2e + 3)p$ for some positive integer p.

$\Delta = -(2e + 1)^3 \left((32p - 8)e^3 + (80p - 12)e^2 + (48p - 6)e - 1\right)$, the discriminant of this quadratic in s is negative.

Non-existence of some families of feasible block graph parameters in Hubaut's paper

Theorem (7)

There is no QSD whose block graph parameters are complement of the family C6 given in Hubaut [ ].

Theorem (8)

There is no QSD whose block graph parameters are complement of the family C7 given in Hubaut [ ].

Theorem (9)

There is no QSD whose block graph parameters are complement of the family C8 given in Hubaut [ ].

Pawale et al [EJC] proved non-existence of QSD whose block graph is pseudo Latin square graph $L_3(n)$ or $L_4(n)$, or their complements.. In present paper, we show

Theorem (10)

There is no QSD whose block graph is the pseudo Latin square graph $L_5(n)$; $n \geq 5$, *with parameters* $\left(n^2, 5(n-1), n+10, 20\right)$.

Theorem (11)

There is no QSD whose block graph is the complement of pseudo Latin square graph $L_5(n), n \geq 5$ *with parameters*

$$\left(n^2, (n-4)(n-1), 28 - 10\,n + n^2, (n-5)(n-4)\right).$$

In Cameron, Goethals and Seidel [1], characterized SRG's attaining Krein bounds in terms of Negative Latin square graph $NL_t(t^2+3t)$. In below, we rule out the possibility of QSD's whose block graph is $NL_t(t^2+3t)$, with $2 \leq t$ or its complement.

Non-existence of QSDs with Negative Latin square block graph parameters

Theorem (12)

There is no QSD whose block graph is the Negative Latin square graph $NL_e(e^2+3e)$; $e \geq 2$, with parameters
$\left(e^2\,(3+e)^2, e\,\left(1+3\,e+e^2\right), 0, e\,(1+e)\right)$ *or its complement.*

Theorem (13)

There is no QSD whose block graph is the Negative Latin square graph $NL_e(e+2)$; $e \geq 2$, *with parameters* $((2+e)^2, e(3+e), e^2+2e-2, e(e+1))$ *or its complement.*

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