4.1.1 Initial Value (IV) and Boundary Value (BV) problems.

- perhaps having initial and/or boundary values will allow us to determine a unique solution to a d.e.
- solve for c's.
- not always the case, as we will see but most time, we can.

Initial Value Problem. (2nd Order)

\[ a_2(x) \frac{d^2y}{dx^2} + a_1(x) \frac{dy}{dx} + a_0(x) y = g(x) \]

Subject to \( y(x_0) = y_0, \ y'(x_0) = y_1 \)

- note # of I.C.'s required = order - 1.
- what the initial conditions imply are:
  - soln curve \( y = f(x) \)
    1. passes through \( x_0, y_0 \)
    2. has a slope of \( y_1 \) at \( x_0 \).
- Theorem 4.1 states:
  - if coefficients \( a(x) \), \( g(x) \) are continuous
  - and \( a_n(x) \neq 0 \)
  - then a unique solution can be found if an initial point \( x = x_0 \) is known.
Example 4.1.1.2

\[ y = c_1 + c_2 \cos x + c_3 \sin x \]
\[ y''' + y' = 0 \quad y'(\pi) = 0 \quad y''(\pi) = -1 \]
\[ y(\pi) = c_1 + c_2 = 0 \quad c_1 = -1 \]
\[ y'(x) = -c_2 \sin x + c_3 \cos x \]
\[ y'('\pi') = -c_3 = 2 \quad c_3 = -2 \]
\[ y''(x) = -c_2 \cos x - c_3 \sin x \]
\[ y''(\pi) = -1 = + c_2 \quad c_2 = 1 \]
\[ y = -1 - \cos x - 2 \sin x \]

Boundary Value Problem

- Picture a guitar string, fixed at two points (boundaries).

\[ a \quad b \]

- A 2nd order DE for such a situation might be,

\[ a_2(x) \frac{d^2y}{dx^2} + a_1(x) \frac{dy}{dx} + a_0(x) y = g(x) \]

Boundary conditions:
\[ y(a) = y_0 \quad y(b) = y_1 \]

If the strings are fixed at these end points,
\[ y(a) = y(b) = 0 \]
Note, for a 2nd order DE, it is possible to specify b.c.s in terms of the derivatives as well, \( y'(a) \) and \( y'(b) \), and to mix b.c.s, e.g.,

\[
y'(a) = y_0 \quad y'(b) = y_1.
\]

However, having a sufficient # of b.c.'s does not ensure that we will have a unique solution.

Example Problem 4.1.14

(c) \( y = C_1 x^2 + C_2 x^4 + 3 \)

\[
y(0) = 3 \quad y(1) = 0
\]

\[
y(0) = 3 = 3
\]

\[
y(1) = 0 = C_1 + C_2 + 3
\]

\[
C_1 + C_2 = -3 \quad \text{or} \quad C_1 = -C_2 - 3
\]

(d) \( y(1) = 3 \quad y(2) = 15 \)

\[
y(1) = 3 = C_1 + C_2 + 3
\]

\[
y(2) = 15 = 4C_1 + 16C_2 + 3
\]

\[
4C_1 + 16C_2 = 12
\]

\[
-4C_2 + 16C_2 = 12
\]

\[
C_2 = 1
\]

\[
C_1 = -1
\]
4.1.2 Homogeneous Equations

\[ g(x) = 0 \]

- We will see that to solve a NH DE we will first need to solve the associated H-DE.
- Going forward we will make the following assumptions about the DE:
  - Coefficients are continuous
  - \( a_n(x) \neq 0 \).

Differential Operator:

\[ D \Rightarrow \text{differentiation.} \]

\[ D y = \frac{d y}{d x}, \quad D (D y) = \frac{d^2 y}{d x^2} \]

- Linear Operator or Polynomial Operator

\[ L = a_2(x) D^2 + a_1(x) D + a_0(x) \]

We could then write

\[ a_2(x) \frac{d^2 y}{d x^2} + a_1(x) \frac{d y}{d x} + a_0(x) y = g(x) \]

\[ L(y) = g(x) \]

- Superposition Principle

Superposition = Sum

Our interest will be in regard to solutions of DE's;

\[ \Rightarrow \] \( \exists \) of 2 or more solutions to a DE

is also a soln

Theorem 4.2
Example Quiz Problem.

\[ y'' - y = 0 \]
\[ y_1 = \frac{1}{2} e^x \]
\[ y_2 = \frac{3}{2} e^{-x} \]

\[ y_1' = y_1' = \frac{1}{2} e^x \]
\[ y_2' = \frac{3}{2} e^{-x} \]

\[ y_1'' - y_1 = y_2'' - y_2 = (y_1'' + y_2'') - (y_1 + y_2) = 0 \]

- Linear Dependence / Independence

\[ c_1 f(x) + c_2 f_2(x) + \ldots + c_n f_n(x) = 0 \]
\[ c_1, c_2, c_n = 0 \]

Another way of looking at this is if 2 functions are linearly dependent, then one is a constant multiple of the other.

Linearly dependent

\[ f(x) = x \]
\[ f_2(x) = 2x \]

Linearly ind.

\[ f_1(x) = e^x \]
\[ f_2(x) = e^{x/2} \]

We are interested in the idea of linear dependence with regard to solutions of D.E.'s. We can test using matrix methods.
Wronskian:
\[ W(f_1, f_2, \ldots) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f'_1 & f'_2 & \cdots & f'_{n-1} \\ \vdots & \vdots & & \vdots \\ f^{(n-1)}_1 & f^{(n-1)}_2 & \cdots & f^{(n-1)}_n \end{vmatrix} \]

If \( W \neq 0 \) then the set of solutions is linearly independent.

1) Fundamental set of solutions:
\[ y_1, y_2, y_3, \ldots y_n \]

2) General solution of HDE
\[ y = C_1 y_1 + C_2 y_2 + C_3 y_3, \ldots + C_n y_n \]

**Problem 4.1.2.16**
Determine if \( 0, x \) and \( e^x \) are linearly independent.

\[
\begin{vmatrix} 0 & x & e^x & 0 & x \\ 0 & 1 & e^x & 0 & 1 \\ 0 & 0 & e^x & 0 & 0 \end{vmatrix}
\]

\[ 0 + 0 + 0 = 0 \]

Therefore, linearly dependent.

\[ y(0) + 0(x) + 0(e^x) = 0 \]

\[ C_1 + C_2 + C_3 = 0 \]
problem 4.1.2.26 verify if $y_1$ and $y_2$ are a solution:

$$y_1 = e^{x/2}, \quad y_2 = xe^{x/2}$$

$$W = \begin{vmatrix} e^{x/2} & xe^{x/2} \\ \frac{1}{2} e^{x/2} & \frac{1}{2} xe^{x/2} + e^{x/2} \end{vmatrix}$$

$$\frac{1}{2} xe^{x} + e^{x} - \frac{1}{2} xe^{x} = e^{x} \neq 0$$

$$y = c_1 e^{x/2} + c_2 xe^{x/2}$$

4.1.3 Nonhomogeneous Equations

Any function $y_p$ that satisfies the DE is said to be a particular solution to the equation.

Ex. $y'' + 9y = 27$

Let $y = 3, \quad y'' = 0$

$0 + 9(3) = 27$  

So $y_p = 3$ is a particular solution.

General Solution to NH DE

$$y = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n + y_p$$

$\uparrow$

Nonhomogeneous solution  
$\uparrow$

Complementary

$\uparrow$

Fit to NH part.

$y_p$ particular
Example 4.1.3.34

\[ 2x^2 \frac{d^2 y}{dx^2} + 5x \frac{dy}{dx} + y = x^2 - x \]

\[ y = C_1 x^{-\frac{3}{2}} + C_2 x^{-1} + \frac{1}{15} x^2 - \frac{1}{6} x \]

\[ y_c = C_1 x^{-\frac{3}{2}} + C_2 x^{-1} \]

\[ y_p = \frac{1}{15} x^2 - \frac{1}{6} x \]

\[ y_p' = -\frac{2}{15} x^{-\frac{1}{2}} \quad y_p'' = \frac{2}{15} \]

By substitution \((y_p)\)

\[ 2x^2 \left( \frac{2}{15} \right) + 5x \left( \frac{2x - \frac{1}{6}}{15} \right) + \left( \frac{1}{15} x^2 - \frac{1}{6} x \right) = x^2 - x \]

\[ \frac{4x^2}{15} + \frac{10}{15} x^2 + \frac{5}{15} x^2 - \frac{5}{6} x - \frac{1}{6} x = x^2 - x \]

\[ y_c' = -\frac{1}{2} x^{-\frac{3}{2}} + C_2 x^{-2} \]

\[ y_c'' = \frac{3}{4} C_1 x^{-\frac{5}{2}} + 2 C_2 x^{-3} \]

\[ 2x^2 \left[ \frac{3}{4} C_1 x^{-\frac{3}{2}} + 2 C_2 x^{-3} \right] + 5x \left[ -\frac{1}{2} C_1 x^{-\frac{3}{2}} - C_2 x^{-1} \right] + C_1 x^{-\frac{3}{2}} + C_2 = \]

\[ \left( \frac{3}{4} C_1 - \frac{1}{2} C_1 + C_1 \right) x^{-\frac{3}{2}} + \left[ 6 + 5 + 17 \right] \frac{C_2}{x} = 0 \]
Homogeneous Linear Equations

Consider a 1st order LDE, homogeneous.

\[ ay' + by = 0 \quad (1) \]

- We solved using SOU or LDE method.
- Could solve another way.

\[ y' = -\frac{b}{a} y = ky \]

More intuitive way:

\( y \) is a function whose derivative is a constant multiple of the function itself.

\( y \) is an exponential function

\( \Rightarrow \) let \( y = e^{mx} \)

\[ y' = me^{mx}, \quad m = \frac{b}{a} \]

Substituting \( y = e^{mx} \) into eqn. (1)

\[ a me^{mx} + b e^{mx} = 0 \]

\[ e^{mx} [am + b] = 0 \]

\( \Rightarrow \) now \( e^{mx} \neq 0 \) so,

\[ am + b = 0 \]

or \( m = -\frac{b}{a} \)

So, solution \( y = e^{-\frac{b}{a}x} \)
Example:

\[ 2y' + 5y = 0 \]
\[ 2m + 5 = 0 \quad m = -\frac{5}{2} \]
\[ y = C_1 e^{-\frac{5}{2}x} \]

Easy to validate by substitution.

- Extend to a 2nd Order Eq.
  
  \[ ay'' + by' + cy = 0 \]

Let solution be of the form \( y = e^{mx} \)

Then:

\[ y' = me^{mx} \]
\[ y'' = m^2 e^{mx} \]

\[ \left[ am^2 + bm + c \right] e^{mx} = 0 \]

Auxiliary Eqn.

- Need to solve quadratic eqn to find \( m_1 \) and \( m_2 \)

\[ b \pm \sqrt{b^2 - 4ac} \]

\[ 2a \]
We get 3 scenarios,

1. **distinct real roots** \( m_1 \) and \( m_2 \) then
   \[
y = c_1 e^{m_1 x} + c_2 e^{m_2 x}
   \]

2. **repeated root** \( m \)
   \[
y = e^{m x} \left[ c_1 + c_2 x \right]
   \]

3. **imaginary root** \( m = \alpha \pm \beta j \)
   \[
y = e^{\alpha x} \left[ c_1 \sin \beta x + c_2 \cos \beta x \right]
   \]

**Example**

4.3.10

\[
3y'' + y = 0
\]

\[
3m^2 + 1 = 0
\]

\[
m^2 = -\frac{1}{3}, \quad m = 0 \pm \frac{\sqrt{3}}{3} j
\]

\[
y = e^{\alpha x} \left[ c_1 \sin \frac{\sqrt{3}}{3} x + c_2 \cos \frac{\sqrt{3}}{3} x \right]
\]
Problem 4.3.32

\[ 4y'' - 4y' - 3y = 0 \]

\[ 4m^2 - 4m - 3 = 0 \]

\[ m = 1.5, \quad -0.5 \]

\[ y = C_1 e^{1.5x} + C_2 e^{-0.5x} \]

\[ y' = 1.5C_1 e^{1.5x} - 0.5C_2 e^{-0.5x} \]

\[ y(0) = (1 = C_1 + C_2) \]

\[ y'(0) = (5 = 1.5C_1 - 0.5C_2) \]

\[ 1 = C_1 + C_2 \]

\[ 10 = 3C_1 - C_2 \]

\[ 11 = 4C_1 \]

\[ C_1 = \frac{11}{4} \quad C_2 = -\frac{7}{4} \]

\[ y = \frac{11}{4} e^{1.5x} - \frac{7}{4} e^{-0.5x} \]
Higher Order Equations:

- Extend auxiliary equation concept
- More roots:
  \[ y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots + c_n e^{m_n x} \]
  
  If roots repeated:
  \[ e^{m_1 x}, x e^{m_2 x}, x^2 e^{m_3 x} \]

Example 4.3.16

\[ y''' - y = 0 \]

\[ m^3 - 1 = 0 \]

\[ m^3 = 1 \]

\[ m_1 = 1, \quad m_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2} i, \quad m_3 = -\frac{1}{2} - \frac{\sqrt{3}}{2} i \]

\[ y = c_1 e^x + e^{-\frac{1}{2} x} \left[ c_2 \sin \frac{\sqrt{3}}{2} x + c_3 \cos \frac{\sqrt{3}}{2} x \right] \]
Solution of Nonhomogeneous Equations

- will use approach different than the book here.
- Method is similar and works for the same groups of functions:
  (a) polynomials
  (b) exponentials
  (c) sines/cosines
- What these functions have in common is that upon taking multiple derivatives, the functions either repeat or go to 0.
- By considering the derivatives we can develop a general form for the particular solution $y_p$. This will be a linear combination of the different forms of the derivatives. Then solve for constants.
- Let's take an example:

  problem 4.5.2

  $y'' - 5y = x^2 - 2x$

  We know that the general solution (from Chapter 4.1) will consist of 2 parts:

  $y = yc + yp$

  Match soln to specific forcing function $f(x) = x^2 - 2x$
\[ m^2 - 5 = 0 \]
\[ m = \pm \sqrt{5} \]
\[ y_c = C_1 e^{\sqrt{5}x} + C_2 e^{-\sqrt{5}x} \]

We will now try to match the solution to \( f(x) = x^2 - 2x \)

See what forms of derivatives that are present:

- \( f(x) = x^2 - 2x \) \( \rightarrow \) \( x^2, x \)
- \( f'(x) = 2x - 2 \) \( \rightarrow \) \( x^- \) (constant)
- \( f''(x) = 2 \) \( \rightarrow \) no new form
- \( f'''(x) = 0 \) \( \rightarrow \) no new form

We can now compose a form of \( Y_p \) with undetermined coefficients. \( Y_p \) will contain all different forms of \( f'(x) \).

\[ Y_p = A + Bx + Cx^2 \]

Now, this \( Y_p \) function must satisfy the DE, so we will substitute it in:

\[ Y_p = A + Bx + Cx^2 \]
\[ Y_p' = B + 2Cx \]
\[ Y_p'' = 2C \]
\[ 2c - 5(A + Bx + Cx^2) = x^2 - 2x \]
\[ y'' \quad y \]

- Finally, equate (match) coefficients on the LHS and RHS.

\[ -5cx^2 = 1x^2 \quad c = -\frac{1}{5} \]
\[ -5bx = -2x \quad b = \frac{2}{5} \]
\[ 2c - 5a = 0 \quad A = -\frac{2}{25} \]
\[ 2 \left( -\frac{1}{5} \right) = 5A \]

\[ \frac{y_p}{y} = -\frac{2}{25} + \frac{2}{5}x - \frac{1}{5}x^2 \]

\[ y = c_1 e^{\sqrt{5}x} + c_2 e^{-\sqrt{5}x} - \frac{1}{5}x^2 + \frac{2}{5}x - \frac{2}{25} \]

Substitute into DE = 0

Substitute into DE = b
4.3.28

\[ y'''' + 2y'' + y' = 10 \]

\[ m^3 + 2m^2 + m = 0 \]

\[ m (m^2 + 2m + 1) = 0 \]

\[ m = 0, m_2 = m_3 = -1 \]

\[ y_c = C_1 e^0 + C_2 e^{-x} + C_3 x e^{-x} \]

\[ y_p = A \]

\[ y_p' = y_p'' = y_p''' = 0 \]

\[ 0 = 10 \]

We note that \( y_p = A \) is contained in the complementary solution. We should not be surprised then that when \( y_p = A \) is substituted into the DE we get 0.

This is a special case. When \( y_c \) contains the same form(s) as \( y_p \) we must multiply \( y_p \) by the lowest power of \( x \) necessary to generate a new form.

In this case

\[ \text{New} = x y_p = A x \]
\[ \mu P = Ax \quad \mu' = A \quad \mu'' = \mu''' = 0 \]

Substituting \( \mu = 10 \) \( \Rightarrow \) \( \mu P = 10x \)

\[ y = y_c + y_p \]

\[ y = c_1 + c_2 e^{-x} + c_3 x e^{-x} + 10x \]

**Problem 4.5.52**

Let's try one with a more complicated \( f(x) = x^2 e^{-x} \)

\[ y'' + 2y' + y = x^2 e^{-x} \]

Find \( y_c \)

\[ m^2 + 2m + 1 = 0 \]

\[ (m + 1)^2 = 0 \quad m = -1 \text{ repeated} \]

\[ y_c = c_1 e^{-x} + c_2 x e^{-x} \]
\[ f(x) = x^2 e^{-x} \]
\[ f'(x) = 2xe^{-x} + xe^{-x} \]
\[ f''(x) = 2e^{-x} - 2xe^{-x} - 2xe^{-x} + xe^{-x} \]
\[ f'''(x) = -2e^{-x} - 2e^{-x} + 2e^{-x} \]

\[ y_p = Ax^2e^{-x} + Bxe^{-x} + Ce^{-x} \]

But it contains 2 form that are
\[ y = e^{-x}, xe^{-x} \]

To get rid of them need to mult by \(x^2\)
\[ y_{pnew} = Ax^4e^{-x} + Bx^3e^{-x} + Cxe^{-x} \]

- Now we need to take derivatives.

\[ y_p' = 4Ax^3e^{-x} - Axe^{-x} + 3Bx^2e^{-x} - Bxe^{-x} \]
\[ + 2Cxe^{-x} - Ce^{-x} \]

\[ y_p'' = 12Ax^2e^{-x} - 4Ax^3e^{-x} \]
\[ - 4Ax^3e^{-x} + 12Ax^2e^{-x} \]
\[ - 3Bx^2e^{-x} + 6Bxe^{-x} \]
\[ + Bxe^{-x} - 3Bxe^{-x} \]
\[ + Cxe^{-x} - 2Cxe^{-x} \]
\[ + 2Cxe^{-x} + 2C \]

\[ 2y_p' = -2Ax^4e^{-x} + 8Ax^3e^{-x} \]
\[ - 2Bx^3e^{-x} + 6Bxe^{-x} + 4Cxe^{-x} \]
\[ - 2Cxe^{-x} \]

\[ y_p = \frac{Ax^4}{Ox^4} + \frac{Bx^3}{OBx^3} + \frac{Cxe^{-x}}{+12Axe^{-x} + 6Bxe^{-x} + 2Ce^{-x}} \]
\[ 12Ae^{-x} + 6Be^{-x} + 2Ce^{-x} = xe^{-x} \]

\[ A = \frac{1}{12}, \quad B = C = 0 \]

\[ y = y_c + y_p \]

\[ y = c_1e^{-x} + c_2xe^{-x} + \frac{1}{12}xe^{-x} \]

Finally let's try an IVP

\[ y'' + 5y' - 6y = 10e^{2x}, \quad y(0) = 1, \quad y'(0) = 1 \]

\[ m^2 + 5m - 6 = 0 \]

\[ (m + 6)(m - 1) = 0 \]

\[ y_c = c_1e^{-x} + c_2e^{-6x} \]

\[ y_p = Ae^{2x} \quad \text{Not same form as} \quad y_c ! \]

\[ y_p' = 2Ae^{2x} \]

\[ y_p'' = 4Ae^{2x} \]

\[ 4A + 10A - 6A = 10 \]

\[ 8A = 10 \]

\[ A = \frac{5}{4} \]
\[ y' = y_c + y_p = c_1 e^x + c_2 e^{-6x} + \frac{5}{4} e^{2x} \]
\[ y' = c_1 e^x - 6c_2 e^{-6x} + \frac{5}{2} e^{2x} \]
\[ y(0) = 1 = c_1 + c_2 + \frac{5}{4} \]
\[ y'(0) = 1 = c_1 - 6c_2 + \frac{5}{2} \]
\[ c_1 + c_2 = -\frac{1}{4} \]
\[ -(c_1 - 6c_2 = -\frac{3}{2}) \]
\[ 7c_2 = \frac{3}{2} - \frac{1}{4} = \frac{5}{4} \]
\[ c_2 = \frac{5}{28} \]
\[ c_1 + \frac{5}{28} = -\frac{1}{4} \]
\[ c_1 = -\frac{12}{28} = -\frac{3}{7} \]
\[ y = -\frac{3}{7} e^x + \frac{5}{28} e^{-6x} + \frac{5}{4} e^{2x} \]
The method of undetermined coefficients is a useful method, but has a severe limitation in that only certain forms work.

\[ a_2(x) y'' + a_1(x) y' + a_0(x) y = g(x) \]

Writing in std. form, \( a_2 \):

\[ y'' + p(x) y' + q(x) y = f(x) \]

We will use the same VOP approach as we used for 1st order DE's:

Recall \( y_p = u_1(x) y_1(x) \)

For a 2nd order DE,

\[ y_p = u_1(x) y_1(x) + u_2(x) y_2(x) \]

Then we take the derivatives,

\[ y_p' = u_1 y_1' + y_1 u_1' + u_2 y_2' + u_2 y_2' \]

\[ y_p'' = u_1 y_1'' + y_1 u_1'' + y_1 u_1'' + u_1 y_1'' + u_2 y_2'' + y_1 u_1'' + y_1 u_1'' + u_2 y_2'' + u_2 y_2'' \]

Substituting \( y_p \) and its derivatives into the std. form of the equation,

\[ y_p'' + p y_p' + q y_p = (u_1 y_1'' + u_2 y_2'') + P (u_1 y_1'' + u_1 y_1' + u_2 y_2'' + u_2 y_2') \]

\[ + (u_1 y_1'' + y_1 u_1'' + y_1 u_1'' + u_1 y_1'' + u_2 y_2'' + y_1 u_1'' + y_1 u_1'' + u_2 y_2'') \]

\[ = u_1 [y_1'' + p y_1' + q y_1] + u_2 [y_2'' + p y_2' + q y_2] \]

\[ + y_1 u_1'' + u_2 y_2'' \]

\[ + P [y_2 u_1' + y_2 u_1'] + y_1 u_1'' + y_2 u_2'' \]

\[ + \frac{d}{dx} (y_1 u_1') + \frac{d}{dx} (y_2 u_2') + P (y_1 u_1' + y_2 u_2') + y_1 u_1' + y_2 u_2' \]
We will assume
\[ y_1 u_1' + y_2 u_2' = 0 \]

This will eliminate the 1st 2 terms of the equation and the equation reduces to
\[ y_1 u_1' + y_2 u_2' = f(x) \]

This system can be solved by determinants.

\[ \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f(x) \end{bmatrix} \]

By Cramer's Rule,
\[ u_1' = \frac{W_1}{W} \quad u_2' = \frac{W_2}{W} \]

\[ W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \quad W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} = y_2 f(x) \]

\[ W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix} = \begin{vmatrix} y_1 f(x) \end{vmatrix} \]
Summary of VOP Method.

1. Solve associated HDE - get $y_1$ and $y_2$

2. Compute Wronskian
   \[ W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \]

3. Determine $u_1$ and $u_2$
   \[ u_1 = \int \frac{-y_2 f(x)}{W} \]
   \[ u_2 = \int \frac{y_1 f(x)}{W} \]

4. Form \( y_p = u_1y_1 + u_2y_2 \)
Problem 4.6.2

\[ y'' + y = \tan x \]

\[ m^2 + 1 = 0 \quad m = \pm j = 0 \pm j \]

\[ y_c = C_1 \sin x + C_2 \cos x \]

\[ y_1 = \sin x \quad y_2 = \cos x \]

\[ W = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\sin^2 x - \cos^2 x = -1 \]

\[ W_1 = -y_2 f(x) = \cos x \frac{\sin x}{\cos x} = -\sin x \]

\[ W_2 = y_1 f(x) = \frac{\sin x}{\cos x} = -\sin x \tan x = \frac{1 - \cos^2 x}{\cos^2 x} \]

\[ u_1' = \sin x \quad u_2' = -\sin x \tan x = \cos x - 1 \]

\[ u = \int \sin x \, dx = -\cos x \quad e^x + e^{-x} = 1 \]

\[ u_2 = \int -\sin x \tan x = \int \frac{(\cos^2 x - 1)}{\cos x} \, dx \]

\[ = \int \cos x \, dx - \int \sec x \, dx \]

\[ u_2 = \sin x - \ln \sec x + \tan x \]

\[ y_p = -\sin x \cos x + \cos x \sin x - \cos x \ln \sec x + \tan x \]

\[ y = C_1 \sin x + C_2 \cos x + \cos x \left[ \sin x - \ln \sec x + \tan x \right] - \sin x \cos x \]
Problem 4.6.20

\[ 2y'' + y' - y = x + 1 \]

\[ y(0) = 1 \]

\[ y'(0) = 0 \]

\[ 2m^2 + m - 1 = 0 \]

\[ (2m-1)(m+1) \quad m = \frac{1}{2}, m = -1 \]

\[ y = c_1 e^{\frac{x}{2}} + c_2 e^{-x} \]

\[ y_1 = e^{\frac{x}{2}} \quad y_2 = e^{-x} \]

\[ W = \begin{vmatrix} e^{\frac{x}{2}} & -e^{-x} \\ \frac{1}{2} e^{\frac{x}{2}} & -e^{-x} \end{vmatrix} = -e^{-x} - \frac{1}{2} e^{\frac{x}{2}} \]

\[ u_1' = -e^{-x} \left( -\frac{3}{2} e^{-x} \right) = \frac{3}{2} e^{-\frac{3x}{2}} \left( \frac{x+1}{2} \right) \]

\[ = \frac{1}{3} e^{-\frac{3x}{2}} (x+1) \]

\[ u_2' = \frac{\left( x+1 \right)}{2} e^{\frac{x}{2}} = -\frac{1}{3} e^{x} (x+1) \]

\[ u_1 = -e^{-\frac{3x}{2}} \left[ \frac{2}{3} x + z \right] \]

\[ u_2 = -\frac{1}{3} xe^x \]

\[ y = c_1 e^{\frac{x}{2}} + c_2 e^{-x} - \left[ \frac{2}{3} x + z \right] - \frac{1}{3} x \]

\[ = c_1 e^{\frac{x}{2}} + c_2 e^{-x} - x - 2 \]

\[ \int e^x = e^x \]

\[ \int e^{-x} = -e^{-x} \]
\[ y'(x) = \frac{1}{2} c_1 e^{\frac{x}{2}} - c_2 e^{-x} - 1 \]

\[ y'(0) = 1 = c_1 + c_2 - 2 \]

\[ y'(0) = 0 = \frac{1}{2} c_1 - c_2 - 1 \]

\[ c_1 + c_2 = 3 \]

\[ \frac{1}{2} c_1 - c_2 = 1 \]

\[ c_1 + \frac{3}{2} c_2 = 4 \]

\[ c_1 = \frac{8}{3} \]

\[ c_2 = -\frac{1}{3} \]

\[ y = \frac{8}{3} e^{\frac{x}{2}} + \frac{1}{3} e^{-x} - x - 2 \]
Reduction of Order:

- Sometimes we have or can find one solution of a 2nd order DE.
- Want to find a second linearly independent soln.
- The basic idea is that we might be able to use the known solution to reduce the 2nd order DE to a 1st order.
- If \( y_1 \) and \( y_2 \) are L.I. then

\[
\frac{y_2}{y_1} = \text{constant}
\]

or

\[
y_2(x) / y_1(x) = u(x)
\]

so

\[
y_2(x) = u(x) y_1(x)
\]

- We will substitute this into the DE and solve.

If we convert the DE to standard form

\[
y'' + Py' + Qy = 0
\]

\[
y' = uy_1' + y_1 u' \quad y'' = uy_1'' + 2y_1'u' + y_1 u''
\]

Substitution yields

\[
uy_1'' + 2y_1'u' + y_1 u'' + P (uy_1' + y_1 u') + Q (uy_1) = 0
\]

\[
u (y_1'' + Py_1' + Qy_1) + 2y_1'u' + y_1 u'' + Py_1 u' = 0
\]

\[
y_1'u'' + (2y_1' + Py_1) u' = 0 \quad \text{let} \ w = u'
\]

\[
y_1 w' + (2y_1' + Py_1) w = 0
\]
\[
\frac{dw}{w} + \left(2y' + p\right) \frac{dx}{y'} = 0
\]
\[
\frac{dw}{w} + 2 \frac{dy}{y'} dx + P dx = 0
\]
\[
\frac{dw}{w} + 2 \frac{dy}{y'} dx + P dx = 0
\]
\[
\int \frac{dw}{w} + 2 \int \frac{dy}{y'} + \int P dx = 0
\]
\[
\ln w + \ln y'^2 + \int P dx = c
\]
\[
\ln (wy')^2 = - \int P dx + c
\]
\[
wy' = c_1 e^{-\int P dx}
\]
\[
w = \frac{c_1 e^{-\int P dx}}{y'}
\]
but \[w = u'
\]
so,
\[
u = c_1 \int e^{-\int P dx} dx + c_2
\]
but \(c_1\)'s are arbitrary \((u's \text{ independent})\)
so choose \(c_1\)'s as \(c_1 = 1, c_2 = 0\)
\[
u = \frac{y'^2}{y'} = \int e^{-\int P dx} dx
\]
\[ y_2 = y_1 \int e^{-s \theta x} \, dx \]

**Example 4.2.2**

\[ y_1 = xe^{-x} \]
\[ y'' + 2y' + y = 0 \]

\[ y_2 = xe^{-x} \int \frac{e^{\frac{-2x}{x^2}}}{x^2} \, dx = xe^{-x} \int \frac{1}{x^2} \, dx \]

\[ y_2 = xe^{-x} \left(-\frac{1}{x}\right) = -e^{-x} \]

**Solution:** \[ c_1 xe^{-x} + c_2 e^{-x} \]

**Example 4.2.10**

\[ y_1 = x^2 \]
\[ y'' + \frac{2}{x} y' - \frac{6}{x^2} = 0 \]

\[ y_2 = x^2 \int \frac{e^{-\frac{2x}{x^2}}}{x^4} \, dx = x^2 \int \frac{e^{\ln (\frac{1}{x^2})}}{x^4} \, dx \]

\[ = x^2 \int \frac{1}{x^6} \, dx = \frac{1}{5} x^2 / x^5 = -\frac{1}{5} x^{-3} \]

\[ y = c_1 x^2 + c_2 / x^3 \]
Example 4.2.17

\[ y'' + y' = 1 \]
\[ u'' + u' = 1 \]
\[ w' + w = 1 \]

\[ \omega e^x = \int e^x + c \]
\[ \omega = 1 + ce^{-x} \]
\[ u = \int (1 + ce^{-x}) = x + ce^{-x} + c_2 \]
\[ y_2 = y, \quad u = 1 u = x + c_1 e^{-x} + c_2 \]

\[ y = y_1 + y_2 = x + c_1 e^{-x} + c_2 \]

Nonhomogeneous
2nd Order Linear Models.

- Consider the situation of free-damped motion. Such a system might be a shock absorber.

**Basis for Model - Newton's 2nd Law.**

\[ \sum F = ma \]

- We will write the model in words:
  \[ F_s + F_d = ma \]

- Note that both the spring force and resistive force oppose the motion. Thus, the signs are (-).

- Often the damping force is found to be proportional to the velocity.
  \[ F_d = -\beta v \]

- From Hooke's Law, the spring force is proportional to the displacement.
  \[ F_s = -kx \]

So,

\[ ma = -kx - \beta v \]
We will write this equation in terms of the position $x$.

$$m \frac{d^2 x}{dt^2} = -k x - \beta \frac{dx}{dt}$$

Then:

$$m \frac{d^2 x}{dt^2} + \beta \frac{dx}{dt} + k x = 0$$

Writing in standard form ($\div m$):

$$\frac{d^2 x}{dt^2} + \frac{\beta}{m} \frac{dx}{dt} + \frac{k}{m} x = 0$$

Let $2\lambda = \frac{\beta}{m}$, $\omega^2 = \frac{k}{m}$

$$\frac{d^2 x}{dt^2} + 2\lambda \frac{dx}{dt} + \omega^2 x = 0$$

Writing the auxiliary equation:

$$m^2 + 2\lambda m + \omega^2 = 0$$

$$m = -\lambda \pm \sqrt{\lambda^2 - \omega^2}$$

$$m = -\lambda \pm \sqrt{\lambda^2 - \omega^2}$$
as we saw before, the \( \sqrt{\lambda^2 - \omega^2} \) determines the form of the solution.

**Case 1** \[ \sqrt{\lambda^2 - \omega^2} > 0 \]

- damping is large when compared with spring force.
- will get
  \[
  m_1 = -\lambda + \sqrt{\lambda^2 + \omega^2} \\
  m_2 = -\lambda - \sqrt{\lambda^2 + \omega^2} \\
  x = e^{-\lambda t} \left[ c_1 e^{\sqrt[2]{\lambda^2 - \omega^2} t} + c_2 e^{-\sqrt[2]{\lambda^2 - \omega^2} t} \right]
  \]
- no oscillation (overdamped)
- motion will exponentially tend due to \( e^{-\lambda t} \) term.

**Case 2** \[ \lambda^2 - \omega^2 = 0 \]

\[ x = e^{-\lambda t} \left[ c_1 + c_2 t \right] \]
- critically - damped
- no oscillation, but faster return to equilibrium

**Case 3** \[ \lambda^2 - \omega^2 < 0 \] imaginary

\[ x = e^{-\lambda t} \left[ c_1 \sin (\lambda^2 - \omega^2) t + c_2 \cos (\lambda^2 - \omega^2) t \right] \]
- oscillation that are damped.
Undamped Resonance

\[ \frac{d^2x}{dt^2} + \omega^2 x = F_0 \sin \omega t \]

\[ m^2 + \omega^2 = 0 \quad m = \sqrt{-\omega^2} = \omega j \]

\[ x_c = C_1 \sin \omega t + C_2 \cos \omega t \]

\[ x_p = t A \sin \omega t + t B \cos \omega t \]

\[ x_p' = \omega A \cos \omega t + A \sin \omega t - B t \omega \sin t + B \cos t \omega \]

\[ x_p'' = -\omega^2 A \sin \omega t + \omega A \cos \omega t + \omega A \cos \omega t - \omega^2 B t \omega \sin \omega t - B \omega \sin \omega t \]

\[ x_p'' = -\omega^2 A \sin \omega t + \omega^2 B t \omega \sin \omega t - 2 \omega A \cos \omega t - 2 B \omega \sin \omega t \]

\[ x_p' + \omega^2 A t \sin \omega t + \omega^2 B t \omega \sin \omega t \]

\[ 2 \omega A t \omega + 2 B \omega \sin \omega t = F_0 \sin \omega t \]

\[ A = 0 \]

\[ x(0) = x'(0) = 0 \]

\[ B = -\frac{F_0}{2 \omega} \]

\[ x = C_1 \sin \omega t + (C_2 + \frac{F_0}{2 \omega}) \cos \omega t \]

\[ C_2 = 0 \]

\[ x' = \omega C_1 \cos \omega t - \frac{F_0}{2 \omega} \left( \cos \omega t - \omega t \sin \omega t \right) \]

\[ 0 = \omega C_1 - \frac{F_0}{2 \omega} \]

\[ C_1 = \frac{F_0}{2 \omega^2} \]

\[ x = \frac{F_0}{2 \omega^2} \sin \omega t - \frac{F_0 t}{2 \omega} \cos \omega t \]
Forced - Damped Motion.

1. Recall the governing equation:
   \[ ma = -bx - \beta v \]

2. We need to add a term to account for the external force that will be in the direction of the motion, \( f(t) \).

3. This has the effect of changing our system to a nonhomogeneous one.

4. Writing the model:
   \[ m \frac{d^2x}{dt^2} + \beta \frac{dx}{dt} + bx = f(t) \]

Let's take a look at a problem.

5.1.30

\[ m = 1 \text{ slug} \quad b = \frac{516}{ft} \quad x_0 = +1 \quad \frac{dx(t)}{dt} \]

\[ \beta = 2 \]

\[ \frac{dx}{dt^2} + 2 \frac{dx}{dt} + 5x = 12 \cos 2t + 3 \sin 2t \]

Solve homogeneous eqn.

\[ m^2 + 2m + 5 = 0 \]

\[ m = -1 \pm \frac{\sqrt{4 - 4(1)(5)}}{2} = -1 \pm 2i \]

\[ m = 1 \pm i \]

\[ x_c = e^{-t} (c_1 \sin 2t + c_2 \cos 2t) \]
\[ x_p = A \sin 2t + B \cos 2t \]
\[ x_p' = 2A \cos 2t - 2B \sin 2t \]
\[ x_p'' = -4A \sin 2t - 4B \cos 2t \]

\[ -4A \sin + 4B \cos + 4A \cos - 4B \sin + 5A \sin + 5B \cos = 12 \cos \]
\[ -4B + A = 3 \]
\[ B + 4A = 12 \]
\[ 17A = 51 \]

\[ x_p = 3 \sin 2t \]

\[ x(t) = e^{-t} \left[ C_1 \sin 2t + C_2 \cos 2t \right] + 3 \sin 2t \]

\[ x(0) = 1 = 1 \left[ 0 + C_2 \right] \]

\[ C_2 = 1 \]

\[ x(t) = e^{-t} \left[ C_1 \sin 2t + \cos 2t \right] + 3 \sin 2t \]

\[ x'(0) = 5 = e^{-t} \left[ 2C_1 \cos 2t - 2 \sin 2t \right] + 6 \cos 2t \]
\[ -e^{-t} \left[ C_1 \sin 2t + \cos 2t \right] \]

\[ 5 = 1 \left[ 2C_1 \right] + 6 + 1 \]
\[ 2C_1 = 0 \]
\[ C_1 = 0 \]

\[ x(t) = e^{-t} \cos 2t + 3 \sin 2t \]

\[ \uparrow \text{ transient} \]
\[ \uparrow \text{ steady state oscillation} \]